

Newton-homotopy analysis method for nonlinear equations

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Abstract

In this paper, we present an efficient numerical algorithm for solving nonlinear algebraic equations based on Newton–Raphson method and homotopy analysis method. Also, we compare homotopy analysis method with Adomian’s decomposition method and homotopy perturbation method. Some numerical illustrations are given to show the efficiency of algorithm.

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1. Introduction

Finding roots of nonlinear equations efficiently has widespread applications in numerical mathematics and applied mathematics. Newton–Raphson method is the most popular technique for solving nonlinear equations. Many topics related to Newton’s method still attract attention from researchers. As is well known, a disadvantage of the methods is that the initial approximation x_0 must be chosen sufficiently close to a true solution in order to guarantee their convergence. Finding a criterion for choosing x_0 is quite difficult and therefore effective and globally convergent algorithms are needed [1]. There is a wide literature concerning such methods, a brief review of which is given in the following.

Adomian’s decomposition method (ADM) [2] is a well-known, easy-to-use tool for nonlinear problems. In 1992, Liao [3] employed the basic ideas of homotopy to propose a general method for nonlinear problems, namely the homotopy analysis method (HAM), and then modified it step by step (see [4–7]). This method has been successfully applied to solve many types of nonlinear problems (for example, please refer to [8–15]). Following Liao, an analytic approach based on the same theory in 1998, which is the so-called “homotopy perturbation method” (HPM), is provided by He [16] and recently HPM is applied for getting Newton-like iteration methods for solving nonlinear equations [17].

In this paper, we propose HAM to solve some nonlinear algebraic equations (the stopping criterion is $|f(x_n)| < 10^{-20}$). The solutions of them are also given by ADM and HPM. We reveal the relationship between

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HAM and other two methods. Furthermore, combining the Newton–Raphson scheme, we construct a more efficient numerical algorithm named Newton-homotopy analysis method (N-HAM). Some examples are tested, and the obtained results suggest that newly improvement technique introduces a promising tool and powerful improvement for solving nonlinear equations.

2. Homotopy analysis method

Consider the nonlinear algebraic equation

$$f(x) = 0, \quad (1)$$

where α is a simple root of it and f is a C^2 function on an interval containing α , and we suppose that $|f'(\alpha)| > 0$. By using Taylor's expansion near x

$$f(x - \delta) = f(x) - \delta f'(x) + \frac{\delta^2}{2} f''(x) + O(\delta^3)$$

and we are looking for δ such as

$$f(x - \delta) = 0 \approx f(x) - \delta f'(x) + \frac{\delta^2}{2} f''(x)$$

giving

$$\delta = \frac{f(x)}{f'(x)} + \frac{\delta^2}{2} \frac{f''(x)}{f'(x)},$$

which can be written as

$$A(\delta) = L(\delta) + N(\delta) = c,$$

where

$$L(\delta) = \delta, \quad N(\delta) = \gamma \delta^2,$$

and

$$\gamma = -\frac{1}{2} \frac{f''(x)}{f'(x)}, \quad c = \frac{f(x)}{f'(x)}.$$

Abbaoui and Cherruault [18] applied the standard Adomian decomposition on simple iteration method to solve the nonlinear algebraic equation (1) and proved the convergence of the series solution. Also, homotopy techniques were applied to find all the roots of [4]. Modified Adomian's decomposition method (MADM) and modified homotopy perturbation method (MHPM) are considered in [19,20], respectively.

In this section, homotopy analysis method (HAM) is proposed to solve (1) by using Newton–Raphson method. Let $q \in [0, 1]$ denotes an embedding parameter, $\hbar \neq 0$ an auxiliary parameter, $H(\delta) \neq 0$ an auxiliary function, and \mathcal{L} an auxiliary linear operator. We construct the zero-order deformation equation [5]

$$(1 - q)\mathcal{L}[v(q) - \delta_0] = q\hbar H(\delta)\{A[v(q)] - c\}, \quad (2)$$

where δ_0 is the initial approximation of δ , and $v(q)$ is a unknown function. It should be emphasized that one has great freedom to choose the initial guess value, the auxiliary linear operator, the auxiliary parameter \hbar , and the auxiliary function $H(\delta)$. Obviously, when $q = 0$ and $q = 1$, it holds

$$v(0) = \delta_0, \quad v(1) = \delta,$$

respectively. When q increases from 0 to 1, $v(q)$ varies from the initial guess δ_0 to the solution δ . Expanding $v(q)$ in Taylor series with respect to the embedding parameter q , one has

$$v(q) = \delta_0 + \sum_{m=1}^{\infty} \delta_m q^m, \quad (3)$$

where

$$\delta_m = \frac{1}{m!} \left. \frac{d^m v(q)}{dq^m} \right|_{q=0}.$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter \hbar , and the auxiliary function are so properly chosen that the series (3) converges at $q = 1$, one has

$$\delta = \delta_0 + \sum_{m=0}^{\infty} \delta_m, \tag{4}$$

which must be one of the solutions of (1), as proved by Liao [21]. It is very important to ensure the convergence of series (3) at $q = 1$, otherwise, the series (4) has no meanings. As $\hbar = -1$ and $H(\delta) = 1$, we obtained “homotopy perturbation method” [22,23].

Setting $\mathcal{L} \equiv L$, and $H(\delta) \equiv 1$, we have the high-order deformation equation [5]

$$L[\delta_m - \chi_m \delta_{m-1}] = \hbar R_m(\delta_0, \dots, \delta_{m-1}),$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1 \end{cases}$$

and

$$R_m(\delta_0, \dots, \delta_{m-1}) = \delta_{m-1} + \gamma \sum_{j=0}^{m-1} \delta_j \delta_{m-1-j} - (1 - \chi_m)c.$$

Hence the following recursive scheme for $m \geq 1$ is obtained

$$\delta_m = (\chi_m + \hbar)\delta_{m-1} + \hbar\gamma \sum_{k=0}^{m-1} \delta_k \delta_{m-1-k} - \hbar(1 - \chi_m)c. \tag{5}$$

Let $\Delta_M = \delta_0 + \dots + \delta_M$ denotes the M -term approximation of δ . Hence, we have the following iterative relations for various M .

For $M = 0$,

$$\begin{aligned} \delta &\approx \Delta_0 = \delta_0 = c = \frac{f(x)}{f'(x)}, \\ \alpha &= x - \delta \approx x - \Delta_0 = x - \frac{f(x)}{f'(x)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}, \end{aligned}$$

which is the Newton–Raphson method and the same as MADM and MHPM [19,20].

For $M = 1$,

$$\begin{aligned} \delta &\approx \Delta_1 = \delta_0 + \delta_1, \\ \alpha &= x - \delta \approx x - \Delta_1 = x - \frac{f(x)}{f'(x)} + \hbar \frac{f^2(x)f''(x)}{2f'^3(x)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} + \hbar \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)}, \end{aligned}$$

which is the Householder’s iteration [24] and the same as MADM and MHPM [19,20] when $\hbar = -1$.

For $M = 2$,

$$\begin{aligned} \delta &\approx \Delta_2 = \delta_0 + \delta_1 + \delta_2, \\ \alpha &= x - \delta \approx x - \Delta_2 = x - \frac{f(x)}{f'(x)} + (2 + \hbar)\hbar \frac{f^2(x)f''(x)}{2f'^3(x)} - \hbar^2 \frac{f^3(x)f''^2(x)}{2f'^5(x)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} + (2 + \hbar)\hbar \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)} - \hbar^2 \frac{f^3(x_n)f''^2(x_n)}{2f'^5(x_n)}, \end{aligned} \tag{6}$$

which is the same as MADM and MHPM [19,20] for $\hbar = -1$.

If one luckily chooses a good enough approximation, one can get accurate results by only a few terms [19,20]. When the initial value x_0 is not good, comparing with the results given by other methods, much fewer iterations are needed by HAM; even if a bad initial approximation is chosen, which leads to divergent results by other method, we can still find the root efficiently (see Tables 1 and 2). It clearly illustrates that, by means of HAM, convergence region and rate of solution can be adjusted and controlled by the auxiliary parameter h . The value of h can be determined by plotting the so-called h -curves, as suggested by Liao [21].

Example 2.1. Equation

$$f(x) = x^2 - e^x - 3x + 2 = 0$$

with solution $\alpha = 0.25753$. Let the initial value $x_0 = 100$. For the convenience of comparison, we choose the different values of h in the iterative formula (6) and obtain solution in Table 1.

Example 2.2. Equation

$$f(x) = xe^{x^2} - \sin^2 x + 3 \cos x + 5 = 0$$

with solution $\alpha = -1.20765$ and the same initial value $x_0 = 10$.

Example 2.3 (Problem 5 in [25]). Equation

$$f(x) = x^{1/3} - 1 = 0$$

with exact solution $\alpha = 1$ and the same initial value $x_0 = 5$.

Above two examples illustrate that HAM can give the solution when Newton, ADM and HPM fail to do that, as shown in Table 2.

In some cases, we can seek all roots with different h , as shown in Table 3.

Example 2.4 (Example 9 in [26]). Equation

$$f(x) = 1/x - \sin x + 1 = 0.$$

Starting with the same initial value $x_0 = -1.3$, we get a number of roots in Table 3.

3. Newton-homotopy analysis method

In homotopy analysis method, for example in (6), we set h as a fixed constant and it can be determined by h -curves. However, it is computationally intensive with computing times for seeking a proper value of h . In Newton-homotopy analysis method (N-HAM), we determine h by Newton–Raphson scheme as follows. Also,

Table 1
Comparison of the iteration number of Example 2.1

Newton	ADM	HPM	HAM(h)		
			-2.0	-3.0	-4.0
103	53	53	38	30	24

Table 2
Comparison of the iteration number of Examples 2.2 and 2.3

	Newton	ADM	HPM	HAM(h)
Example 2.2	>1000	Divergent	Divergent	97(-0.125)
Example 2.3	Divergent	Divergent	Divergent	5(-0.5)

Table 3
Multiple-roots given by HAM

Root	\hbar	Iteration number
-0.629446	-1/4	6
-3.988573	-1/144	12
-5.334676	-1/140	12
-10.55680	-1/137	12
-11.41723	-1/136	14
-16.93337	-1/134	14
-29.58439	-1/133	14
-42.19335	-1/132	12

we can use another iterative schemes. Let x_0 and \hbar_0 are the initial values for α and \hbar , respectively. From (6), we have

$$x_{n+1} = a_n + \hbar_n b_n + \hbar_n^2 c_n,$$

where

$$a_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$b_n = \frac{f^2(x_n)f''(x_n)}{f'^3(x_n)},$$

$$c_n = \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)} - \frac{f^3(x_n)f''^2(x_n)}{2f'^5(x_n)}.$$

After computing x_{n+1} , we want to renew \hbar_n by Newton–Raphson scheme on $g(\hbar) = f(a_{n+1} + \hbar b_{n+1} + \hbar^2 c_{n+1}) = 0$. Hence

$$\hbar_{n+1} = \hbar_n - \frac{g(\hbar_n)}{g'(\hbar_n)} = \hbar_n - \frac{f(a_{n+1} + \hbar_n b_{n+1} + \hbar_n^2 c_{n+1})}{f'(a_{n+1} + \hbar_n b_{n+1} + \hbar_n^2 c_{n+1})[2\hbar_n c_{n+1} + b_{n+1}]}, \quad (7)$$

where

$$\hbar_0 = -\frac{f(a_0)}{b_0 f'(a_0)}.$$

Therefore, this method does not need to choose \hbar and has better numerical behaviour. Some results are listed in Tables 4 and 5.

Example 3.1 (Example 3.2 in [19,20]). Equation

$$f(x) = x - 2 - e^{-x} = 0$$

with solution $\alpha = 2.120028239$. Comparison among Newton method, ADM, HPM and Newton-HAM are listed in Table 4.

Example 3.2. Equation

$$f(x) = \cos x - x = 0$$

Table 4
Comparison of the iteration number of Example 3.1

Initial value	-10	-40	-100
Newton	15	45	105
ADM [19]	8	23	53
HPM [20]	8	23	53
N-HAM	8	11	17

Table 5
Comparison of the iteration number of Example 3.2

Initial value	2	3	4
Newton	4	7	Divergent
ADM	4	8	Divergent
HPM	4	8	Divergent
N-HAM	3	4	6

Table 6
The iteration procedure for \tilde{h}_n of Example 3.1

IN	$x_0 = 2$		$x_0 = -5$	
	\tilde{h}_n	$f(x_n)$	\tilde{h}_n	$f(x_n)$
0	-0.487238	-0.135335	-1.01685	-155.41316
1	-0.743565	-0.000222	-1.97808	-22.112262
2	-0.871783	-1.550×10^{-13}	-1.69447	-0.7299097
3	-0.935891	-1.889×10^{-23}	-1.34792	-0.0026254
4	-0.967946	-7.015×10^{-50}	-1.17396	-3.966×10^{-8}
5	-0.983973	-2.419×10^{-103}	-1.08698	-2.277×10^{-18}
6	-0.991986	-7.190×10^{-211}	-1.04349	-1.876×10^{-39}
7			-1.02175	-3.186×10^{-82}
8			-1.01087	-2.296×10^{-168}

with solution $\alpha = 0.739085$. Some result can be found in Table 5.

What we emphasize here is that \tilde{h}_n tends to the same value -1 when the root is obtain, as shown in Table 6.

4. Conclusion

To increase the range of initial value and the efficiency of convergence, this paper present a new implementation scheme based on HAM. We found that HAM logically contains ADM. Besides, if the same initial value and the same auxiliary linear operator are chosen, the approximations given by HPM are exactly a special case of those given by HAM when $\tilde{h} = -1$ and $H = 1$. Therefore, these two methods can be unified in the frame of HAM. In HAM, we set \tilde{h} as a fixed constant and it can be determined by \tilde{h} -curves. In this work, we give a new approach to determine the value of \tilde{h} using Newton–Raphson scheme. Their efficiency is demonstrated by numerical experiments.

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