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On peaked solitary waves of the Degasperis-Procesi equation

LIAO ShiJun*

State Key Laboratory of Ocean Engineering, Dept. of Mathematics, School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai 200240, China

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The Degasperis-Procesi (DP) equation describing the propagation of shallow water waves contains a physical parameter ω, and it is well-known that the DP equation admits solitary waves with a peaked crest when $\omega = 0$. In this article, we illustrate, for the first time, that the DP equation admits peaked solitary waves even when $\omega \neq 0$. This is helpful to enrich our knowledge and deepen our understandings about peaked solitary waves of the DP equation.

peaked solitary waves, discontinuity, Degasperis-Procesi equation

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1 Introduction

The solitary surface wave was first discovered by Russell [1] in 1834. Since then, many models for shallow water waves have been developed, such as the Boussinesq equation [2], the Korteweg & de Vries (KdV) equation [3] and the Benjamin-Bona-Mahony (BBM) equation [4], and so on. All of them admit solitary waves with smooth crest. However, in contrast to the KdV, Boussinesq and BBM equations, the celebrated Camassa-Holm (CH) equation [5]

$$u_t + 2\omega u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1}$$

can model both phenomena of soliton interaction and wave breaking (see ref. [6]), where $u(x,t)$ denotes the wave elevation, x, t are the temporal and spatial variables, ω is a constant related to the critical shallow water wave speed, and the subscript denotes the partial differentiation, respectively. The CH equation (1) is integrable and bi-Hamiltonian. Especially, when $\omega = 0$, the CH equation (1) has the peaked solitary wave [5]

$$u(x,t) = c\exp(-|x - ct|),$$

which has a discontinuity at crest. This is rather unusual.

Degasperis and Procesi [7] investigated a generalized equation

$$u_t + 2\omega u_x - u_{xxt} + (\beta + 1)uu_x = \beta u_x u_{xx} + uu_{xxx} \tag{2}$$

and found that it is completely integrable only when $\beta = 2$, corresponding to the CH equation (1), and $\beta = 3$, corresponding to the so-called Degasperis - Procesi (DP) equation

$$u_t + 2\omega u_x - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \tag{3}$$

respectively. The DP equation (3) can model the propagation of shallow water waves with small amplitude and large wavelength. It is interesting that, like the CH equation (1), the DP equation (3) also admits solitary waves with a peaked crest when $\omega = 0$.

Currently, Liao [8] gained the closed-form solutions of the peaked solitary waves of the KdV equation, the modified KdV equation, the Boussinesq equation and the BBM equation. Besides, by means of the Mathematica package BVPh 1.0 that is based on the homotopy analysis method (HAM) [9–12], a powerful analytic method for highly nonlinear equations, Liao [13] found that the CH equation (1)

*Corresponding author (email: sjliao@sjtu.edu.cn)

also admits peaked solitary waves even in the case of $\omega \neq 0$. All of these suggest that most of mainstream models of shallow water waves might admit peaked solitary waves, if such kind of peaked waves indeed have physical meanings.

In this article, using the HAM-based Mathematica package BVPh 1.0 for nonlinear boundary value problems in a similar way, we illustrate that the DP equation (3) also admits peaked solitary waves even when $\omega \neq 0$. In addition, we further illustrate that eq. (2) for arbitrary constant β and the more generalized equation

$$u_t + 2\omega u_x - u_{xxt} + \alpha u u_x = \beta u_x u_{xx} + u u_{xxx} \tag{4}$$

for arbitrary constants α and β also admit peaked solitary waves even when $\omega \neq 0$, although they are not integrable in general cases.

2 Peaked solitary waves of eq. (4) in the case of $\omega \neq 0$

Let us first consider the propagating solitary waves of the DP equation (3) with permanent form, corresponding to $\alpha = 4$ and $\beta = 3$ of eq. (4). To avoid the repeat, eq. (4) is used to describe the basic ideas of our analytic approach. Writing $\xi = x - c\ t$ and $w(\xi) = c\ u(x,t)$, the original equation (4) becomes

$$w''' - \left(1 - \frac{2\omega}{c}\right) w' + \alpha w w' = \beta w' w'' + w w''', \tag{5}$$

subject to the boundary conditions

$$w \to 0, w' \to 0, w'' \to 0, \quad \text{as } |\xi| \to +\infty, \tag{6}$$

where the prime denotes the differentiation.

The corresponding linearized equation

$$w''' - \left(1 - \frac{2\omega}{c}\right) w' = 0 \tag{7}$$

has the peaked solitary wave

$$w(\xi) = A\ \exp(-\mu|\xi|), \quad \mu = \sqrt{1 - \frac{2\omega}{c}}, \tag{8}$$

provided $\omega < c/2$, where $A = w(0)$. The solution of eq. (5) can be expressed in the form

$$w(\xi) = \sum_{n=1}^{+\infty} a_n\ \exp(-n\mu|\xi|),$$

where a_n is a consant to be determined. Using the homotopy analysis method (HAM) [9–12], an analytic technique for highly nonlinear differential equations, we gain the series solution

$$w(\xi) = w_0(\xi) + \sum_{m=0}^{+\infty} w_m(\xi). \tag{9}$$

Here

$$w_0(\xi) = A \exp(-\mu|\xi|)$$

is the initial guess, w_m for $m \geqslant 1$ is governed by

$$\mathcal{L}\left[w_m(\xi) - \chi_m\ w_{m-1}(\xi)\right] = c_0\ \delta_{m-1}(\xi), \tag{10}$$

subject to the boundary condition

$$w_m(0) = 0,\ w_m \to 0, \quad \text{as } |\xi| \to +\infty, \tag{11}$$

where

$$\mathcal{L} f = f''' - \mu^2 f'$$

is an auxiliary linear operator, and

$$\delta_n = w_n''' - \mu^2 w_n' + \sum_{j=0}^{n} \Big[\beta\, w_j\, w_{n-j}' - \alpha\, w_j'\, w_{n-j}'' - w_j\, w_{n-j}'''\Big], \tag{12}$$

$$\chi_n = \begin{cases} 1 & \text{when } n > 1, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Note that $c_0 \neq 0$ is an auxiliary parameter, called the convergence-control parameter, which provides us a convenient way to guarantee the convergence of the approximation series. For details, please refer to Liao [9, 11, 12]. In fact, directly using the HAM-based Mathematica package BVPh 1.0 (see Part II of Liao's book [12]) for nonlinear boundary-value/eigenvalue problems, it is straightforward to gain high-order analytic approximations of eqs. (5) and (6). For details, please refer to the Appendix.

The accuracy of the mth-order approximation of $w(\xi)$ is defined by the averaged residual square of the governing equation (5) in the domain $\xi \in [0, a]$:

$$\mathcal{E}_m = \frac{1}{a} \int_0^a \left[\mathcal{N}\left(\sum_{j=0}^{m} w_j\right)\right]^2 \mathrm{d}\xi, \tag{14}$$

where

$$\mathcal{N} w = w''' - \mu^2 w' + \alpha\ w w' - \beta\ w' w'' - w w'''.$$

We choose $a = 10$ in this article, because the wave elevation decays exponentially.

For simplicity, we investigate the case $c = 1$ in this article. Without loss of generality, we first consider the case of $\omega = 1/4$ and $A = \pm 1/10$ for the DP equation (3). As shown in Table 1, the averaged residual squares of the 10th-order analytic approximations decrease to 5.2×10^{-20} in the case of $A = 1/10$ and to 3.5×10^{-20} in the case of $A = -1/10$, respectively. Besides, the corresponding values of $u'(0_+)$ (the limit is taken as $\xi \to 0$ from the right) quickly converge to -0.065105 and 0.074782, respectively. All of these clearly indicate that the series given by the Mathematica package BVPh 1.0 converge to the solution of the DP equation (3). The solutions for larger values of $|A|$ can be gained in a similar way. All of these solitary waves have a peaked crest, as shown in Figure 1. These clearly illustrate that, like the CH equation (1), the DP equation (3) also admits peaked solitary waves even when $\omega \neq 0$. Besides, like the CH equation (1),

Table 1 $u'(0_+)$ of the mth-order analytic approximations and the residual squares of the DP equation (3) in the case of $c = 1, \omega = 1/4$, and $A = \pm 1/10$ given by the HAM-based Mathematica package BVPh 1.0 with the convergence-control parameter $c_0 = -1$

	$A = 1/10$		$A = -1/10$	
m	$u'(0_+)$	$\mathcal{E}_m$	$u'(0_+)$	$\mathcal{E}_m$
2	−0.065250	3.8×10^{-9}	0.074675	4.8×10^{-9}
4	−0.065110	2.0×10^{-12}	0.074779	1.6×10^{-12}
6	−0.065105	9.2×10^{-15}	0.074782	5.9×10^{-15}
8	−0.065105	2.4×10^{-17}	0.074782	1.6×10^{-17}
10	−0.065105	5.2×10^{-20}	0.074782	3.5×10^{-20}

Table 2 $u'(0_+)$ of the mth-order analytic approximations and the residual squares of eq. (2) in the case of $c = 1, \omega = 1/4, A = \pm 1/10$ and $\beta = 5/2$ given by the HAM-based Mathematica package BVPh 1.0 with the convergence-control parameter $c_0 = -1$

	$A = 1/10$		$A = -1/10$	
m	$u'(0_+)$	$\mathcal{E}_m$	$u'(0_+)$	$\mathcal{E}_m$
2	−0.066002	2.1×10^{-9}	0.074251	2.6×10^{-9}
4	−0.065904	5.7×10^{-13}	0.074323	5.3×10^{-13}
6	−0.065901	1.6×10^{-15}	0.074325	1.1×10^{-15}
8	−0.065901	3.0×10^{-18}	0.074325	2.2×10^{-18}
10	−0.065901	4.4×10^{-21}	0.074325	3.2×10^{-21}

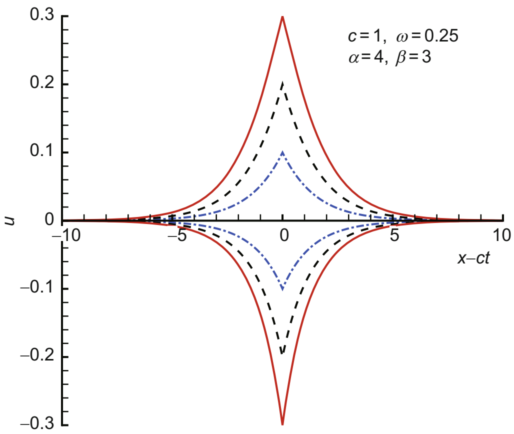

Figure 1 (Color online) The peaked solitary waves $u(x, t)$ of the DP equation (3) in the case of $\omega = 1/4$ and $c = 1$. Solid line: $A = \pm 3/10$; dashed line: $A = \pm 1/5$; dash-dotted line: $A = \pm 1/10$

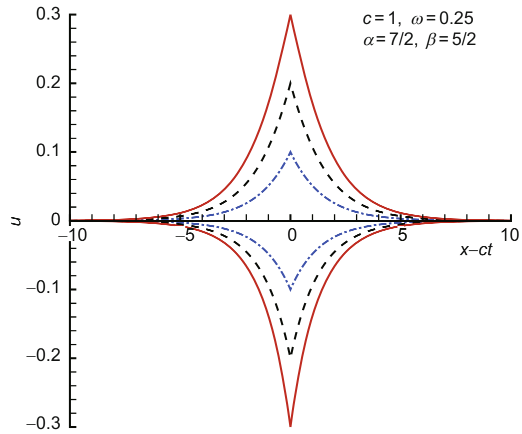

Figure 2 (Color online) The peaked solitary waves $u(x, t)$ of eq. (2) in the case of $\omega = 1/4$, $c = 1$ and $\beta = 5/2$, corresponding to $\alpha = 7/2, \beta = 5/2$ of eq. (4). Solid line: $A = \pm 3/10$; Dashed line: $A = \pm 1/5$; Dash-dotted line: $A = \pm 1/10$.

the phase speed of the peaked solitary waves of the DP equation (3) has nothing to do with the wave amplitude A. This is quite interesting.

As pointed out by Degasperis and Procesi [7], the generalized equation (2) is completely integrable only when $\beta = 2$ or $\beta = 3$, corresponding to the CH equation (1) or the DP equation (3), respectively. Does eq. (2) admit peaked solitary waves when $\beta \neq 2$ and $\beta \neq 3$?

Without loss of generality, let us consider the case of $c = 1, \omega = 1/4, \beta = 5/2$ and $A = \pm 1/10$ for eq. (2), corresponding to $\alpha = 7/2, \beta = 5/2$ for eq. (4). Similarly, by means of the HAM-based Mathematica package BVPh 1.0, we gain the convergent approximations with $c_0 = -1$. As shown in Table 2, the averaged residual squares of the 10th-order analytic approximations decrease to 4.4×10^{-21} in the case of $A = 1/10$ and to 3.2×10^{-21} in the case of $A = -1/10$, respectively. Similarly, using the Mathematica package BVPh 1.0, we gain the convergent approximations for larger values of A, as shown in Figure 2. Note that all of these solitary waves have a peaked crest! Thus, the generalized equation (2) in the case of $\omega \neq 0$ also admits peaked solitary waves, even when $\beta \neq 2$ and $\beta \neq 3$ so that it is not integrable!

How about the more generalized equation (4)?

Without loss of generality, let us consider the generalized equation (4) in the case of $c = 1, \omega = 1/4, \alpha = 11/3, \beta = 5/2$ and $A = \pm 1/10$. Similarly, by means of the HAM-based Mathematica package BVPh 1.0, we gain the convergent approximations with $c_0 = -1$. As shown in Table 3, the averaged residual squares of the 10th-order analytic approximations decrease to 1.1×10^{-20} in the case of $A = 1/10$ and to 7.8×10^{-21} in the case of $A = -1/10$, respectively. Similarly, using the Mathematica package BVPh 1.0, we gain the convergent approximations for larger values of $|A|$, as shown in Figure 3. It should be emphasized that all of these solitary waves have a peaked crest. These illustrate that the generalized equation (4) also admits peaked solitary waves for arbitrary constants α and β even when $\omega \neq 0$, although the corresponding equation is *not* integrable in general cases. Therefore, the existence of peaked solitary waves might be a common property of the generalized equation (4).

As pointed out by Constantin and Molinet [14], all of these peaked solitary waves should be understood mathematically as weak solutions of the generalized equation (4). However, physically, this kind of discontinuity of wave elevation widely appears in fluid mechanics, such as dam break [15] in hydrodynamics and shock waves in aerodynamics, which

Table 3 $u'(0_+)$ of the mth-order analytic approximations and the residual squares of the generalized equation (4) in the case of $c = 1, \omega = 1/4, A = \pm 1/10, \alpha = 11/3$ and $\beta = 5/2$ given by the HAM-based Mathematica package BVPh 1.0 with the convergence-control parameter $c_0 = -1$

	$A = 1/10$		$A = -1/10$	
m	$u'(0_+)$	$\mathcal{E}_m$	$u'(0_+)$	$\mathcal{E}_m$
2	−0.065541	2.3×10^{-9}	0.074576	2.9×10^{-9}
4	−0.065428	9.8×10^{-13}	0.074658	7.6×10^{-13}
6	−0.065425	3.3×10^{-15}	0.074660	2.2×10^{-15}
8	−0.065425	6.6×10^{-18}	0.074660	4.6×10^{-18}
10	−0.065425	1.1×10^{-20}	0.074660	7.8×10^{-21}

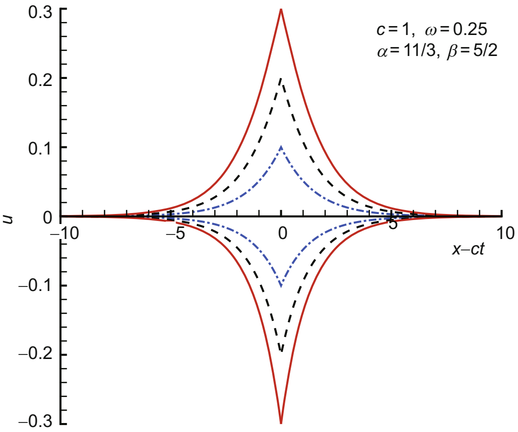

Figure 3 (Color online) The peaked solitary waves $u(x, t)$ of (4) in the case of $\omega = 1/4, c = 1, \alpha = 11/3$ and $\beta = 5/2$. Solid line: $A = \pm 3/10$; dashed line: $A = \pm 1/5$; dash-dotted line: $A = \pm 1/10$.

have clear physical meanings. Note also that such kind of discontinuous problems belong to the so-called Riemann problem [15–18], a classical field of fluid mechanics.

3 Concluding remarks

In this paper, using the HAM-based Mathematica package BVPh 1.0 as a useful tool, we illustrate that, like the CH equation (1), the DP equation (3) also admits peaked solitary waves even when $\omega \neq 0$. Furthermore, we illustrate in a similar way that the generalized equations (2) and (4) also admit peaked solitary waves even when $\omega \neq 0$ and the corresponding equation is not integrable! The work of Liao [8] suggests that nearly all mainstream models of shallow water waves might admit peaked solitary waves, if they indeed have physical meanings. This article further confirms this viewpoint. This is mainly because the fully nonlinear water wave equations might admit peaked solitary waves, as discussed by Liao [19]. Therefore, mathematically speaking, peaked solitary waves might widely exist and might be a common property of models for water waves. However, it should be carefully investigated whether or not such kind of peaked solitary waves indeed could be regarded as weak solutions and besides have physical meanings.

Appendix

The use of HAM-based Mathematica package BVPh 1.0

The Mathematica package BVPh 1.0 for nonlinear boundary-value/eigenvalue problems is developed and issued by Liao [12] (Part II), which is based on the HAM and is free available online. Using the BVPh 1.0, it is straightforward to gain the analytic approximations of the peaked solitary waves of eqs. (5) and (6) for given $w(0) = A$. Here, we briefly describe how to do it in case of $c = 1, \omega = 1/4$, $\alpha = 4, \beta = 3$ and $A = 1/10$, corresponding to the DP equation (3).

1. First, download the BVPh 1.0 (the code file is named by BVPh_1.0.txt) online (http://numericaltank.sjtu.edu.cn/BVPh.htm) and save it in a directory such as C:/math/DP as an example.

2. Then, run the computer algebra system Mathematica, and type the following command one by one:

```
SetDirectory["C:\math\DP"];
<<InputDP.txt
```

The file named InputDP.txt contains the following Mathematica commands and necessary definitions for BVPh 1.0:

```
(* Install the BVPh 1.0 *)
<<BVPh1_0.txt;

(* Define the physical and control parameters *)
TypeEQ = 1;
ApproxQ = 0;
ErrReq = 10^(-30);
zRintegral = 10;

(* Define the governing equation *)
ALPHA = 4 ;
BETA = 3;
mu2 = 1-2*omega/c;
f[z_,u_,lambda_] := D[u,z,3]-mu2*D[u,z] \
+ALPHA*u*D[u,z]-BETA*D[u,z]*D[u,z,2]-u*D[u,z,3];
If[BETA == 2 && ALPHA == 3, Print["CH equation"]];
If[BETA == 3 && ALPHA == 4, Print["DP equation"]];

(* Define Boundary conditions *)
zR = Infinity;
OrderEQ = 3;
BC[1,z_,u_,lambda_] := Limit[u-A, z -> 0 ];
BC[2,z_,u_,lambda_] := Limit[u, z -> zR ];
BC[3,z_,u_,lambda_] := Limit[D[u,z], z -> zR ];

(* Define initial guess *)
mu = Sqrt[mu2];
u[0] = A*Exp[-mu*z];

(* Define output term *)
output[z_,u_,k_]:=Print["output=",D[u[k],z]/.z->0//N];

(* Defines the auxiliary linear operator *)
L[u_] := D[u,z,3] - mu2 * D[u,z];
```

```
(* Print input and control parameters *)
PrintInput[u[z]];

(* Set convergence-control parameter c0 and physical parameters *)
c0 = -1 ;
A = 1/10;
omega = 1/4;
c = 1;
Print[" c0=", c0, " omega=",omega, " c=", c, " A=",A];

(* Gain up to 10th-order HAM approxiamtion *)
BVPh[1,10];

(* Get results in the whole domain *)
For[k=0,k<=10,k++,W[k] = U[k] /. z− > Abs[x]];

(* Show the 5th and 10th-order approximation *)
Plot[W[5],W[10],x,-10,10,PlotRange− >Min[A,0],Max[A,0]]
```

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