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Ultra-chaotic property of Navier–Stokes turbulence

Shijie Qin [a,b], Kun Xu [a], Shijun Liao [b,c,*]

[a] Department of Mathematics, Hong Kong University of Science and Technology, Hong Kong, China
[b] State Key Laboratory of Ocean Engineering, Shanghai 200240, China
[c] School of Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai 200240, China

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ABSTRACT

A chaotic system is called ultra-chaos when its statistics have sensitivity dependence on initial condition and/or other small disturbances. In this paper, using two-dimensional turbulent Kolmogorov flow as an example, we illustrate that tiny variation of initial condition of Navier–Stokes equations can lead to huge differences not only in spatiotemporal trajectory but also in flow symmetry and its statistics. Here, to avoid the influence of artificial numerical noise, we apply "clean numerical simulation" (CNS) proposed by Liao in 2009. Different from direct numerical simulation (DNS), CNS can guarantee that the numerical noise is reduced to such a desired low level that it is negligible in a time interval that is long enough for calculating statistics. This discovery highly suggests that the Navier–Stokes turbulence (i.e. turbulence governed by the Navier–Stokes equations) should be an ultra-chaos, say, small disturbances must be considered even from viewpoint of statistics. Some fundamental characteristics of turbulence model are discussed and suggested.

1. Introduction

The Navier–Stokes (NS) equations are widely used as a mathematical model to describe turbulent flows. This model of turbulence is so important in physics and engineering that it has been listed as one of the seven Millennium Prize Problem [1]. Many researchers, such as Deissler [2], Bffetta & Musacchio [3], Berera & Ho [4], and so on, reported that spatiotemporal trajectories of the NS equations have sensitivity dependence on initial conditions, corresponding to "trajectory instability". In 2023, Ge, Rolland and Vassilicos [5] pointed out that the uncertainty of a three-dimensional NS turbulence (i.e. turbulent flow governed by the NS equations) grows exponentially. All of these provide us rigorous evidences that the NS turbulence is chaotic, say, all tiny (natural/artificial) stochastic disturbances increase exponentially to a macroscopic level of spatiotemporal trajectories due to the famous "butterfly-effect" of chaos [6].

It is well known that, for turbulent flows, statistical results are more important than spatio-temporal trajectories. Normally, although the trajectory of a chaotic system is sensitive to small disturbance, its statistic results are stable. Such kind of chaos is called "normal-chaos". However, it has been found that statistical results of some chaotic systems are even sensitive to small disturbances, corresponding to "statistic instability", called "ultra-chaos" [7–10]. Obviously, ultra-chaos has higher disorder than normal-chaos.

In this paper, using the two-dimensional (2D) turbulent Kolmogorov flow as an example, we illustrate in Section 3 that the NS turbulence might be ultra-chaos, say, even its statistic results might have sensitivity dependence on initial condition. It means that small disturbances *cannot* be neglected for the Navier–Stokes turbulence even from the viewpoint of *statistics*. But unfortunately, tiny stochastic disturbances are in fact indeed neglected in the Navier–Stokes equations. Some fundamental characteristics of a turbulence model in general are discussed in Section 4.

2. Mathematical model of Navier–Stokes turbulence

Consider the 2D incompressible Kolmogorov flow [11–13] in a square domain $[0, L]^2$ with the periodic boundary condition in space, under the so-called Kolmogorov forcing that is stationary, monochromatic, and periodically varying in space with an integer n_K describing the forcing scale and χ the forcing amplitude per unit mass of fluid, respectively. Using the length scale $L/2\pi$ and the time scale $\sqrt{L/2\pi\chi}$, the governing equation of this 2D Kolmogorov flow, say, the non-dimensional Navier–Stokes equation in the form of stream function ψ, reads

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) + \frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} - \frac{1}{Re}\nabla^4\psi + n_K\cos(n_K y) = 0, \tag{1}$$

subject to the periodic boundary condition

$$\psi(x, y, t) = \psi(x \pm 2\pi, y, t) = \psi(x, y \pm 2\pi, t) = \psi(x \pm 2\pi, y \pm 2\pi, t), \tag{2}$$

* Corresponding author at: School of Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai 200240, China.
E-mail address: sjliao@sjtu.edu.cn (S. Liao).

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where

$$Re = \frac{\sqrt{\chi}}{\nu}\left(\frac{L}{2\pi}\right)^{\frac{3}{2}}$$

is the Reynolds number, ν denotes the kinematic viscosity, ψ is the stream function defined by

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x},$$

u and v represent horizontal and vertical velocities, t denotes the time, $x, y \in [0, 2\pi]$ are horizontal and vertical coordinates, ∇^2 is the Laplace operator, $\nabla^4 = \nabla^2\nabla^2$, and

$$\frac{\partial(a,b)}{\partial(x,y)} = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}$$

is the Jacobi operator, respectively. In order to investigate a relatively strong state of turbulent flow, we choose $n_K = 16$ and $Re = 2000$ for all cases considered in this paper.

To investigate its sensitivity dependence on initial condition, let us consider the following three different initial conditions:

$$\begin{aligned}\Phi_{1,0}(x,y) = \psi(x,y,0) &= -\frac{1}{2}\left[\cos(4x+4y)+\cos(4x-4y)+\sin(4x+4y)\right]\\ &\quad + 10^{-10}\left[\sin(x+y)+\sin(x-y)\right], \end{aligned} \tag{3}$$

$$\begin{aligned}\Phi_{2,0}(x,y) = \psi(x,y,0) &= -\frac{1}{2}\left[\cos(4x+4y)+\cos(4x-4y)+\sin(4x+4y)\right]\\ &\quad + 10^{-10}\left[\sin(2x+y)+\sin(2x-y)\right], \end{aligned} \tag{4}$$

$$\begin{aligned}\Phi_{3,0}(x,y) = \psi(x,y,0) &= -\frac{1}{2}\left[\cos(4x+4y)+\cos(4x-4y)+\sin(4x+4y)\right]\\ &\quad + 10^{-10}\left[\sin(4x+y)+\sin(4x-y)\right], \end{aligned} \tag{5}$$

corresponding to the three turbulent Kolmogorov flows with different spatial symmetries, as described below. Note that the deviations between these three initial conditions $\Phi_{1,0}(x,y)$, $\Phi_{2,0}(x,y)$, and $\Phi_{3,0}(x,y)$ are quite small over the whole field, i.e.

$$\left|\Phi_{m,0}(x,y) - \Phi_{n,0}(x,y)\right| \leq 4\times 10^{-10}, \qquad x,y \in [0,2\pi], \tag{6}$$

where $m, n = 1, 2, 3$ but $m \neq n$. Note that the difference is so small that these three initial conditions are *almost the same*. The tiny number 4×10^{-10} has no special meanings in physics, since, due to the butterfly-effect of chaos, no matter how small these disturbances are, they certainly increase to a macro-level sooner or later, as illustrated in [14, 15]. The spatiotemporal evolutions of such kind of tiny deviations are neglected by the traditional numerical algorithms (even including the direct numerical simulation, i.e. DNS), but can be accurately obtained by clean numerical simulation (CNS) [15–21], which will be briefly described in Section 3.

It should be emphasized that the initial condition $\Phi_{1,0}(x,y)$ given by (3) has the spatial translation symmetry A:

$$\mathcal{S}(x,y) = \mathcal{S}(x+\pi, y+\pi), \tag{7}$$

the initial condition $\Phi_{2,0}(x,y)$ given by (4) has the spatial translation symmetry B:

$$\mathcal{S}(x,y) = \mathcal{S}(x+\pi/2, y+\pi), \tag{8}$$

and the initial condition $\Phi_{3,0}(x,y)$ given by (5) has the spatial translation symmetry C:

$$\mathcal{S}(x,y) = \mathcal{S}(x+\pi/2, y), \tag{9}$$

respectively. It should be emphasized that the above-mentioned three spatial symmetries are quite different. Note that the vorticity $\omega = \nabla^2\psi$ retains the same spatial symmetry as the stream function ψ. It is well-known that turbulence looks rather disorderly. Can the spatial symmetry of the initial conditions maintain for the turbulent Kolmogorov flows under consideration?

3. Ultra-chaotic property of Navier–Stokes turbulence

As pointed out by many researchers [2–5], Navier–Stokes turbulence (i.e. turbulent flows governed by NS equations) are chaotic. It is a pity that numerical noises are avoidable for all algorithms including DNS. So, due to the butterfly-effect of chaos [6], spatiotemporal trajectories given by the traditional numerical algorithms including DNS should be quickly polluted by numerical noises, i.e. far away from the true solution of NS turbulence. To obtain an accurate enough spatiotemporal trajectory of the NS turbulence, whose numerical noise is much smaller than that of the true solution and thus is negligible in a long enough interval of time, we use the CNS [15–21] to solve Eq. (1) subject to the periodic boundary condition (2) and one initial condition among (3)–(5). The three CNS results are named by Flow CNS-1 corresponding to (3), Flow CNS-2 corresponding to (4), and Flow CNS-3 corresponding to (5), respectively.

For the sake of simplicity, the basic ideas of the CNS [14–21] are briefly described here. Firstly, to decrease the spatial truncation error to a small enough level, we use an uniform mesh $N^2 = 1024^2$ and adopt the Fourier pseudo-spectral method for spatial approximation with the $3/2$ rule for dealiasing. In this way, the corresponding spatial resolution is fine enough for the considered three kinds of turbulent Kolmogorov flows: the grid spacing is less than the averaged Kolmogorov scale and enstrophy dissipative scale, according to Pope [22] and Boffetta & Ecke [23], which is carefully checked and confirmed later in this paper. But, unlike DNS, in order to decrease the temporal truncation error to a small enough level, we use the Mth-order Taylor expansion with the time step $\Delta t = 10^{-3}$. Especially, we use *multiple-precision* with N_s significant digits for all physical/numerical variables and parameters so as to decrease the round-off error to a small enough level. In addition, another CNS result is given by the same CNS algorithm but with even smaller numerical noise, i.e. using even larger M and N_s than those mentioned above, which guarantees (by comparison) that the numerical noise of the former CNS result is rigorously negligible throughout the whole time interval $t \in [0, 300]$ so that it can be used as a "clean" benchmark solution of the NS turbulence. It is found that we need to adopt $M = 60$ & $N_s = 110$ for Flow CNS-1, $M = 100$ & $N_s = 230$ for Flow CNS-2, and $M = 120$ & $N_s = 430$ for Flow CNS-3 in the same time interval $t \in [0, 300]$, respectively, because the three different turbulent flows have different noise-growing exponents so that different values of M and N_s in the frame of CNS are necessary to decrease their respective numerical noises to the corresponding small enough levels. Note that the self-adaptive strategy [24] and parallel computing are adopted to dramatically increase the computational efficiency of the CNS algorithm. For further details about the CNS algorithm mentioned above, please refer to Qin et al. [14] and Liao's book [17]. Note that the related code of CNS can be downloaded via Github (https://github.com/sjtu-liao/Ultrachaos-2D-Kolmogorov). The detailed comparisons of the spatial symmetry and the statistical results between the three different turbulent flows, i.e. Flow CNS-1, Flow CNS-2, and Flow CNS-3, are described below in details.

Firstly, let us compare the spatial symmetry of the flow field. It is found that the vorticity field ω of Flow CNS-1 remains the *same* spatial translation symmetry (7) as its initial condition (3) throughout the *whole* time interval $t \in [0, 300]$, for example at $t = 100$ as shown in Fig. 1(a). Similarly, the vorticity field ω of Flow CNS-2 remains the same spatial translation symmetry (8) as the initial condition (4) throughout the whole time interval $t \in [0, 300]$, and the vorticity field ω of Flow CNS-3 remains the same spatial translation symmetry (9) as the initial condition (5) throughout the whole time interval $t \in [0, 300]$, for examples at $t = 100$ as shown in Fig. 1(b) and (c), respectively. Here, we had to emphasize that, as shown by (6), the deviations between the three initial conditions (3)–(5) are quite small, which however, due to the butterfly-effect, lead to three completely distinct turbulent flows with different spatial symmetries. This clearly illustrates that the spatial

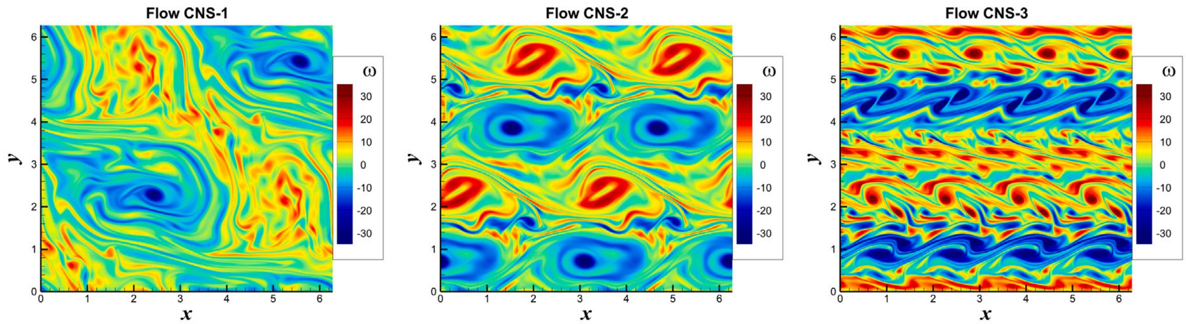

Fig. 1. Vorticity fields ω at $t = 100$ of the 2D turbulent Kolmogorov flow governed by (1) and (2) in the case of $n_K = 16$ and $Re = 2000$, given by CNS subject to the initial conditions (3) (left, marked by Flow CNS-1), (4) (middle, marked by Flow CNS-2), and (5) (right, marked by Flow CNS-3), respectively. (See the supplementary movie for the temporal evolution of vortex structure on the website https://github.com/sjtu-liao/Ultrachaos-2D-Kolmogorov/tree/main/movies).

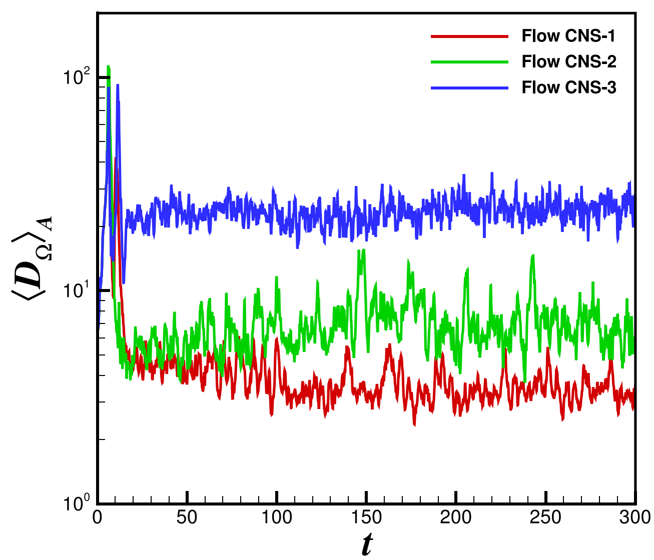

Fig. 2. Comparison of time histories of the spatially averaged enstrophy dissipation rate $\langle D_\Omega\rangle_A$ of the 2D turbulent Kolmogorov flow governed by (1) and (2) in the case of $n_K = 16$ and $Re = 2000$, given by Flow CNS-1 (red line), Flow CNS-2 (green line), Flow CNS-3 (blue line), respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

symmetry of the Navier–Stokes turbulence has sensitivity dependence on the initial condition.

Here it should be emphasized that, Liao [25] currently proved such a mathematical theorem of spatial symmetry invariance for Kolmogorov flow that, if the initial condition of the Kolmogorov flow under periodic boundary condition has a spatial symmetry, its exact solution maintains the same spatial symmetry *forever*, i.e. for all $t > 0$. Note that a similar theorem for the three-dimensional incompressible Navier–Stokes equations is proved in the appendix of [25]. Certainly, this mathematical theorem provide a simple way to examine the correctness and accuracy of numerical simulations of Navier–Stokes turbulence. It should be emphasized that, all of our CNS results **agree** well with this mathematical theorem [25], but all of our DNS results **violate** it as mentioned later.

Secondly, let us compare some statistic results. Fig. 2 compares the time histories of the spatially averaged enstrophy dissipation rate $\langle D_\Omega\rangle_A$ given by Flow CNS-1, Flow CNS-2, and Flow CNS-3. It is found that the obvious deviations between these time histories appear at $t \approx 10$ when the tiny disturbances caused by $10^{-10}[\sin(x+y)+\sin(x-y)]$ in the initial condition (3), by $10^{-10}[\sin(2x+y)+\sin(2x-y)]$ in the initial condition (4), and by $10^{-10}[\sin(4x+y)+\sin(4x-y)]$ in the initial condition (5) just increase to the macro level. Because of the different spatial symmetries of the three turbulent flows, there are obvious deviations between the probability density functions (PDFs) of the kinetic energy $E(x,y,t)$, enstrophy $\Omega(x,y,t)$, kinetic energy dissipation $D(x,y,t)$, and enstrophy dissipation rate $D_\Omega(x,y,t)$, as illustrated in Fig. 3(a) to (d), respectively. For calculation of these PDFs, the spatial and temporal intervals of integration are $x, y \in [0, 2\pi]$ and $t \in [100, 300]$ corresponding to a relatively stable state of turbulence (as illustrated in Fig. 2). Besides, as shown in Fig. 4(a) and (b), the spatiotemporally averaged kinetic energy dissipation rate $\langle D\rangle_{x,t}(y)$ and enstrophy dissipation rate $\langle D_\Omega\rangle_{x,t}(y)$ of Flow CNS-1, Flow CNS-2, and Flow CNS-3 are also quite different. Moreover, as illustrated in Fig. 5(a) to (d), there exist the significant deviations between the results of Flow CNS-1, Flow CNS-2, and Flow CNS-3 for the spatiotemporally averaged horizontal velocity $\langle u\rangle_{x,t}(y)$ and Reynolds-stress associated terms $\langle u'u'\rangle_{x,t}(y)$, $\langle v'v'\rangle_{x,t}(y)$, and $\langle u'v'\rangle_{x,t}(y)$, which are of great significance in turbulence modeling. Here, the definitions of $\langle\ \rangle_{x,t}$ and other statistic operators are given in Appendix.

The above-mentioned deviations can be confirmed once again by comparing the temporal averaged kinetic energy spectra $\langle E_k\rangle_t$ of Flow CNS-1, Flow CNS-2, Flow CNS-3, as shown in Fig. 6: the three turbulent flows have different kinetic energy spectra, say, different distributions of kinetic energy. It is very interesting that Flow CNS-3 contains more kinetic energy than Flow CNS-2, and Flow CNS-2 contains more kinetic energy than Flow CNS-1, although all the three turbulent flows satisfy the $-5/3$ power law when the wavenumber k is less than the forcing scale n_K, which corresponds to the inverse cascade of energy flux in 2D turbulence [23,26]. It is worth noting that, for $k > n_K$, there exists the direct cascade of energy flux where the spectra (as shown in Fig. 6) are steeper than the Kraichnan prediction, i.e., -3 power law, which is due to the effect of finite resolution and agrees with the previous investigation [26].

According to Liao's mathematical theorem of the spatial symmetry invariance for the 2D Kolmogorov flow [25], Flow CNS-1, Flow CNS-2, and Flow CNS-3 should have different spatial symmetries. Thus, their statistics certainly must be different! This confirms the correctness of our CNS results once again.

Here we must emphasize that the deviations between the three initial conditions considered in this investigation are rather tiny, as shown by (6), which are several orders of magnitude smaller than natural/artificial stochastic disturbances. However, these tiny deviations can lead to macroscopic, obvious differences in the spatial symmetries and statistic results of the corresponding turbulent flows given by CNS, whose numerical noises are negligible compared with their true solutions throughout a long enough interval of time $t \in [0, 300]$. All of these results provide us rigorous evidence that the corresponding Navier–Stokes turbulences of the 2D Kolmogorov flows are ultra-chaotic [7, 8], say, their small disturbances *cannot* be neglected even from the viewpoint of *statistics*.

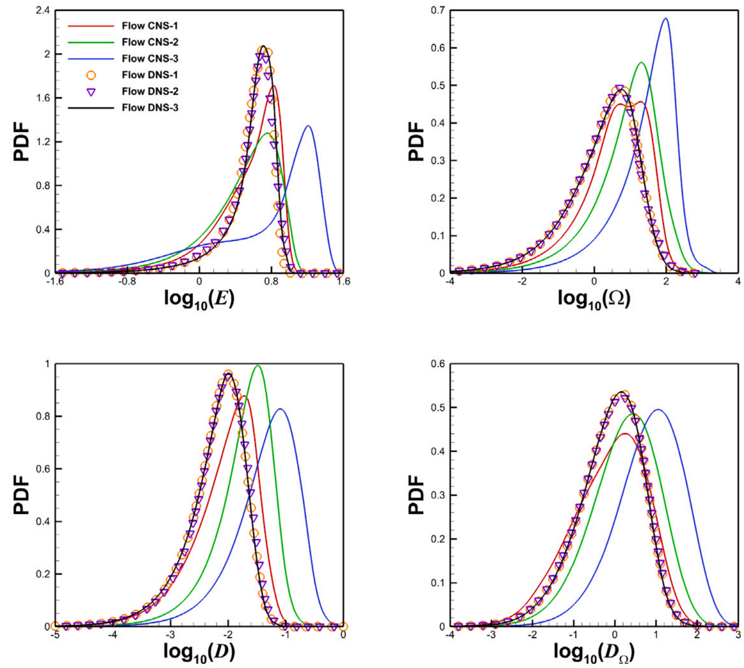

Fig. 3. Comparison of probability density functions (PDFs) of the (top-left) kinetic energy $E(x, y, t)$, (top-right) enstrophy $\Omega(x, y, t)$, (bottom-left) kinetic energy dissipation $D(x, y, t)$, and (bottom-right) enstrophy dissipation rate $D_\Omega(x, y, t)$ of the 2D turbulent Kolmogorov flow, governed by (1) and (2) in the case of $n_K = 16$ and $Re = 2000$, given by Flow CNS-1 (red line), Flow CNS-2 (green line), Flow CNS-3 (blue line), Flow DNS-1 (orange circle), Flow DNS-2 (purple inverted triangle), Flow DNS-3 (black line), respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

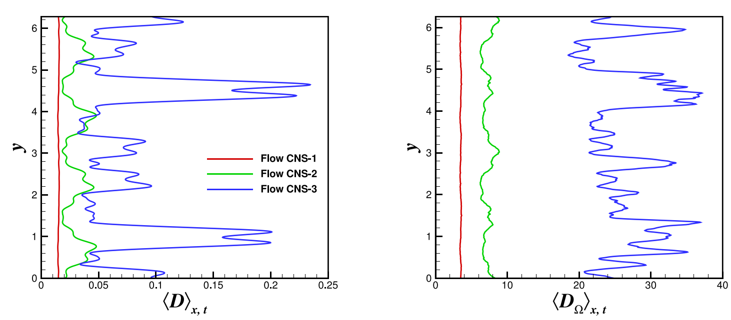

Fig. 4. Comparison of the spatiotemporally averaged (left) kinetic energy dissipation rates $\langle D\rangle_{x,t}(y)$ and (right) enstrophy dissipation rates $\langle D_\Omega\rangle_{x,t}(y)$ of the 2D turbulent Kolmogorov flow governed by Eqs. (1) and (2) in the case of $n_K = 16$ and $Re = 2000$, given by Flow CNS-1 (red line), Flow CNS-2 (green line), Flow CNS-3 (blue line), respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Note that all results reported above are given by CNS, which remain the same spatial symmetry as that of the corresponding initial condition. In other words, all of our CNS results agree well with Liao's mathematical theorem of the spatial symmetry invariance of the Kolmogorov flow [25]: this fact guarantees the correctness of our CNS results. To further demonstrate the advantage of the CNS, let us make a direct comparison with the conventional direct numerical simulation (e.g., DNS using the Runge–Kutta method and the single/double precision). Using the same initial conditions, the same boundary condition, and the same physical parameters as described above, the governing Eq. (1) is solved numerically by a traditional DNS algorithm, i.e., the fourth-order Runge–Kutta's method using the time step $\Delta t = 1 \times 10^{-4}$ with double precision. We name the DNS results subject to the three initial conditions (3), (4), and (5) as follows: Flow DNS-1, Flow DNS-2, and Flow DNS-3, respectively. It is found that Flow DNS-1, Flow DNS-2, and Flow DNS-3 quickly lose the corresponding spatial symmetry completely, for example as shown in Fig. 7 at $t = 100$. This is completely different from the CNS results, as shown in Fig. 1, which remain the spatial symmetry as the corresponding initial conditions. Thus, these DNS results obviously violate the mathematical theorem of spatial symmetry invariance of the Kolmogorov flow: this is a rigorous evidence that these DNS results are indeed quickly polluted by numerical noises badly. In addition, the PDFs of $E(x, y, t)$, $\Omega(x, y, t)$, $D(x, y, t)$, and $D_\Omega(x, y, t)$ given by all DNS results are the same, but different from those of CNS results, as shown in Fig. 3, where the spatial and temporal intervals of integration are $x, y \in [0, 2\pi]$ and $t \in [100, 300]$. Thus, the DNS results lose the fundamental property of spatial symmetry invariance of the Kolmogorov flows and thus **cannot** reveal the truth. Note that, from the viewpoint of DNS, the small disturbances in the three initial conditions could not lead to distinct deviations in statistics so that the turbulent Kolmogorov flows should be a normal-chaos, but not an ultra-chaos: unfortunately all of these are wrong, as mentioned above. In other words, these DNS results are quite misleading: using DNS, one **cannot** discover the ultra-chaotic property of the turbulent Kolmogorov flow.

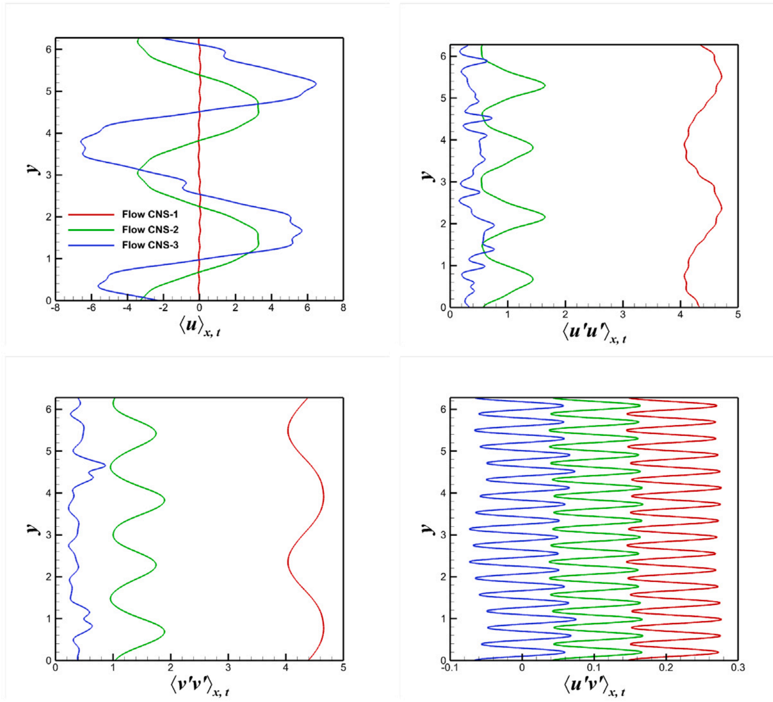

Fig. 5. Comparison of the spatiotemporally averaged (top-left) horizontal velocities $\langle u \rangle_{x,t}(y)$, (top-right) normal stresses $\langle u'u' \rangle_{x,t}(y)$, (bottom-left) normal stresses $\langle v'v' \rangle_{x,t}(y)$, and (bottom-right) shear stresses $\langle u'v' \rangle_{x,t}(y)$ of the 2D turbulent Kolmogorov flow, governed by (1) and (2) in the case of $n_K = 16$ and $Re = 2000$, given by Flow CNS-1 (red line), Flow CNS-2 (green line), Flow CNS-3 (blue line), respectively. These stresses are defined in Appendix. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

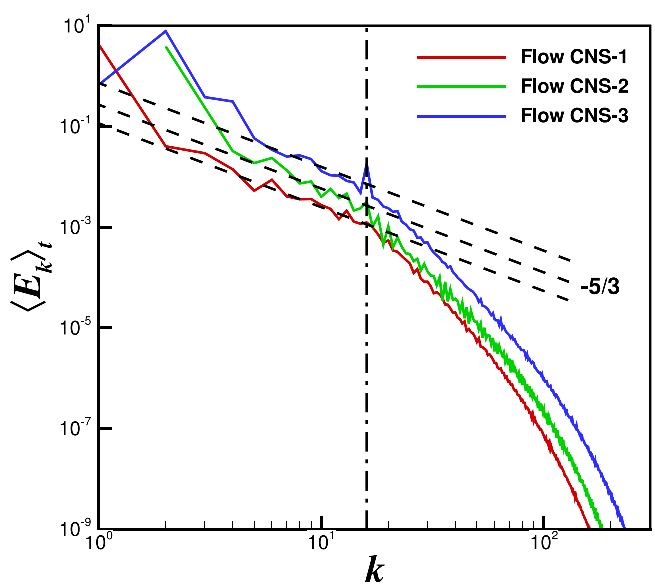

Fig. 6. Comparison of the temporal averaged kinetic energy spectra $\langle E_k \rangle_t$ of the 2D turbulent Kolmogorov flow governed by (1) and (2) in the case of $n_K = 16$ and $Re = 2000$, given by Flow CNS-1 (red line), Flow CNS-2 (green line), Flow CNS-3 (blue line), respectively, where the black dashed line corresponds to $-5/3$ power law and the black dash-dot line denotes $k = n_K = 16$. The spectrucm is defined in Appendix. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

4. Concluding remarks and discussions

In this paper, using the 2D turbulent Kolmogorov flow as an example, we illustrate that the Navier–Stokes turbulence (i.e. a turbulent flow governed by Navier–Stokes equations subject to initial/boundary conditions) is ultra-chaotic, say, *not only* its spatiotemporal trajectory *but also* its statistical results have sensitivity dependence on initial condition. Obviously, ultra-chaos has higher disorder than normal-chaos.

Currently, it has been reported [27] that the final state and statistics of a 3D centrifugal convection has sensitivity dependence on initial condition, which can be regarded as another example of the ultra-chaotic property of turbulence.

Before the following discussions, we should emphasize here that the "real turbulence" in practice is one thing, the "Navier–Stokes turbulence" governed by Navier–Stokes equations (i.e. a mathematical model of turbulence) is a completely **different** thing. More importantly, due to the butterfly-effect of chaos, the "exact" solution of the Navier–Stokes turbulence is one thing, unfortunately its "numerical simulation" might be a completely **different** thing. It seems that, as long as we investigate turbulent flows by numerical methods, we **artificially** add an influence on spatiotemporal trajectory that might be **non-negligible** when time-interval is long enough. Whether or not a mathematical model of turbulence can well describe a real turbulence needs a lots of mathematical, numerical and experimental evidences.

The discovery that the Navier–Stokes turbulence is ultra-chaotic has important meanings in theory. Firstly, from mathematical viewpoint, the situation of Navier–Stokes turbulence at an arbitrary time $t = \tau > 0$ can be regarded as a new starting point. Thus, for the ultra-chaotic Navier–Stokes turbulence, its statistic results should be sensitive to small disturbances at *all* time $t \geq 0$, therefore all of the natural and/or artificial, small stochastic noises should be considered. But unfortunately, Navier–Stokes turbulence in fact neglects *all* stochastic disturbances for $t > 0$. Secondly, from physical viewpoint, tiny stochastic natural/artificial noises are unavoidable in practice. Therefore, from both of mathematical and physical viewpoints, it is unreasonable to neglect the influences of small stochastic disturbances on the Navier–Stokes turbulence, which in some cases could greatly affect statistics of turbulent flow.

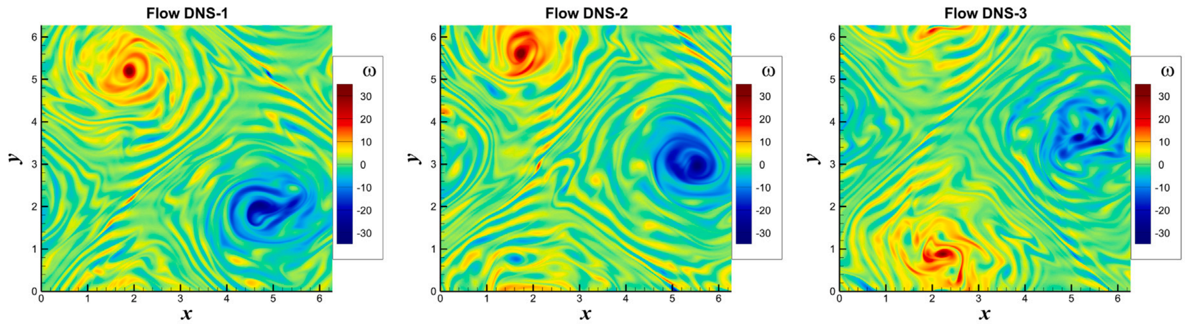

Fig. 7. Vorticity fields ω at $t = 100$ of the 2D turbulent Kolmogorov flow governed by (1) and (2) in the case of $n_K = 16$ and $Re = 2000$, given by DNS subject to the initial conditions (3) (left, marked by Flow DNS-1), (4) (middle, marked by Flow DNS-2), and (5) (right, marked by Flow DNS-3), respectively.

What fundamental characteristics should a reasonable model of turbulence in general have? This is an important question that we had to answer, since the Navier–Stokes turbulence as a mathematical model is now a Millennium Prize Problem [1] which has attracted the attention of many mathematicians. According to the ultra-chaotic property of the Navier–Stokes turbulence illustrated in this paper, we suggest that a reasonable turbulence model in general should have the following three fundamental characteristics:

(A) physical laws of conservation should be satisfied;
(B) small stochastic natural/artificial disturbances should be considered;
(C) solution should be non-differentiable.

Note that the so-called Landau–Lifshitz–Navier–Stokes (LLNS) equations [28], which include the influences of stochastic thermal fluctuation, have these three fundamental characteristics. The stochastic differential equations (such as LLNS equations) are fundamentally different from the Navier–Stokes equations: solution of the former one is continue but unsmooth, and solution of the latter one is smooth and differentiable. So, the LLNS equations might be a more reasonable model of turbulence in physics than the Navier–Stokes equations. Note that Landau–Lifshitz–Navier–Stokes (LLNS) equations are stochastic differential equations, but Navier–Stokes equations are deterministic: they are fundamentally different.

Recently, a multi-scale approach, wave-particle turbulence simulation (WPTS) method [29–31], is developed for the numerical simulation of turbulent flows. A uniform framework of laminar and turbulence flow can be achieved through wave-particle decomposition and the coupled evolution, where the wave stands for the cell-resolved large-scale flow structure, while the unresolved turbulent dynamics under the employed coarse grid is modeled by the stochastic particles, carrying the evolution of turbulent kinetic energy during their movement. Of course, this approach needs and is worth further investigation. Nonetheless, it satisfies the forgoing three fundamental characteristics without doubt.

Finally, we would like to emphasize the importance of Liao's mathematical theorem of spatial symmetry invariance of the Kolmogorov flow and NS turbulence [25], which guarantees that the turbulent flows given by the three almost same initial conditions (3), (4), and (5) are completely different in spatial symmetry: this fact certainly leads to their distinct deviations in statistics, say, the Navier–Stokes turbulence is ultra-chaotic, according to the definition of ultra-chaos [7,8]. Note that, all DNS results given by the three almost same initial conditions (3), (4), and (5) quickly violate the mathematical theorem of spatial symmetry invariance of the Kolmogorov flow, and besides these DNS results are the same in statistics, indicating that the corresponding flows are normal-chaos: this however is completely wrong, and thus these DNS results are quite misleading. Obviously, CNS as a powerful tool can provide us reliable numerical simulations of turbulent flows and chaotic systems, and can give us some enlightenments to the truth for better and deeper understandings. Certainly, more mathematical, physical and experimental investigations should be done in the future to give more evidences to support and/or modify our above-mentioned viewpoints.

CRediT authorship contribution statement

Shijie Qin: Writing – original draft, Visualization, Software, Methodology, Formal analysis, Data curation. **Kun Xu:** Writing – review & editing, Project administration, Funding acquisition. **Shijun Liao:** Writing – review & editing, Funding acquisition, Conceptualization.

Declaration of competing interest

We declare that we have no conflict of interest. We have no financial and personal relationships with other people or organizations that can inappropriately influence our work, also there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled.

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Appendix. Some definitions and measures

For the definitions of the operators such as $\langle\ \rangle_A, \langle\ \rangle_{x,t}, \langle\ \rangle$ and so on, please refer to Liao and Qin [15]. The stream function can be expanded as the Fourier series

$$\psi(x,y,t) \approx \sum_{m=-\lfloor N/3 \rfloor}^{\lfloor N/3 \rfloor} \sum_{n=-\lfloor N/3 \rfloor}^{\lfloor N/3 \rfloor} \Psi_{m,n}(t) \exp(\mathbf{i}\, m x) \exp(\mathbf{i}\, n y), \tag{10}$$

where m, n are integers, $\lfloor\ \rfloor$ stands for a floor function, $\mathbf{i} = \sqrt{-1}$ denotes the imaginary unit, and for dealiasing $\Psi_{m,n} = 0$ is imposed for wavenumbers outside the above domain $\sum$. Note that for the real number ψ, $\Psi_{-m,-n} = \Psi^*_{m,n}$ must be satisfied, where $\Psi^*_{m,n}$ is the conjugate of $\Psi_{m,n}$. Therefore, the kinetic energy spectrum is defined as

$$E_k(t) = \sum_{k-1/2 \le \sqrt{m^2+n^2} < k+1/2} \frac{1}{2}\,(m^2+n^2)\,|\,\Psi_{m,n}(t)\,|^2, \tag{11}$$

where the wave number k is a non-negative integer.

By Reynolds decomposition, the velocity $\mathbf{u} = (u, v)$ is divided into mean and fluctuating quantities

$$\mathbf{u} = \langle \mathbf{u} \rangle_{x,t} + \mathbf{u}', \tag{12}$$

considering the homogeneity in x direction because of the form of external force, which also leads to $\langle v \rangle_{x,t} = 0$.

Data availability

All data are available by sending requirement to the corresponding author. The CNS codes and the movies about Flow CNS-1, Flow CNS-2 and Flow CNS-3 in $t \in [0, 100]$ can be free download via the website https://github.com/sjtu-liao/Ultrachaos-2D-Kolmogorov.

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