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A family of new solutions on the wall jet

Hang Xu [a,*], Shi-Jun Liao [a], Guo-Xiong Wu [b]

[a] *School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai 200030, China*
[b] *Department of Mechanical Engineering, University College London, Torrington Place, London WC1E 7JE, UK*

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Abstract

The fluid flow problem within the wall jet created by fluid hitting on a solid surface at the right angle is solved based on the homotopy analysis method (HAM). A new family of solutions for jet with injection/suction which has been overlooked so far are obtained. Numerical evidence seems to suggest that these solutions decay algebraically far away from the wall.

Keywords: Wall jet; Boundary layer; Injection/suction; Similarity solution; Algebraic decay

1. Introduction

Based on the definition of Glauert [1], a wall jet is a thin layer of fluid spreading over a surface as a result of the fluid hitting the surface at the right angle. The flow within the jet can be described through the boundary layer theory. Within the framework of this theory, Glauert [1] found that similarity solutions would be possible under certain conditions. In particular, he was able to obtain close-form similarity solutions for a rigid and fixed wall. Merkin and Needham [2], and Needham and Merkin [3] subsequently extended the work of Glauert by including the wall motion and the effects of injection and suction of fluid through the wall. They drew two conclusions. The first one was that the Glauert type of solutions would not be possible when only one of the following two modifications was introduced: (i) the wall was moving and (ii) injection and suction through the wall were applied. The second conclusion was that when the motion of the wall and the injection or suction were combined properly, the Glauert type of solutions would still be possible. More recently, Magyari and Keller [5] showed that the first conclusion of Merkin and Needham [2] and Needham and Merkin [3] was valid only when the wall jet was an 'e-jet', i.e., the jet flow far away from the wall would decay exponentially, as in the solution of Glauert. They then showed that when suction alone was introduced similarity solutions would still be possible, but only in the form of 'a-jet', i.e., the flow would decay algebraically.

Another interesting result in the work of Glauert [1] was that under assumption of no reverse flow he found that for the rigid and fixed wall, the two dimensional similarity solution would be possible in the form of $u \propto x^a$, $\delta \propto x^b$, only if $a = -1/2$ and $b = 3/4$, where u is the velocity and x is the axis along the wall surface, and δ is the jet thickness. This led to the coefficient in his ordinary differential equation $\gamma = (2b-1)/(1-b) = 2$. This was the number used

* Corresponding author.
E-mail addresses: hangxu@sjtu.edu.cn (H. Xu), sjliao@sjtu.edu.cn (S.-J. Liao), gx_wu@meng.ucl.ac.uk (G.-X. Wu).

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in the above papers of Merkin and Needham [2], Needham and Merkin [3] and Magyari and Keller [5]. Thus all their conclusions corresponded to $\gamma = 2$. Cohen, Amity and Bayly [4] then showed when there was injection or suction through the wall, γ did not have to be equal to 2 to allow solutions to exist. Unlike the first conclusion of Merkin and Needham [2] and Needham and Merkin [3] corresponding to $\gamma = 2$, Cohen, Amity and Bayly [4] demonstrated that the solutions could exist with either injection or suction alone when γ was chosen properly.

One other interesting point about the wall jet flow is that the similarity solution in general is not unique. Glauert [1] noticed that there was an undetermined coefficient at infinity. He used that coefficient for normalization and obtained his close-form solution. In the work of Magyari and Keller [5] when suction was included and the flow decayed algebraically, they found that the solution was also non-unique but for different reasons. They showed that a family of solutions were possible corresponding to the different skin friction on the wall. By altering γ, Cohen, Amity and Bayly [4] extended Glauert's analysis and obtained a family of similarity solutions.

In the present paper, we shall first demonstrate that in the model of Cohen, Amity and Bayly [4], further solutions are possible, which seems to have been overlooked so far. The homotopy analysis method (HAM) [6–13] will then be used to solve the wall jet flow problem. By introducing an embedding parameter q the nonlinear ordinary differential equation is converted to a linear differential equation at $q = 0$. When q evolves, the differential equation becomes the original one at $q = 1$. The method has been used in a variety of problems and the details can be found in Liao [6,7]. Here the method is first used to solve the problem based on the models of Glauert [1] and Magyari and Keller [5] and the results are found to be in agreement with those obtained from the methods in their papers. The HAM is then used to find the solution for the model of Cohen, Amity and Bayly [4]. Apart from those solutions in their paper, additional solutions are also found.

2. Mathematical description

We consider a two dimensional laminar jet flowing over a fixed plane wall in the presence of a lateral suction/injection. The governing equations can be given based on the steady boundary layer theory

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

where x and y are the axises parallel and normal to the wall surface, respectively, u and v are the corresponding velocity components, ν is the kinematic viscosity.

The boundary conditions on the wall and at infinity can be written as

$$u = 0, \quad v = V_w(x), \quad \text{at } y = 0, \qquad u \to 0 \quad \text{as } y \to \infty. \tag{3}$$

Let ψ denote the stream function, satisfying

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{4}$$

By introducing the following non-dimensional variables

$$u = U\hat{u}, \quad v = U\hat{v}, \quad x = \nu\hat{x}/U, \quad y = \nu\hat{y}/U, \quad \psi = \nu\hat{\psi}, \tag{5}$$

where U is a constant reference velocity, Eq. (2) can be written as

$$\frac{\partial \hat{\psi}}{\partial \hat{y}}\frac{\partial^2 \hat{\psi}}{\partial \hat{x}\partial \hat{y}} - \frac{\partial \hat{\psi}}{\partial \hat{x}}\frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} = \frac{\partial^3 \hat{\psi}}{\partial \hat{y}^3}. \tag{6}$$

To find similarity solutions, it is assumed that the jet streamwise velocity $\hat{u} \propto \hat{x}^a$ and the jet thickness $\delta \propto \hat{x}^b$, as in Glauert [1]. we then introduce function f and its variable η as

$$\hat{\psi} = \hat{x}^{a+b} \cdot f(\eta), \qquad \eta = d \cdot \hat{y}/\hat{x}^b, \tag{7}$$

where d is a positive constant. Substituting (7) into (6), we have

$$af'^2 - (a+b)ff'' = d \cdot \hat{x}^{-a-2b+1} f'''. \tag{8}$$

For similarity solutions to exist, it obviously requires

$$a + 2b = 1, \tag{9}$$

d here is to adjust the scale of η and can be chosen as $d = (1-b) > 0$, as in Glauert [1], and in Cohen, Amitay and Bayly [4]. Eq. (8) then becomes

$$f''' + ff'' + \gamma f'^2 = 0, \tag{10}$$

subject to the following boundary conditions

$$f(0) = f_w, \qquad f'(0) = f'(\infty) = 0, \tag{11}$$

where $\gamma = (2b-1)/(1-b)$ and $f_w = \hat{x}^b V_w(x)/(Ub - U)$ is a suction/injection coefficient. For similarity solutions to hold, $V_w(x) \propto \hat{x}^{-b}$ is assumed.

Glauert [1] considered the case of an impermeable wall with $f_w = 0$. He found that when there was no reverse flow ($f''(0) \geqslant 0$), similarity solutions would be possible only when $\gamma = 2$. He noticed that when $f(\eta)$ was a solution of Eq. (10), $Af(A\eta)$ would also be a solution. Thus without loss of generality, he obtained the following close-form solution for $f(\infty) = 1$

$$\eta = 3^{1/2} \arctan\left[\frac{(3f)^{1/2}}{2 + f^{1/2}}\right] + \ln\left[\frac{(1 + f + f^{1/2})^{1/2}}{1 - f^{1/2}}\right]. \tag{12}$$

This leads to

$$f'(\eta) \to \sqrt{12} \exp\left(\frac{\pi}{\sqrt{12}} - \eta\right), \quad \eta \to \infty, \tag{13}$$

which decays exponentially.

Magyari and Keller [5] showed that when injection was applied, no similarity solutions would be possible when $\gamma = 2$. However, when suction was applied, although no similarity solutions which decayed exponentially would be possible (Merkin and Needham [2], and Needham and Merkin [3]), similarity solutions which decayed algebraically were possible. In fact, it is obvious if

$$f'(\eta) \to \beta\eta^{-2/3} \quad \text{as } \eta \to \infty, \tag{14}$$

where β is a constant, Eq. (10) can be satisfied at $\eta \to \infty$. The behavior of $f'(\eta)$ in Eq. (14) is therefore very much different from that in (13). Magyari and Keller [5] then showed that the boundary conditions in Eq. (11) did not give a unique solution. They introduced a condition on $f''(0)$ which was related to the skin friction on the wall. Numerical solutions were then obtained for different f_w with the skin friction equal to that in Glauert [1].

Cohen, Amitay and Bayly [4] obtained the similarity solutions with $\gamma \neq 2$ and $f_w = 0$. They showed that the solution would be possible if (i) $f_w > 0$, $\gamma > 2$, (ii) $f_w = 0$, $\gamma = 2$, and (iii) $f_w < 0$, $1 < \gamma < 2$. Here we shall demonstrate that solutions are also possible when these two parameters are outside these ranges. As in Magyari and Keller [5], we may assume

$$f(\eta) \sim \beta^* \eta^{\alpha^*} + C. \tag{15}$$

Substituting Eq. (15) into (10) and equating the leading terms, we obtain

$$\alpha^* = \frac{1}{1+\gamma}. \tag{16}$$

This gives

$$f'(\eta) \sim \frac{\beta^*}{1+\gamma} \eta^{-\frac{\gamma}{1+\gamma}} \quad \text{as } \eta \to \infty. \tag{17}$$

We shall show that this form of solutions exists even when f_w and γ are outside the ranges given by Cohen, Amitay and Bayly [4].

3. Analytical approximations

To seek the new solutions discussed above, we introduce the following transformation with a constant $\lambda > 0$

$$f(\eta) = g(\xi)/\lambda, \quad \xi = 1 + \lambda\eta, \tag{18}$$

which is to shift the origin and to adjust the scale of the axis. Eq. (10) then takes the form

$$\lambda^2 g''' + gg'' + \gamma g'^2 = 0, \tag{19}$$

and the boundary conditions become

$$g(1) = \lambda f_w, \quad g'(1) = 0, \quad g'(\infty) = 0. \tag{20}$$

Because of the behavior of $g(\xi)$ at infinity in Eq. (20) and the principle discussed in Liao [7], we assume that the solution can be written as

$$g(\xi) = b_{1,0}\xi^{\alpha^*} + \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} b_{m,n}\xi^{n\alpha^*-m}. \tag{21}$$

Based on the general form in Eq. (21) and the boundary conditions in Eq. (20), we choose

$$g_0(\xi) = \frac{-2f_w\lambda + 4\alpha^* f_w\lambda - 2\alpha^{*2} f_w\lambda - \sigma}{\alpha^* - 2}\xi^{\alpha^*} + \frac{-2\alpha^* f_w\lambda + 2\alpha^{*2} f_w\lambda + \sigma}{\alpha^* - 1}\xi^{\alpha^*-1} + \frac{-\alpha^* f_w\lambda + \alpha^{*2} f_w\lambda + \sigma}{(\alpha^* - 2)(\alpha^* - 1)}\xi^{2\alpha^*-2}, \tag{22}$$

as the initial approximation to the solution, where $\sigma = f''(0)/\lambda$ is a constant. We then define a linear operator as

$$\mathcal{L}[\phi] = \xi^3\phi''' - 2(2\alpha^* - 3)\xi^2\phi'' + (\alpha^* - 1)(5\alpha^* - 6)\xi\phi' - 2\alpha^*(\alpha^* - 1)^2\phi, \tag{23}$$

which has the following property

$$\mathcal{L}\left[C_0\xi^{\alpha^*} + C_1\xi^{\alpha^*-1} + C_2\xi^{2\alpha^*-2}\right] = 0, \tag{24}$$

where C_0, C_1 and C_2 are constants. A nonlinear operator in the HAM can also be defined as

$$\mathcal{N}[\Phi(\xi;q)] = \lambda^2\frac{\partial^3\Phi(\xi;q)}{\partial\xi^3} + \Phi(\xi;q)\frac{\partial^2\Phi(\xi;q)}{\partial\xi^2} + \gamma\left[\frac{\partial\Phi(\xi;q)}{\partial\xi}\right]^2, \tag{25}$$

where $q \in [0, 1]$ is an embedding parameter. Following these definitions, we can construct the HAM deformation equation as

$$(1 - q)\mathcal{L}\left[\Phi(\xi;q) - g_0(\xi)\right] = q\hbar H(\xi)\mathcal{N}\left[\Phi(\xi;q)\right], \tag{26}$$

which is subject to the boundary conditions below

$$\Phi(1;q) = \lambda f_w, \quad \left.\frac{\partial\Phi(\xi;q)}{\partial\xi}\right|_{\xi=1} = 0, \left.\frac{\partial^2\Phi(\xi;q)}{\partial\xi^2}\right|_{\xi=1} = \sigma, \quad \left.\frac{\partial\Phi(\xi;q)}{\partial\xi}\right|_{\xi=\infty} = 0, \tag{27}$$

where $\Phi(\xi;q)$ is a mapping of $g(\xi)$, $\hbar \neq 0$ is an auxiliary parameter and its role is to adjust to the rate of evolution process, $H(\xi) \neq 0$ is an auxiliary function and its role is to produce a prescribed series solution form of Eq. (19). Here, we choose $H(\xi) = \xi$ to obtain the *solution form* of Eq. (30). We can also set $H(\xi) = 1$ or $H(\xi) = \xi^2$, etc., but in these cases, the final solution form will be changed accordingly. The values of $\hbar$ may be also changed for these cases. Note that the HAM deformation equation (26) is non-unique because the linear operator $\mathcal{L}$ can be changeable. For example, we can use the linear operator

$$\mathcal{L} = \xi^3\phi''' - 3(\alpha^* - 2)\xi^2\phi'' + 3(\alpha^* - 1)(\alpha^* - 2)\xi\phi' - \alpha^*(\alpha^* - 1)(\alpha^* - 2)\phi \tag{28}$$

and the auxiliary function $H(\xi) = 1$ to derive another *solution form* of Eq. (19)

$$g(\xi)=\sum_{k=0}^{+\infty}\sum_{n=1}^{k+1}\sum_{m=1}^{k+1}A^k_{n,m}\xi^{n\alpha^*-m},\tag{29}$$

which is obviously different from the present *solution form* given in Eq. (30).

For more details on how to choose a linear operator for a nonlinear problem and how to get the higher deformation equation, readers are referred to Liao [7] or Xu et al. [12].

Using the homotopy analysis method, we finally derive the series solution of the present problem in the following form

$$g(\xi)=\sum_{k=0}^{+\infty}\sum_{n=1}^{2k+2}\sum_{m=n-1+\chi_n}^{2k+1+[n/2]}A_k^{n,m}\xi^{n\alpha^*-m},\tag{30}$$

where $[x]$ stands for the integer part of a real number x, and the coefficients $A_k^{n,m}$ are given by

$$A_k^{n,m}=\frac{\hbar(\lambda^2B_k^{n,m}+C_k^{n,m})}{(n\alpha^*-m-\alpha^*)(n\alpha^*-m-\alpha^*+1)(n\alpha^*-m-2\alpha^*+2)}+\chi_{2k+2-n}\chi_{2k+1+[n/2]-m}A_{k-1}^{n,m},\quad n\alpha^*-m\neq\alpha^*,\alpha^*-1,\alpha^*-2\tag{31}$$

$$B_k^{n,m}=(n\alpha^*-m+2)(n\alpha^*-m+1)(n\alpha^*-m)\chi_{2k+2-n}\chi_{2k+3-m+[n/2]}\chi_{m+1-n-\chi_n}A_{k-1}^{n,m-2},\tag{32}$$

$$\begin{aligned}C_k^{m,s}=&\sum_{n=0}^{k-1}\sum_{p=\max\{1,m+2n-2k\}}^{\min\{2n+2,m-1\}}\sum_{q=\max\{p-1+\chi_p,s+2n-2k-[(m-p)/2]\}}^{\min\{2n+1+[p/2],s+p-m-\chi_{m-p}\}}\chi_m\\&\times\chi_{2k-s+3+[p/2]+[(m-p)/2]}\chi_{s+3-m-\chi_p-\chi_{m-p}}\big[(m-p)\alpha^*+q-s+1\big]\\&\times\big\{\big[(m-p)\alpha^*+q-s\big]+\gamma(p\alpha^*-q)\big\}A_n^{p,q}A_{k-1-n}^{m-p,s-q-1}\end{aligned}\tag{33}$$

with

$$A_k^{1,0}=\frac{2(\alpha^*-1)^2\delta_0+(4-3\alpha^*)\delta_1+\delta_2}{\alpha^*-2},\tag{34}$$

$$A_k^{1,1}=-2\alpha^*\delta_0+3\delta_1-\frac{\delta_2}{\alpha^*-1},\tag{35}$$

$$A_k^{2,2}=\frac{\alpha^*(1-\alpha^*)\delta_0-2(1-\alpha^*)\delta_1-\delta_2}{(\alpha^*-1)(\alpha^*-2)},\tag{36}$$

$$A_0^{1,0}=\frac{-2f_w\lambda+4f_w\lambda-2\alpha^{*2}f_w\lambda-\sigma}{-2+\alpha^*},\tag{37}$$

$$A_0^{1,1}=\frac{-2\alpha^*f_w\lambda+2\alpha^{*2}f_w\lambda+\sigma}{-1+\alpha^*},\tag{38}$$

$$A_0^{2,2}=\frac{-\alpha^*f_w\lambda+\alpha^{*2}f_w\lambda+\sigma}{(1-\alpha^*)(2-\alpha^*)},\tag{39}$$

and $\sigma=g''(0)=\lambda f''(0)$. Other parameters in these equations are defined as

$$\delta_0=\sum_{n=1}^{2k+2}\sum_{m=n-1+\chi_n}^{2k+1+[n/2]}A_k^{n,m}\varepsilon^{n,m},\tag{40}$$

$$\delta_1=\sum_{n=1}^{2k+2}\sum_{m=n-1+\chi_n}^{2k+1+[n/2]}A_k^{n,m}(n\alpha^*-m)\varepsilon^{n,m},\tag{41}$$

$$\delta_2=\sum_{n=1}^{2k+2}\sum_{m=n-1+\chi_n}^{2k+1+[n/2]}A_k^{n,m}(n\alpha^*-m)(n\alpha^*-m-1)\varepsilon^{n,m},\tag{42}$$

where

$$\varepsilon^{n,m} = \chi_{(n-1)^2+m^2+1}\chi_{(n-1)^2+(m-1)^2+1}\chi_{(n-2)^2+(m-2)^2+1}, \tag{43}$$

and

$$\chi_k = \begin{cases} 0, & k = 1, \\ 1, & k > 1. \end{cases} \tag{44}$$

Once $g(\xi)$ is obtained, Eq. (18) gives

$$f(\eta) = \frac{1}{\lambda}\sum_{k=0}^{+\infty}\sum_{n=1}^{2k+2}\sum_{m=n-1+\chi_n}^{2k+1+[n/2]} A_k^{n,m}(1+\lambda\eta)^{n\alpha^*-m}. \tag{45}$$

The two parameters λ and $\hbar$ have been introduced to improve the solution procedure. For λ, the residual error of the initially assumed solution $g_0(\xi)$ can be expressed by

$$E_0(\lambda) = \int_1^{\infty} \left(R_1[\vec{g}_0]\right)^2 d\xi. \tag{46}$$

Letting

$$\frac{dE_0}{d\lambda} = 0, \tag{47}$$

we can get an optimal value corresponding to the least error. The value of $\hbar$ is chosen by the following way. The series solution (45) is the function of the auxiliary parameter $\hbar$ and the variable η. Given any a prescribed value of $\eta = \eta^*$, the analytic approximations (45) and its derivatives are only depend on $\hbar$ now. At certain order HAM approximation, we can get the curves of $f(\eta)$ versus $\hbar$, or $f'(\eta)$ versus $\hbar$, or $f''(\eta)$ versus $\hbar$, etc. If all of these curves looks smooth and they have the common region $\hbar_{\min} \leqslant \hbar \leqslant \hbar_{\max}$, we can obtain the proper value of $\hbar$ from the region $[\hbar_{\min}, \hbar_{\max}]$. For more details, the readers are referred to Liao [7] or Xu et al. [12].

4. Results and discussions

It is important to ensure that the solution converges when more and more terms are used in Eq. (30). We thus consider the case of Glauert [1], or $\gamma = 2$ and $f_w = 0$ in Eqs. (10) and (11). Fig. 1 gives the results obtained with $\hbar = -4$ and $\lambda = 1/2$, and the series in Eq. (45) is truncated at $k = 30$. They are compared with the solutions which are obtained from 50,000 step numerical integration over $\eta = 0$ to $\eta = 100$ using the fourth order Runge–Kutta method together with the Newton–Raphson technique. Very good agreement can be found. Fig. 2 gives the similar results for $f_w = 1$. This is the case considered by Magyari and Keller [5] and the solution decays algebraically at infinity. The figure shows that the results from HAM and the numerical integration are in agreement. Also error in $|f'(\eta) - \beta\eta^{-2/3}|/\eta^{2/3}$ is checked and is found to be less than 1×10^{-6} when $\eta = 100$, where $\beta = [f''(0)f_w/9]^{1/3}$, as shown in [5].

As has been discussed above, we reemphasize here that Cohen, Amitay and Bayly [4] found that the solution of Eq. (10) would be possible if (i) $f_w > 0$, $\gamma > 2$, (ii) $f_w = 0$, $\gamma = 2$ and (iii) $f_w < 0$, $1 < \gamma < 2$. If we check their proof procedure carefully however, we can see that the conclusion is valid only when (1) $\lim_{\eta\to\infty} f(\eta)f'(\eta) = 0$ in Eq. (18) and (2) $\lim_{\eta\to\infty} f(\eta)\int_\eta^\infty f'^2(\eta)\,d\eta = 0$ in Eq. (20) in their paper. These two conditions are certainly valid, if $f(\eta)$ decays exponentially. For a solution in a form in Eq. (52), we can use the leading term in the expansion and find the conditions are satisfied when $\gamma > 2$. It is, however, important to point out that the proof by Cohen, Amitay and Bayly cannot rule out that solution can exist when f_w and γ are outside of all these three categories if conditions (1) and (2) are not met.

We consider a case with $f_w = 1$ and $\gamma = 1.5$ which are outside of the three categories of Cohen, Amitay and Bayly [4]. Fig. 3 gives the obtained solution from HAM. To demonstrate the accuracy of the solution, we define an error function as

$$E_f(\eta) = f''' + ff'' + \gamma f'^2. \tag{48}$$

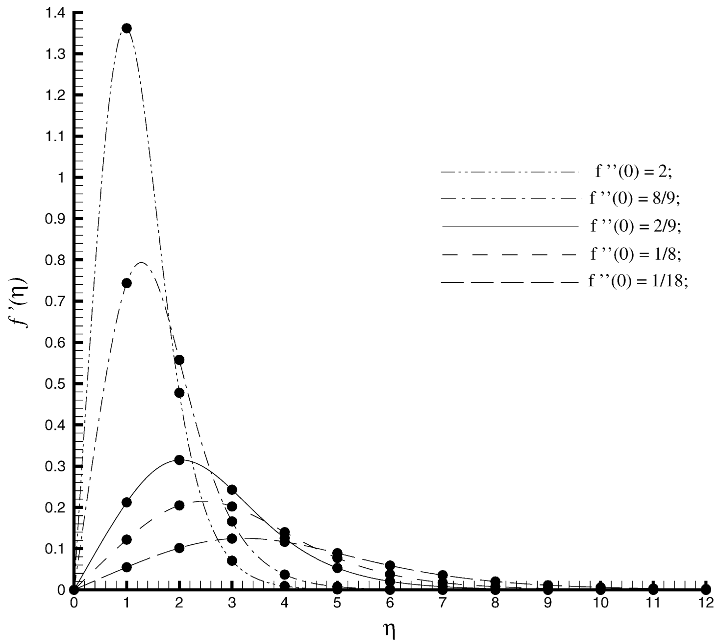

Fig. 1. The 30th order approximations of $f'(\eta)$ given by the homotopy analysis method for different values of $f''(0)$ when $f_w = 0$. Line: HAM approximations; Filled circles: Numerical results.

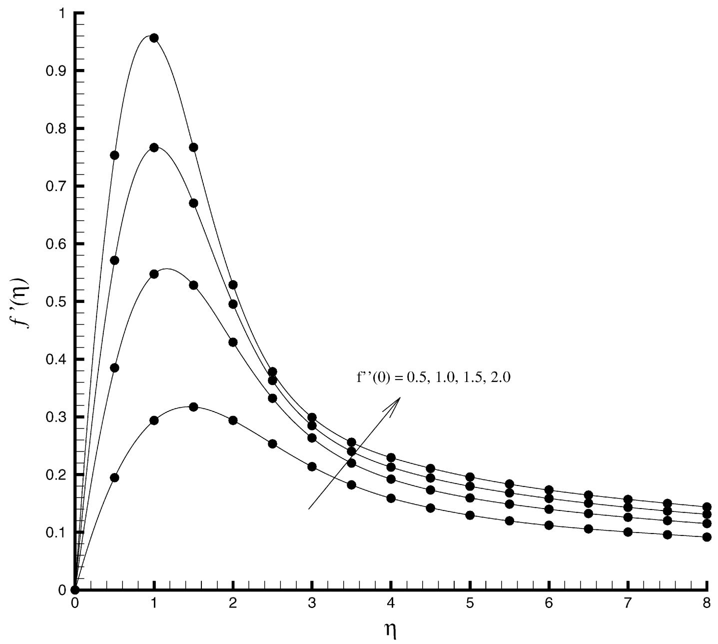

Fig. 2. The comparison of 30th order approximations of $f'(\eta)$ with the numerical results for some values of σ when $f_w = 1$. Solid line: 30th order HAM approximations; Filled circles: Numerical results.

Substituting the expansion of $f(\eta)$ with the obtained coefficients into Eq. (48), we find that errors tend to zero as k increases, $|E_f(\eta)|$ at the 80th order approximation is less than 5×10^{-6}, as shown in Fig. 4. This is clearly a new solution which has been overlooked so far.

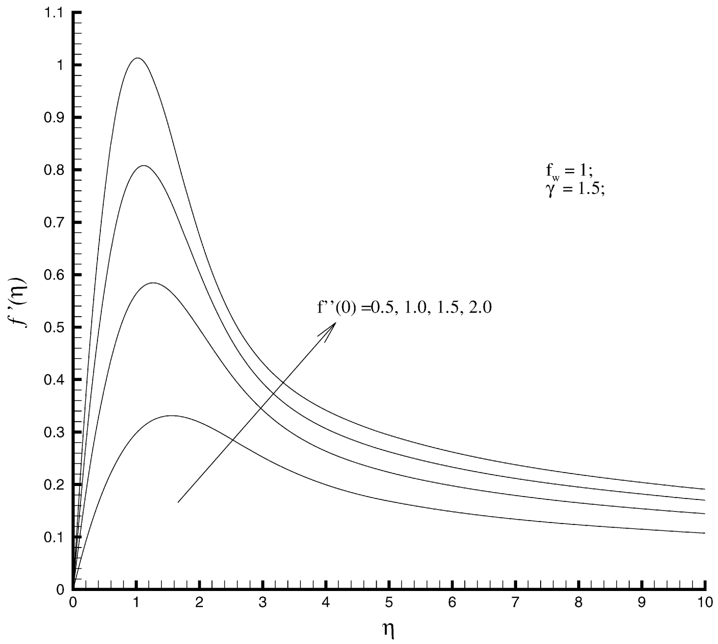

Fig. 3. 30th order approximations of $f'(\eta)$ given by the homotopy analysis method for some values of $f''(0)$ when $f_w = 1$ and $\gamma = 1.5$.

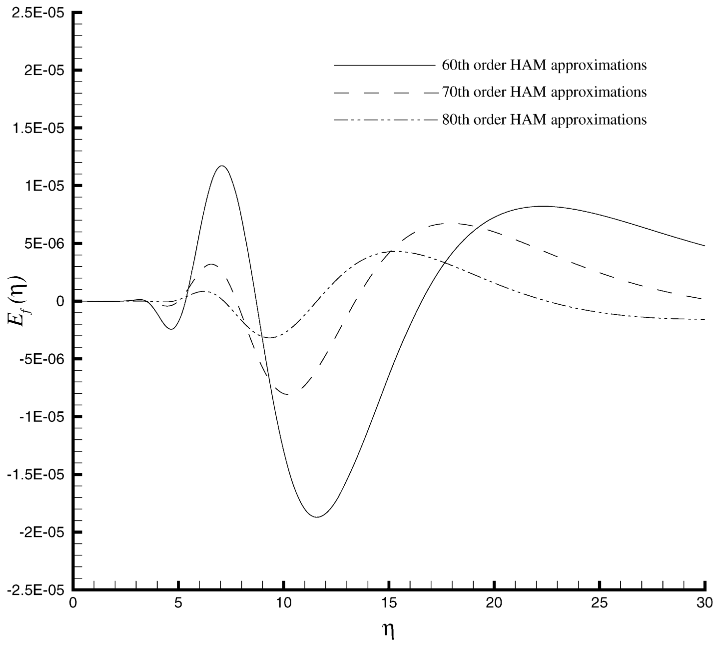

Fig. 4. Several order approximations of $E_f(\eta)$ given by the homotopy analysis method when $f_w = 1$, $\gamma = 1.5$ and $f''(0) = 1$.

We now consider cases with different γ at $f_w = 1$. The results are given in Fig. 5. The case for $\gamma > 2$ clearly belongs (i) above and the existence of the solution is expected. However, it is interesting to note that the solution in this case seems to still decay algebraically. In fact we can rewrite Eq. (17) as

$$\frac{\beta^*}{\alpha^*} = \lim_{\eta \to \infty} \frac{f'(\eta)}{\eta^{\alpha^*-1}}. \tag{49}$$

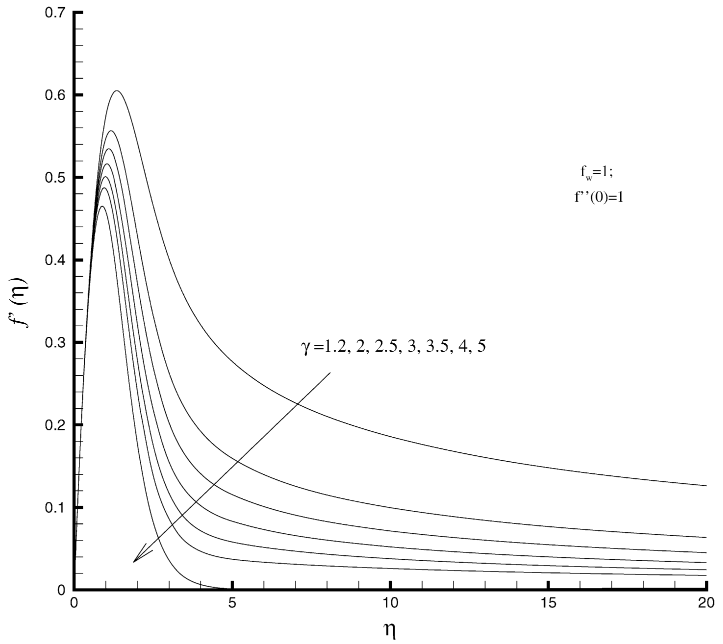

Fig. 5. 30th order approximations of $f'(\eta)$ given by the homotopy analysis method for some values of γ when $f_w = 1$ and $f''(0) = 1$.

Table 1
Numerical results for Eq. (49) at $f_w = 1$ and $f''(0) = 1$ when $\eta \to \infty$

γ	30th order	40th order	50th order	60th order	Exact result
1.5	0.58221	0.582279	0.582270	0.582270	
2	0.480245	0.480780	0.480750	0.480749	0.480750
3	0.337486	0.337430	0.337434	0.337433	
4	0.240066	0.240170	0.240207	0.240206	

Substituting the solution in Eq. (45) into (49), we can work out the limit numerically as shown in Table 1 in which the exact solution at $\gamma = 2$ is obtained from $\beta^*/\alpha^* = [f''(0) f_w/9]^{1/3}$, as in [5]. The table shows that the result tends to a constant when k increases. This gives strong evidence that these solutions decay algebraically. It is appropriate at this stage to compare the HAM and the numerical method. It may be possible to obtain the same new solution using the numerical method. However, it is not so easy to get the insight of the way the function decays in the numerical method. HAM on the other hand gives the strong indication through Table 1.

Fig. 6 gives the results for several values of $f''(0)$ when $\gamma = 3$ and $f_w = 1$ which belongs to case (i) above. The convergence and accuracy of the results have been checked in a way similar to that used above. Cohen, Amitay and Bayly [4] have used a normalization and gave solution for $f(\infty) = 1$. However it is obvious that unlike the case of Glauert [1], when $f_w \neq 0$, $Af(A\eta)$ is no longer a solution even when $f(\eta)$ is the solution of Eq. (10). Thus solutions for different $f''(0)$ have therefore been given individually.

Although different solutions can be obtained by choosing different $f''(0)$ for a fixed f_w, for some choices of $f''(0)$, however, solutions cannot be obtained, in the sense that the error defined by Eq. (48) does not decay beyond a limit no matter how many terms are used in the HAM expansion and what values are chosen for λ and $\hbar$. It is then found through systematic investigation that for a given $f''(0)$, there exists a minimum value $f_w^{\min}$. When $f_w < f_w^{\min}$, no solution can be found through HAM despite extensive search. While $f_w > f_w^{\min}$ on the hand, solutions can always be found. Figs. 7 and 8 give the solution for different f_w at given $f''(0)$ and γ. The figures show that when f_w tends to $f_w^{\min}$ the solution seems to decay much faster. For many boundary flows (refer to [1,2,11]) have the exponentially decaying property at infinity, we speculate that these wall jet flows have the similar property when $f_w = f_w^{\min}$.

To verify that the existence of $f_w^{\min}$ is not a false result due to the limitation of the HAM, Runge–Kutta method has also been used to solve the problem. It is found that for given $f''(0)$ and γ, the method gives a solution when

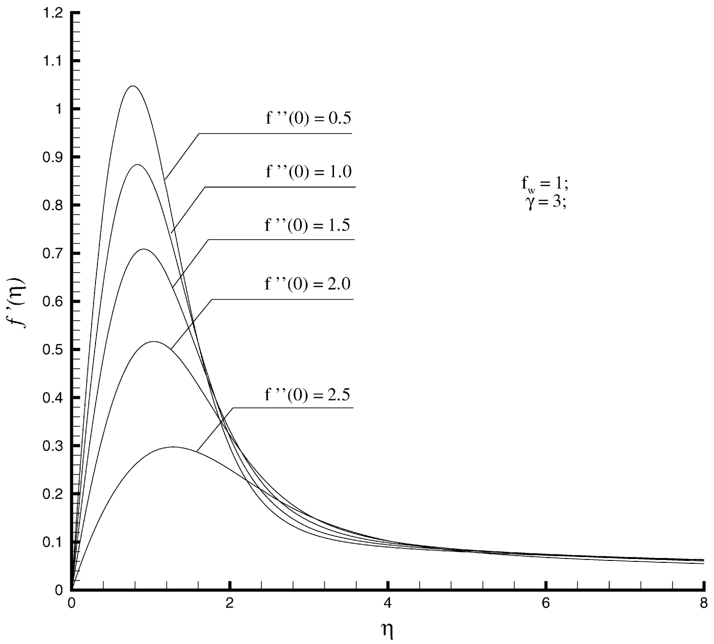

Fig. 6. 30th order approximations of $f'(\eta)$ given by the homotopy analysis method for some values of $f''(0)$ when $f_w = 1$ and $\gamma = 3$.

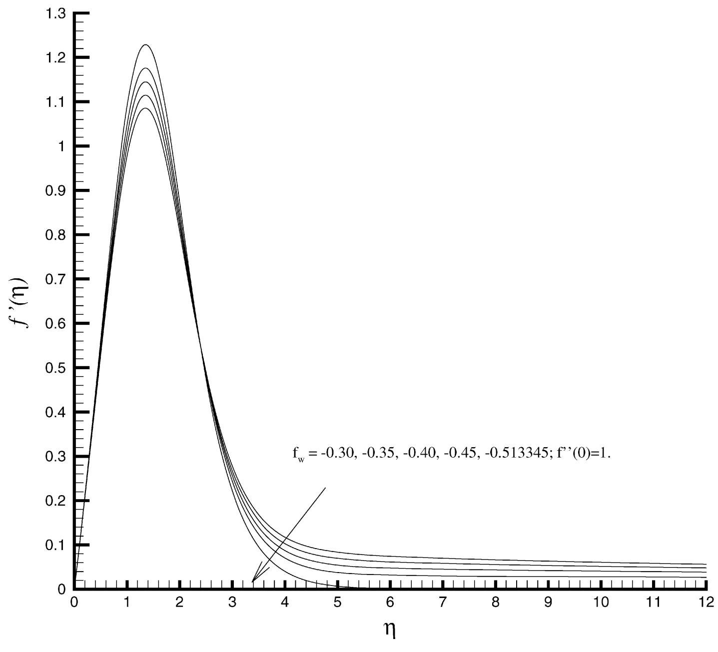

Fig. 7. The 60th order HAM approximations of $f'(\eta)$ for $f''(0) = 1$ and $\gamma = 1.5$ as f_w tends to $f_w^{\min}$.

$f_w = f_w^{\min} + \epsilon$, where ϵ is a small and positive number. When $f_w = f_w^{\min} - \epsilon$ on the other hand, the numerical method shows $f'(\infty) \to \infty$. This remains to be true even when $\epsilon \approx 10^{-5}$. Thus the result from Runge–Kutta method is clearly consistent with what is found from the HAM and the existence of $f_w^{\min}$ is unlikely to be a false result of HAM. Fig. 9 gives some typical curves which show the relationship between $f''(0)$ and $f_w^{\min}$ at different γ. It has to be emphasized however that this is merely a conjecture based on the numerical evidence. Solid proof is still required before a definite conclusion can be made.

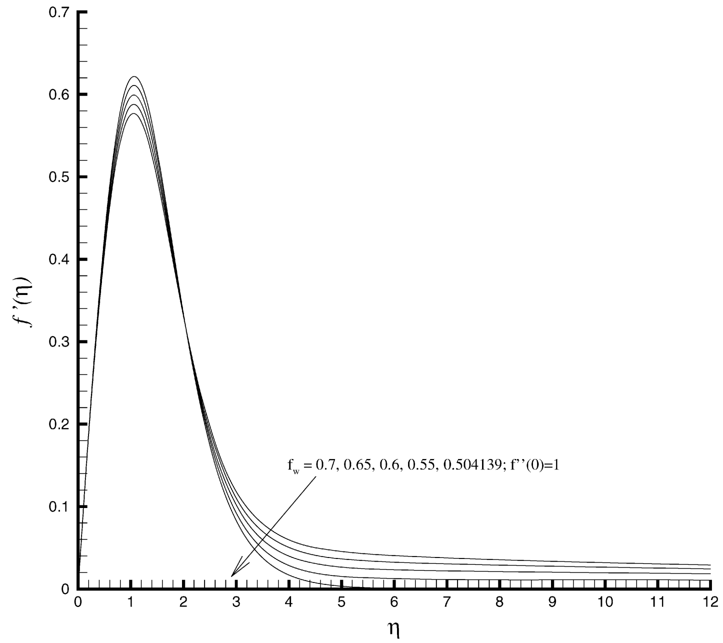

Fig. 8. The 60th order HAM approximations of $f'(\eta)$ for $f''(0) = 1$ and $\gamma = 3$ as f_w tends to $f_w^{\min}$.

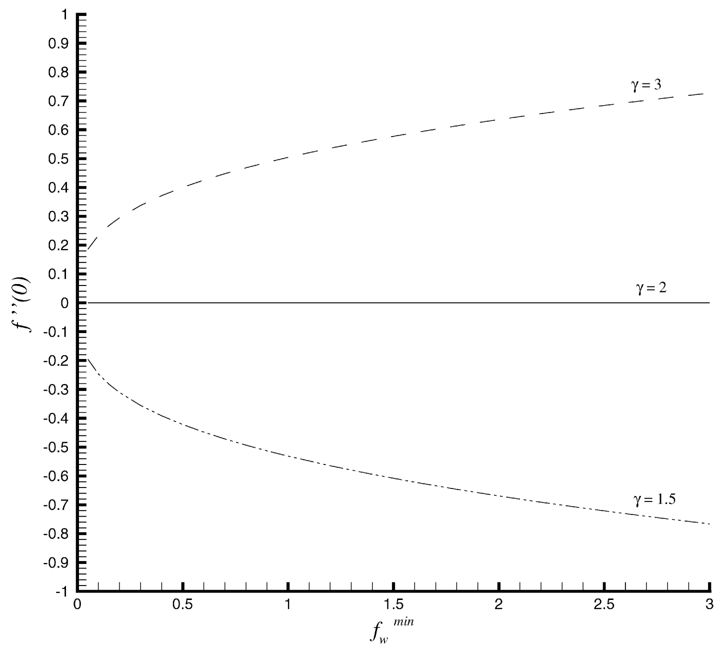

Fig. 9. The minimum values of f_w versus $f''(0)$ for some values of γ.

In fact, it is easily to analyze the physical nature of these new solutions. Rewrite the continuity equation (1) in the following integral form

$$\frac{\partial}{\partial x}\int_0^\infty\int_y^\infty u^2\,dy'\,dy = v(0)\int_0^\infty u^2\,dy. \tag{50}$$

For the Glauert problem without suction (v(0) = 0), this provides us an integral constraint that permits the immediate identification of the similarity parameters. In this case, it requires $3a + 2b = 0$. With suction, this constraint is lost, thus γ need not equal to 2.

For convince, we rewrite boundary condition (15) in the following form

$$f'(\infty) \to \beta\eta^{\alpha}, \tag{51}$$

where $\alpha = -\gamma/(1+\gamma)$ and $\beta = \beta^*/(1+\gamma)$.

From Eq. (7), we further have

$$\hat{u} = d \cdot \hat{x}^{a} f'(\eta), \tag{52}$$

$$\hat{v} = -\hat{x}^{a+b-1}\big[(a+b)f(\eta) - b\eta f'(\eta)\big]. \tag{53}$$

Substituting the boundary conditions (51) into Eqs. (52) and (53), we have

$$\hat{u} = \beta \cdot d \cdot \hat{x}^{a}\eta^{\alpha}, \tag{54}$$

$$\hat{v} = -\beta \cdot \hat{x}^{a+b-1}\eta^{\alpha+1}\left(\frac{a+b}{\alpha+1} - b\right) - \hat{x}^{a+b-1}C(a+b). \tag{55}$$

It is known that the flow will be unrealistic if $\hat{v}_{\infty}$ tends to infinity, this requires

$$\frac{a+b}{\alpha+1} - b = 0. \tag{56}$$

From Eqs. (9) and (56), they hold

$$a = \frac{\alpha}{\alpha+2}, \qquad b = \frac{1}{\alpha+2}, \tag{57}$$

which gives

$$\gamma = -\frac{\alpha}{\alpha+1}, \tag{58}$$

the same result can be obtained from Eq. (15).

Integrating Eq. (10) once from $\eta = 0$ to ∞ one obtains

$$\lim_{\eta\to\infty}\big[f''(\eta) + f(\eta)f'(\eta)\big] - f''(0) + (\gamma - 1)\int_0^{\infty}\big[f'(\eta)\big]^2\,d\eta = 0. \tag{59}$$

From Eq. (51), it holds

$$f''(\eta) = \alpha\beta\eta^{\alpha-1}, \tag{60}$$

$$f(\eta)f'(\eta) = \frac{\beta^2}{\alpha+1}\eta^{2\alpha+1} + \beta C\eta^{\alpha}. \tag{61}$$

Taking into account the boundary conditions (59), Eqs. (60) and (61), and Eq. (15), it is found that when $\alpha < -1/2$, $f''(0) = (\gamma - 1)\int_0^{\infty}[f'(\eta)]^2\,d\eta$, we have $\gamma > 1$ and $f''(0) > 0$. When $\alpha = -1/2$, it reads $f''(0) = \beta^2/(\alpha+1)$ and $\gamma = 1$. When $\alpha > -1/2$, from Eq. (61), we knows that $\lim f(\infty)f'(\infty)$ tends to infinity, which results in $\int_0^{\infty}[f'(\eta)]^2\,d\eta$ tends to infinity. For this case, the flows are obviously unpractical.

For the similarity flow to exist, it requires that $a = \alpha/(\alpha+2)$, $b = 1/(\alpha+2)$ and $V_w(x) \propto \hat{x}^{-b}$. These basic assumptions guarantee that the flows are realized physically.

As mentioned above, these flows have an outstanding feature that they decay algebraically at infinity. They can be categorized into the wall-bounded flow. The kind of flows have been reported by Kuiken [15,16], Weidman et al. [14] Magyari et al. [5,17], etc. It should be noted that such power-law velocity profile for fully developed turbulent flows has been put on firm theoretical ground by Barenblatt [18]. In this critical paper it is shown that the exponent α is proportional to the inverse of the Reynolds number $Re = \bar{u}d/v$, where $\bar{u}$ is the local average streamwise velocity, d is the characteristic dimension of the duct, and ν is the kinematic viscosity of the fluid.

5. Conclusion

The two dimensional flow within wall jet is analyzed by the homotopy analysis method. It is found that the ranges given by Cohen, Amitay and Bayly [4] in which solutions can exist are valid only under certain conditions which were not explicitly given in their paper. New solutions outside these ranges have been obtained, which seem to decay algebraically and have been overlooked so far. The success of the present work has further demonstrated the effectiveness of HAM, which has the potential to be used to understand many new phenomena in science and engineering.

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