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A Simple Approach of Enlarging Convergence Regions of Perturbation Approximations

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Abstract. In this paper, a simple approach to enlarging convergence regions of perturbation approximations is proposed. Based on the so-called general Taylor theorems, this approach has a solid mathematical foundation. Moreover, it is rather simple to apply. Two nonlinear equations, the Riccati equation and the Van der Pol equation are used as examples to illustrate the validity and the great potential of this approach.

Keywords: Homotopy analysis method, enlarging convergence region, general Taylor theorem, Van der Pol equation, Riccati equation.

1. Introduction

Although perturbation techniques are widely applied to analyze nonlinear problems in science and technology, they are in most cases strongly dependent on small parameters and therefore valid only for weakly nonlinear problems. Thus, it is necessary and worthwhile developing some new analytic techniques which do not depend upon small parameters at all. Some attempts in this direction have been made and some progress has been achieved. However, due to the fact that perturbation techniques have been widely applied, it should be beneficial to propose a simple approach to enlarge the convergence regions of perturbation approximations.

Liao [6–10] proposed a new kind of analytic technique for nonlinear problems, namely the homotopy analysis method (HAM). Based on homotopy which is an important part of topology, the validity of the homotopy analysis method does *not* depend on whether or not equations under consideration contain small parameters. This is in essence quite different from perturbation techniques. As reported by Liao [6–10], the homotopy analysis method is valid even for those strongly nonlinear problems whose governing equations and boundary conditions do not contain small parameters. Liao [9, 10] successfully applied the homotopy analysis method to give, the first time (to our knowledge), explicit, uniformly valid, and purely analytic solutions of viscous flows over a semi-infinite flat plate governed by Blasius' or Falkner–Skan's equations. Also, based on the homotopy analysis method, Liao [11] greatly generalized the boundary element method and proposed a so-called general boundary element method which largely increases the application regions of the boundary element method as a numerical tool. All of these verify the validity and great potential of the homotopy analysis method.

Liao [8] considered a 2D Blasius viscous flow over a semi-infinite flat plate governed by a nonlinear differential equation

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0, \quad \eta \in [0, +\infty), \tag{1}$$

with boundary conditions

$$f(0) = f'(0) = 0, \quad f'(+\infty) = 1, \tag{2}$$

where the prime denotes derivative with respect to η. It is well known that Blasius gave a solution to the above problem in power series, i.e.

$$f(\eta) = \sum_{k=0}^{+\infty} \left(-\frac{1}{2}\right)^k \frac{A_k \sigma^{k+1}}{(3k+2)!} \eta^{3k+2}, \tag{3}$$

where

$$A_0 = A_1 = 1, \quad A_k = \sum_{r=0}^{k-1} \binom{3k-1}{3r} A_r\, A_{k-r-1} \quad (k \geq 2) \tag{4}$$

and $\sigma = f''(0) = 0.33206$ has to be numerically given. As pointed out by Liao [8], the power series (3) can be also obtained by perturbation techniques. However, it converges in a rather restricted region $|\eta| \leq 5.690$. Liao [8] applied the homotopy analysis method to solve the foregoing Blasius flow and obtained a *family* of power series:

$$f(\eta) = \lim_{m \to +\infty} \sum_{k=0}^{m} \left[\left(-\frac{1}{2}\right)^k \frac{A_k \sigma^{k+1}}{(3k+2)!} \eta^{3k+2} \right] \Phi_{m,k}(\mu), \quad \eta \in [0, +\infty), -2 < \mu < 0, \tag{5}$$

where A_k $(k \geq 0)$ is given by Equation (4), $\sigma = f''(0)$ and the real function $\Phi_{m,n}(\mu)$ is defined by

$$\Phi_{m,n}(\mu) = \begin{cases} 0 & (n > m), \\ (-\mu)^n \sum_{k=0}^{m-n} \binom{m}{m-n-k} \binom{n+k-1}{k} \mu^k & (1 \leq n \leq m), \\ 1 & (n \leq 0). \end{cases} \tag{6}$$

The power series (5) is convergent in the region

$$-\rho_0 < \eta < \rho_0 \left[\frac{2}{|\mu|} - 1 \right]^{1/3} \quad (-2 < \mu < 0), \tag{7}$$

which becomes larger and larger as $|\mu|$ $(-2 < \mu < 0)$ gets smaller and smaller, where $\rho_0 = 5.690$ is the convergence radius of the Blasius power series (3). Therefore, the power series (5) may be valid for the *whole* domain $\eta = [0, +\infty)$ as $|\mu|$ $(-2 < \mu < 0)$ tends to zero! Moreover, the Blasius power series (3) is only a special case of (5) when $\mu = -1$, because Liao [8] pointed out that the real function $\Phi_{m,n}(\mu)$ has such an interesting property that $\Phi_{m,n}(-1) = 1$ for $n \leq m$. Therefore, the power series (5) given by the homotopy analysis method is more general than the perturbation approximation (3), because the former contains the latter in logic. This example illustrates well the validity and the great potential of the homotopy analysis method. Liao [8] further studied the properties of the approaching function $\Phi_{m,n}(\mu)$ in general and gave a related mathematical theorem, namely the General Taylor Theorem, by which the convergence region of a traditional Taylor series can be greatly enlarged in most cases. Our current studies indicate that the so-called approaching function has more general meanings and can be further generalized, which in many cases can be applied to enlarge convergence regions of perturbation approximations. Therefore, it provides us with a simple but effective approach to improve perturbation results.

In this paper we use two nonlinear problems as examples to show this point. Firstly, we describe the perturbation approximations of two nonlinear problems. Then, we simply cite the so-called general Taylor theorems, which can provide us with a solid foundation for the proposed approach. Finally, we illustrate how to simply apply the general Taylor theorems to greatly enlarge the convergence regions of related perturbation approximations so as to verify the validity and potential of this approach.

2. Perturbation Approximations of Two Nonlinear Equations

Perturbation approximations are in general strongly dependent on small parameters. Moreover, in most cases, perturbation approximations are valid in rather restricted regions of considered small parameters. For instance, we consider here perturbation approximations of the following two nonlinear problems:

EXAMPLE 1 (Riccati's equation). First, let us consider Riccati's equation

$$y'(t) + \varepsilon y^2(t) = t, \quad y(0) = 0, \tag{8}$$

where $t \geq 0$, $\varepsilon > 0$ and $y(t)$ is a real function. *Suppose* that t is small enough and $y(t)$ can be expressed in the form

$$y(t) = \sum_{k=0}^{+\infty} \Psi_k(t)\varepsilon^k. \tag{9}$$

Substituting Equation (9) into Equation (8) and then equalizing its coefficients gives the perturbation approximation

$$y(t) = \frac{1}{2}t^2 - \frac{\varepsilon}{20}t^5 + \frac{\varepsilon^2}{160}t^8 - \frac{7\varepsilon^3}{8800}t^{11} + \cdots = \sum_{k=0}^{+\infty} C_k t^{3k+2}, \tag{10}$$

where

$$C_0 = \frac{1}{2}, \quad C_m = -\frac{\varepsilon}{3m+2}\sum_{k=0}^{m-1} C_k C_{m-1-k}. \tag{11}$$

The convergence radius of the power series (10) is

$$\bar{\rho}_0 = \left(\frac{7.83735}{\varepsilon}\right)^{1/3}. \tag{12}$$

Obviously, $\bar{\rho}_0$ is large enough only if ε is sufficiently small. However, as ε increases, the convergence radius of power series (10) decreases. In fact, for $\varepsilon > 1$, Equation (10) is almost useless. This is a simple but typical example showing that the validity of perturbation approximations is strongly dependent on the value of the so-called small parameters under consideration.

EXAMPLE 2 (Van der Pol's equation). Secondly, let us consider the well-known Van der Pol equation

$$u''(t) + u(t) = \varepsilon[1 - u^2(t)]u'(t), \tag{13}$$

which describes a typical nonlinear self-excited system, where $u(t)$ is a real function, t denotes time, ε is a real number and the prime denotes the derivative with respect to t. For more details, refer to [1–5, 12, 13].

There are many ways to obtain perturbation approximations of Van der Pol's equation (13). Here, we exactly follow Chen et al. [3] to give the related perturbation approximations. The difference is only that we apply the automatic derivation software MATHEMATICA and high-performance computers to get higher-order approximations.

Chen et al.'s approach [3] is classical. Let ω denote the frequency of vibration. Then, by means of transformation $x = \omega t$, Equation (13) becomes

$$\omega^2 \frac{d^2 u(x)}{dx^2} + u(x) = \varepsilon\omega[1 - u^2(x)]\frac{du(x)}{dx}. \tag{14}$$

Supposing ε is a small parameter, we rewrite the frequency ω in the form

$$\omega = b_0 + \sum_{k=1}^{+\infty} b_k \varepsilon^{2k} \tag{15}$$

and the related limit cycle as

$$u(x) = \sum_{k=0}^{+\infty} \varphi_k(x)\varepsilon^k, \tag{16}$$

where

$$\varphi_{2m}(x) = \sum_{k=0}^{2m} c_{2k+1,2m} \cos[(2k+1)x] \quad (m = 0, 1, 2, 3, \ldots), \tag{17}$$

$$\varphi_{2m+1}(x) = \sum_{k=0}^{2m+1} c_{2k+1,2m+1} \sin[(2k+1)x] \quad (m = 0, 1, 2, 3, \ldots). \tag{18}$$

Substituting the above expressions into Equation (14) and then balancing the coefficients, by means of the software MATHEMATICA, we obtain the related coefficients of perturbation approximations one after another in order, say,

$$c_{1,0} = 2,$$

$$c_{1,1} = \frac{3}{4}, \quad c_{3,1} = -\frac{3}{4},$$

$$c_{1,2} = -\frac{1}{8}, \quad c_{3,2} = \frac{3}{16}, \quad c_{5,2} = -\frac{5}{96},$$

$$c_{1,3} = -\frac{7}{256}, \quad c_{3,3} = \frac{21}{256}, \quad c_{5,3} = -\frac{35}{576}, \quad c_{7,3} = \frac{7}{576},$$

$$c_{1,4} = \frac{73}{12288}, \quad c_{3,4} = -\frac{47}{1536}, \quad c_{5,4} = \frac{1085}{27648}, \quad c_{7,4} = -\frac{2149}{110592}, \quad c_{9,4} = \frac{61}{20480},$$

$$c_{1,5} = -\frac{12971}{4423680}, \quad c_{3,5} = -\frac{2591}{294912}, \quad c_{5,5} = \frac{52885}{2654208}, \quad c_{7,5} = -\frac{110621}{6635520}, \quad c_{9,5} = \frac{7457}{1228800},$$

$$\ldots,$$

and moreover,

$$b_0 = 1, \quad b_1 = -\frac{1}{16}, \quad b_2 = \frac{17}{3072}, \quad b_3 = \frac{35}{884736}, \quad b_4 = -\frac{678899}{5096079360},$$

$$b_5 = \frac{28160413}{2293235712000} \approx 1.227977 \times 10^{-5},$$

$$\ldots,$$

and so on. Andersen and Geer [1] gave the perturbation expression (15) to $O(\varepsilon^{162})$ and listed the related values of b_k. The amplitude A of the self-excited vibration is given by the series

$$A = \sum_{j=0}^{+\infty} a_{2j} \varepsilon^{2j}, \tag{19}$$

where the coefficients a_{2j} are given in detail by Dadfar et al. [4].

However, as reported in [1, 4], the convergence of the series (15) and (19) are limited by the presence of two pairs of complex conjugate singularities located (approximately) at the point

$$\varepsilon^2 = R_0^2\, \mathrm{e}^{\pm 2\beta_0 i}, \tag{20}$$

where $R_0 = 1.85$, $\beta = 0.897$. So, the convergence radius ρ of the power series (15) for the frequency ω and the series (19) for the amplitude A are unfortunately only about 1.85. Therefore, for ε greater than 1.85, such as $\varepsilon = 2.5$, $\varepsilon = 3$, $\varepsilon = 4$, $\varepsilon = 5$ and so on, both of the perturbation power series (15) and (19) are divergent.

Also, Dadfar et al. [4] pointed out that the convergence radius of the limit cycle (16) varies with x and is somewhat smaller for values of x near $\pi/2$ and $3\pi/2$ than for other values of x. Their analysis indicated that, for values of x between 0 and about $\pi/4$, there exists only two pairs of 'fixed' complex conjugate branch singularity points rather close to those indicated by Equation (20). As x increases to values greater than $\pi/4$, a 'new' singularity leaves its position in the second quadrant and moves toward the imaginary axis and crosses that axis at $x = \pi/2$. As x increases above $\pi/2$, the moving singularity enters the first quadrant and moves toward the position of the 'fixed' singularity in that quadrant. For $\pi/4 < x < 3\pi/4$, the moving singularity becomes the dominant singularity, because it moves closer to the origin than the fixed singularity. At $x = \pi/2$, the distance from the moving singularity to the origin reaches a minimum of about 1.65. Therefore, the perturbation expressions (16) of the limit cycle is divergent when $\varepsilon > 1.65$.

The foregoing two simple but typical examples illustrate well the strong dependence of straightforward perturbation approximations on related small parameters. Obviously, it is necessary and worthwhile to propose a simple but effective approach to improving perturbation approximations by means of enlarging their convergence regions.

3. The General Taylor Theorem

While applying the homotopy analysis method to solve the two-dimensional Blasius flow problem in fluid mechanics, Liao [8] obtained a so-called approaching function $\Phi_{m,k}(\mu)$ defined by Equation (6). Liao [8] further seriously studied the so-called approaching function and found that it has some interesting properties such as

$$\Phi_{m+1,n}(\hbar) - \Phi_{m,n}(\hbar) = \binom{m}{n-1} (-\hbar)^n (1+\hbar)^{m-n+1}, \quad n \le m, \tag{21}$$

$$\Phi_{m,n}(-1) = 1, \quad n \le m, \tag{22}$$

$$\lim_{m \to +\infty} \Phi_{m,\nu}(\hbar) = 1, \quad |1+\hbar| < 1, \tag{23}$$

and so on, where $\hbar$ can be a complex number and $\nu \ge 0$ is a finite positive integer. Furthermore, Liao [11] gave such a related mathematical theorem:

THEOREM 1. *Let $\hbar$ be a complex number and ν a finite positive integer. If the complex function $f(z)$ is analytic at $z = z_0$, it holds that*

$$\begin{aligned} f(z) &= \lim_{m \to +\infty} \sum_{k=0}^{m} \left[\frac{f^{(k)}(z_0)}{k!} (z-z_0)^k \right] \Phi_{m,k}(\hbar) \\ &= \lim_{m \to +\infty} \sum_{k=0}^{m+\nu} \left[\frac{f^{(k)}(z_0)}{k!} (z-z_0)^k \right] \Phi_{m,k-\nu}(\hbar) \\ &= \lim_{m \to +\infty} \sum_{k=0}^{m} \left[\frac{f^{(k)}(z_0)}{k!} (z-z_0)^k \right] \Phi_{m+\nu,k+\nu}(\hbar) \end{aligned} \tag{24}$$

in the region determined by

$$\bigcap_{k \in I} \left| 1 + \hbar - \hbar \left(\frac{z - z_0}{\xi_k - z_0} \right) \right| < 1, \quad |1+\hbar| < 1, \tag{25}$$

where $\xi_k (k \in I)$ denotes all singularities of $f(z)$ and $\Phi_{m,k}(\hbar)$ is defined by Equation (6).

Owing to Equation (22), the traditional Taylor series of $f(z)$ at $z = z_0$, i.e.

$$\lim_{m \to +\infty} \sum_{k=0}^{m} \left[\frac{f^{(k)}(z_0)}{k!} (z-z_0)^k \right],$$

is only a special case of the series (24) when $\hbar = -1$. Besides, the above theorem can be further generalized. Currently, the author has rigorously proved a more general theorem:

THEOREM 2. *Let p, z, z_0 and $\hbar \ne 0$ be complex numbers, $\mathcal{A}(p)$ and $\mathcal{B}(p)$ be analytic complex functions in the region $|p| \le 1$ whose Maclaurin series are $\sum_{k=1}^{+\infty} \alpha_{1,k} p^k$ and $\sum_{k=1}^{+\infty} \beta_{1,k} p^k$, respectively. Besides, $\mathcal{A}(p) = \mathcal{B}(p) = 1$. Define*

$$\alpha_{m,k} = \sum_{i=m-1}^{k-1} \alpha_{m-1,i} \alpha_{1,k-i}, \quad m \ge 2, k \ge m, \tag{26}$$

$$\beta_{0,0} = 1, \quad \beta_{0,k} = 0, \quad k \geq 1, \tag{27}$$

$$\beta_{m,k} = \sum_{i=m-1}^{k-1} \beta_{m-1,i}\, \beta_{1,k-i}, \quad m \geq 2, k \geq m, \tag{28}$$

and

$$\begin{aligned} \chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B}) \;=\; & (-\hbar)^k \sum_{n=0}^{m-k} \sum_{r=0}^{n} \binom{k+r-1}{r} \\ & \times\; (1+\hbar)^r \sum_{s=0}^{n-r} \alpha_{k,k+s}\, \beta_{r,n-s}, \quad m \geq 1, 1 \leq k \leq m. \end{aligned} \tag{29}$$

If a complex function $f(z)$ is analytic at z_0 but singular at ξ_k ($k = 1, 2, \ldots, N$), where N may be infinity, the series

$$f(z_0) + \sum_{k=1}^{+\infty} \left[\frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] \chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B}), \tag{30}$$

converges to $f(z)$ in the region $D = \bigcap_{k=0}^{M} S_k$, where $S_k = \{z : |\omega_k| > 1\}$, and the complex numbers ω_k ($k = 0, 1, 2, 3, \ldots, M$, M may be infinity) are the solutions of one of the following equations:

$$\mathcal{B}(\omega_0) = (1+\hbar)^{-1},$$

$$1 - (1+\hbar)\mathcal{B}(\omega_k) + \hbar \left(\frac{z - z_0}{\xi_n - z_0} \right) \mathcal{A}(\omega_k) = 0, \quad 1 \leq n \leq N.$$

We call $\mathcal{A}(p)$, $\mathcal{B}(p)$ *the imbedding function*. It can be proved that the approaching function $\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B})$ has such an interesting property that, if $\mathcal{A}(p) = \mathcal{B}(p) = p$, then

$$\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B}) = \Phi_{m,k}(\hbar), \tag{31}$$

and for definite positive integer k,

$$\lim_{m \to +\infty} \chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B}) = 1, \quad |1+\hbar| < 1. \tag{32}$$

It means that $\Phi_{m,k}(\hbar)$ is only a special case of $\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B})$ when $\mathcal{A}(p) = \mathcal{B}(p) = p$. So, we call $\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B})$ the *general approaching function*. Therefore, Theorem 1 is only a special case of Theorem 2 when $\mathcal{A}(p) = \mathcal{B}(p) = p$. Thus, the traditional Taylor theorem is only a special case of Theorems 1 and 2. So, we call Theorems 1 and 2 ‘the general Taylor theorem’ and the series (24) and (30) ‘the general Taylor series’ of $f(z)$ at $z = z_0$. In general, by the foregoing general Taylor theorems, the convergence region of a classical power series can be greatly enlarged. We will illustrate this point in the next section.

4. The Simple Approach to Enlarge Convergence Regions

4.1. Riccati's Equation

Let μ be a nonzero real number and rewrite Equation (10) in the form

$$y(t) = \lim_{m\to+\infty} \sum_{k=0}^{m} C_k t^{3k+2} \Phi_{m,k}(\mu), \quad -2 < \mu < 0, \tag{33}$$

where $\Phi_{m,k}(\mu)$ and $C_k (k = 0, 1, 2, 3, \ldots)$ are defined by Equations (6) and (11), respectively. Let z be a complex number. The related classical power series

$$\sum_{k=0}^{+\infty} C_k z^k \tag{34}$$

converges in the region $|z| < \bar{\rho}_0^3$, where $\bar{\rho}_0$ is given by Equation (12). However, we do not exactly know the distribution of the singularities. Notice that the limit

$$\lim_{k\to+\infty} \frac{C_k}{C_{k+1}} = -\frac{7.83735}{\varepsilon}, \quad \varepsilon > 0, \tag{35}$$

exists and is a real number. So, the simplest case is that singularities of Equation (34) are only on real negative axis. If so, according to the above-mentioned general Taylor theorem (Theorem 1), the following power series

$$\lim_{m\to+\infty} \sum_{k=0}^{m} C_k x^k \Phi_{m,k}(\mu), \tag{36}$$

where μ and $x = \mathrm{Re}\{z\}$ are real numbers, would be valid in the region

$$-\bar{\rho}_0^3 < x < \bar{\rho}_0^3 \left[\frac{2}{|\mu|} - 1\right], \quad -2 < \mu < 0. \tag{37}$$

Our numerical calculations verify that this is indeed true. So, setting $x = t^3$ into Equations (36) and (37), we have the conclusion that Equation (33) is valid in the region

$$-\bar{\rho}_0 < t < \bar{\rho}_0 \left[\frac{2}{|\mu|} - 1\right]^{1/3}, \quad -2 < \mu < 0. \tag{38}$$

The curves given by the series (33) for $\varepsilon = 1$ and different values of μ are as shown in Figure 1. Clearly, the smaller the value of $|\mu|(-1 \le \mu < 0)$, the larger the convergence region of the series (33). What we would like to emphasize here is that, due to Equation (38), as μ $(-2 < \mu < 0)$ tends to zero, Equation (33) may be valid in the region $-\bar{\rho}_0 < t < +\infty$. Therefore, Equation (33) may be valid in a region much larger than that of the corresponding perturbation approximation (10).

Besides, due to the general Taylor Theorem 2, we can rewrite Equation (10) in the form

$$y(t) = \lim_{m\to+\infty} \sum_{k=0}^{m} C_k t^{3k+2} \chi_{m,k}(\mu, \mathcal{A}, \mathcal{B}), \quad -2 < \mu < 0, \tag{39}$$

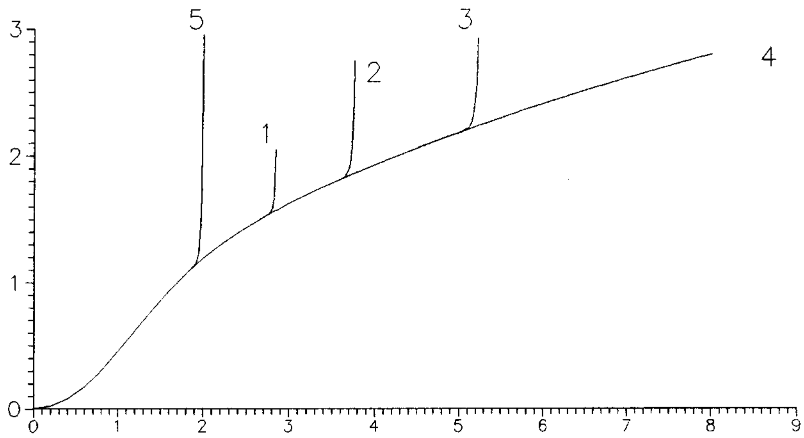

Figure 1. Comparisons of the series (33) with the perturbation solutions (10) of Riccati's equation (8) when $\varepsilon = 1$. Curve 1: series (33) when $\mu = -1/2$; curve 2: series (33) when $\mu = -1/4$; curve 3: series (33) when $\mu = -1/10$; curve 4: numerical solution; curve 5: perturbation solution (10). Horizontal axis: t, vertical axis: $y(t)$.

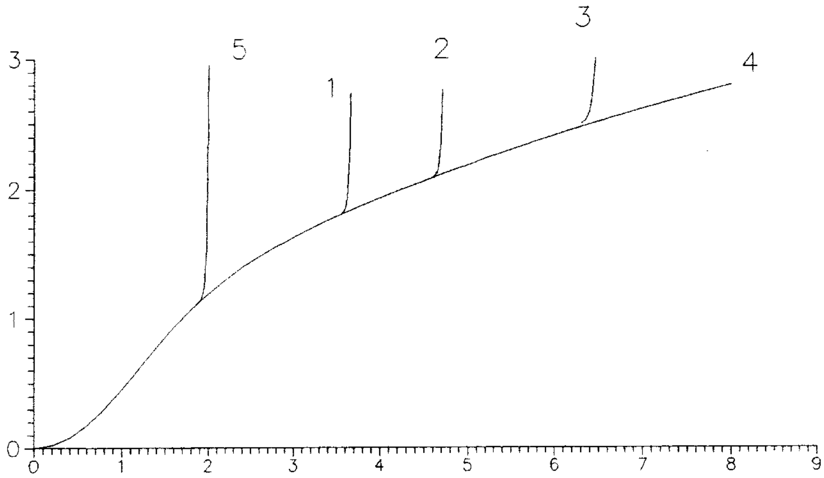

Figure 2. Comparisons of the series (39) with the perturbation solutions (10) of Riccati's equation (8) when $\varepsilon = 1$. Curve 1: series (39) when $\mu = -1/2$; curve 2: series (39) when $\mu = -1/4$; curve 3: series (39) when $\mu = -1/10$; curve 4: numerical solution; curve 5: perturbation solution (10). Horizontal axis: t, vertical axis: $y(t)$.

where, without loss of generality, we set

$$\mathcal{A}(p) = \mathcal{B}(p) = (e^p - 1)/(e - 1). \tag{40}$$

The curves given by the series (39) for $\varepsilon = 1$ and different values of μ are as shown in Figure 2. Comparing Figure 1 with Figure 2, we note that for the same values of μ $(-1 < \mu < 0)$, the convergence radius of the series (39) is larger than that of the series (33). And when $-1 < \mu < 0$, the convergence radius of the series (33) and (39) is always greater than that of the traditional perturbation power series (10). This example illustrates well that we can greatly enlarge the convergence region of a perturbation approximation by simply applying the approaching functions $\Phi_{m,n}(\mu)$ and $\chi_{m,k}(\mu, \mathcal{A}, \mathcal{B})$.

4.2. Van der Pol's Equation

Using such a variable transformation that

$$w = \frac{\varepsilon^2}{\sqrt{R_0^4 - 2R_0^2\varepsilon^2\cos\beta_0 + \varepsilon^4}},$$

Andersen and Geer [1] and Dadfar et al. [4] enlarged the convergence radius of Equations (15) and (19) to infinity. Besides, by using the variable transformation

$$w = \frac{\varepsilon}{\sqrt[6]{(R_0^4 - 2R_0^2\varepsilon^2\cos\beta_0 + \varepsilon^4)(1.65^2 + \varepsilon^2)}},$$

Dadfar et al. [4] recasted the limit cycle (16) into a new series in w whose minimum radius of convergence is about 0.953 corresponding to a radius of convergence in the ε-plane of about 3.97.

Here we show that, not using any of new variable transformations, we can simply apply the above-mentioned general Taylor theorems to enlarge the convergence of the series (16). Let μ be a real number. Through Equation (15), we give the following $(2m)$th-order HAM approximation of frequency ω in the a form

$$\omega_{2m} = b_0 + \sum_{k=1}^{m} b_k \varepsilon^{2k}\Phi_{m,k}(\mu), \quad -2 < \mu < 0, \tag{41}$$

so that

$$\omega = \lim_{m\to+\infty}\omega_{2m} = b_0 + \lim_{m\to+\infty}\sum_{k=1}^{m} b_k \varepsilon^{2k}\Phi_{m,k}(\mu), \quad -2 < \mu < 0, \tag{42}$$

where $\Phi_{m,k}(\mu)$ is defined by Equation (6).

Let z be a complex number. We consider such a related complex power series

$$W(z) = b_0 + \lim_{m\to+\infty}\sum_{k=1}^{m} b_k z^k \Phi_{m,k}(\mu) \quad (-2 < \mu < 0). \tag{43}$$

According to Andersen and Geer's [1] work, the closest singularities of $\lim_{m\to+\infty}\sum_{k=0}^{m} b_k z^k$ to the origin are about $R_0^2\,\mathrm{e}^{\pm\beta_0^i}$, where $R_0 = 1.85$, $\beta_0 = 0.897$. Let $\zeta = \mathrm{Re}\{z\}$ denote the real part of z and μ be a real number. Then, according to Theorem 1, the convergence region of the real series

$$b_0 + \lim_{m\to+\infty}\sum_{k=1}^{m} b_k \zeta^k \Phi_{m,k}(\mu) \quad (-2 < \mu < 0) \tag{44}$$

must be in such a form that

$$-\frac{\sqrt{1-(1+\mu)^2\sin^2(2\beta_0)} + (1+\mu)\cos(2\beta_0)}{|\mu|} < \frac{\zeta}{R_0^2} < \frac{\sqrt{1-(1+\mu)^2\sin^2(2\beta_0)} - (1+\mu)\cos(2\beta_0)}{|\mu|}. \tag{45}$$

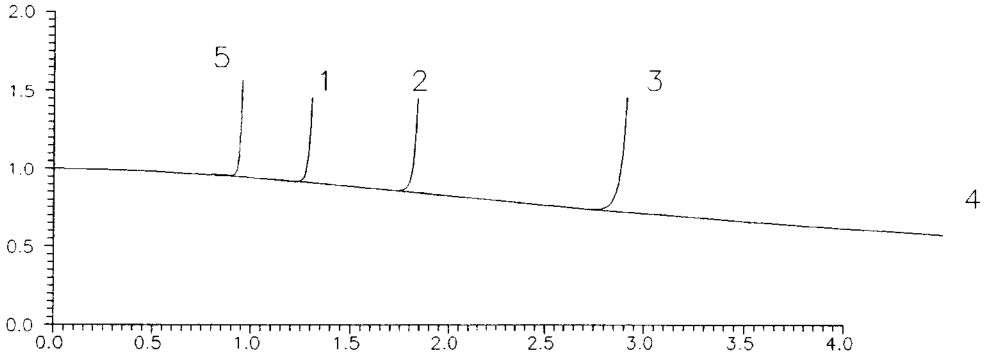

Figure 3. Comparisons of the series (42) with the perturbation solutions (15) of Van der Pol's equation (13). Curve 1: series (42) when $\mu = -1/2$; curve 2: series (42) when $\mu = -1/4$; curve 3: series (42) when $\mu = -1/10$; curve 4: numerical solution; curve 5: perturbation solution (15). Horizontal axis: ε, vertical axis: frequency ω.

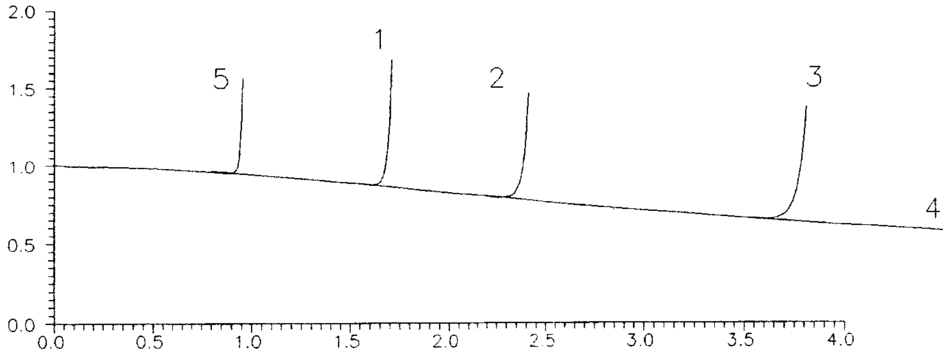

Figure 4. Comparisons of the series (47) with the perturbation solutions (15) of Van der Pol's equation (13). Curve 1: series (47) when $\mu = -1/2$; curve 2: series (47) when $\mu = -1/4$; curve 3: series (47) when $\mu = -1/10$; curve 4: numerical solution; curve 5: perturbation solution (15). Horizontal axis: ε, vertical axis: frequency ω.

Thus, substituting $\zeta = \varepsilon^2$, $R_0 = 1.85$, $\beta_0 = 0.897$ into Equation (45), we get that Equation (42) is valid in the region

$$|\varepsilon| < R_0\sqrt{\frac{\sqrt{1-0.952(1+\mu)^2+0.221(1+\mu)}}{|\mu|}}, \quad -2 < \mu < 0. \tag{46}$$

Our numerical calculations verify that the above expression is indeed right, as shown in Figure 3. Notice that the smaller the value of $|\mu|$ $(-1 \leq \hbar < 0)$ the larger the convergence region of the series (42). We emphasize that, as μ $(-2 < \mu < 0)$ tends to zero, the above expression gives $|\varepsilon| < +\infty$ so that formula (42) may be valid for *any* values of ε. Therefore, Equation (42) may be seen as a closed-form analytic solution of frequency ω of Van der Pol's equation (13). Notice that the corresponding perturbation approximation (15) is valid only in a rather restricted region $|\varepsilon| < 1.85$!

Also, to apply Theorem 2, we should rewrite Equation (15) in the form

$$\omega_{2m} = b_0 + \sum_{k=1}^{m} b_k \varepsilon^{2k} \chi_{m,k}(\mu, \mathcal{A}, \mathcal{B}), \quad -2 < \mu < 0, \tag{47}$$

where we select $\mathcal{A}(p) = \mathcal{B}(p) = (e^p - 1)/(e - 1)$ for the general approach function $\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B})$. The curves of the series (47) for different values of μ are as shown in Figures 4. Comparing Figure 4 with Figure 3, we note that for the same values of $\mu(-1 < \mu < 0)$, the convergence region of the series (47) is even larger than that of the series (42).

Similarly, we can apply the general Taylor theorems to enlarge the convergence region of the series (19) for the amplitude A of the self-excited vibration. To do so, we simply rewrite Equation (19) in the form

$$A = \lim_{m\to+\infty} \sum_{j=0}^{m} a_{2j}\varepsilon^{2j}\Phi_{m,j}(\mu), \quad -2 < \mu < 0. \tag{48}$$

In the similar way as mentioned above, we can conclude that the above series converges in the region

$$|\varepsilon| < R_0\sqrt{\frac{\sqrt{1 - 0.952(1+\mu)^2} + 0.221(1+\mu)}{|\mu|}}, \quad -2 < \mu < 0. \tag{49}$$

Notice that the convergence region increases as the value of $|\mu|$ $(-1 \le \mu < 0)$ decreases, as shown in Figure 5. Especially, we emphasize that, due to Equation (49), as $|\mu|$ $(-1 < \mu < 0) \to 0$, the series (48) is convergent for *any* values of ε.

Similarly, we can also apply the general approach function $\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B})$ to increase the convergence region of the series (19). To do so, we rewrite Equation (19) in the form

$$A = \lim_{m\to+\infty} \sum_{j=0}^{m} a_{2j}\varepsilon^{2j}\chi_{m,j}(\mu, \mathcal{A}, \mathcal{B}), \quad -2 < \mu < 0, \tag{50}$$

where $\mathcal{A}(p) = \mathcal{B}(p) = (e^p - 1)/(e - 1)$. The comparison of the series (50) for different values of μ with the perturbation series (19) is as shown in Figure 6. Comparing Figure 6 with Figure 5, we note that, for the same values of μ, the convergence region of the series (50) is larger than that of the series (48). Also, in all cases under consideration, the perturbation series (19) possesses the smallest convergence region. This means that the approach functions $\Phi_{m,k}(\hbar)$ and $\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B})$ can indeed increase the convergence region of a perturbation power series.

As mentioned in Section 2, the perturbation approximation (16) for the limit cycle of Van der Pol's equation is divergent when $\varepsilon > 1.65$. By means of the general Taylor theorems, we can also enlarge the convergence region of (16) in a similar way. For example, we can rewrite the $(2m)$th-order HAM approximation of the limit cycle in the form

$$u_{2m}(x) = \varphi_0(x) + \sum_{k=1}^{m}[\varphi_{2k-1}(x)\varepsilon^{2k-1} + \varphi_{2k}(x)\varepsilon^{2k}]\Phi_{m,k}(\mu), \tag{51}$$

where μ is a real number, $\varphi_k(x)$ is defined by Equations (17) and (18), and $\Phi_{m,k}(\mu)$ is defined by Equation (6). Notice that Equation (16) is not exactly a power series so that the general Taylor theorems cannot be directly applied to it. Even so, our calculations illustrate that Equation (51) can indeed converge in regions of ε larger than 1.65 for properly selected values of μ. For instance, in the case of $\varepsilon = 2.0$, the 36th-order HAM approximation given by Equation (51) when $\mu = -1/2$ agrees very well with the corresponding numerical result, as shown in Figure 7. In fact, even the 24th-order approximation is accurate enough. Moreover,

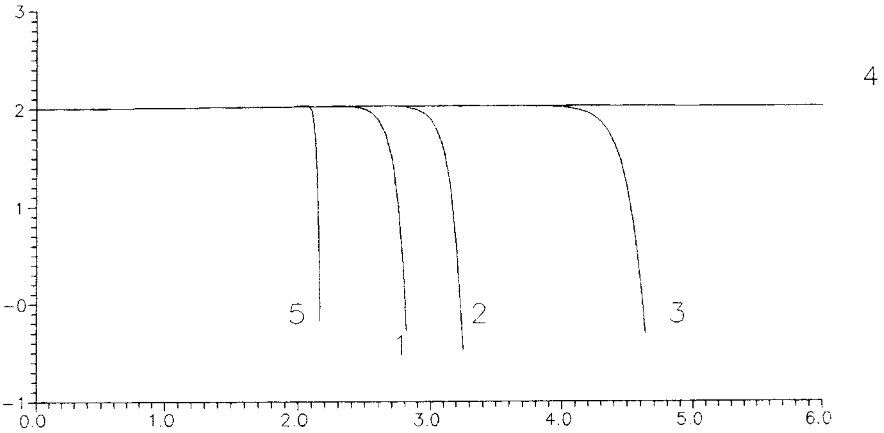

Figure 5. Comparisons of the series (48) with the perturbation solutions (19) of Van der Pol's equation (13). Curve 1: series (48) when $\mu = -1/2$; curve 2: series (48) when $\mu = -1/4$; curve 3: series (48) when $\mu = -1/10$; curve 4: numerical solution; curve 5: perturbation solution (19). Horizontal axis: ε, vertical axis: amplitude A.

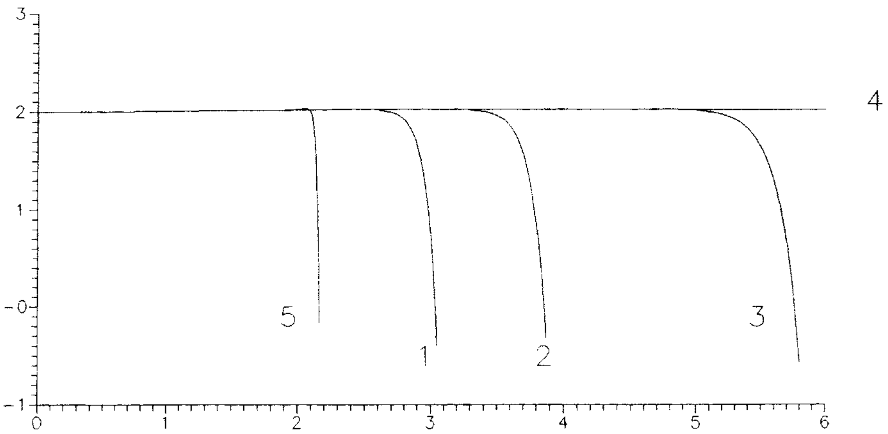

Figure 6. Comparisons of the series (50) with the perturbation solutions (19) of Van der Pol's equation (13). Curve 1: series (50) when $\mu = -1/2$; curve 2: series (50) when $\mu = -1/4$; curve 3: series (50) when $\mu = -1/10$; curve 4: numerical solution; curve 5: perturbation solution (19). Horizontal axis: ε, vertical axis: amplitude A.

in the case of $\varepsilon = 2.5$, the HAM approximations given by Equation (51), when $\mu = -1/5$, converge to the exact limit cycle as the order of approximation goes up, as shown in Figure 8. For further, larger ε, more terms are needed to propose a satisfactory approximation of limit cycle. We emphasize that the perturbation approximation (16) of the limit cycle is divergent for $\varepsilon > 1.65$ but Equation (51) can be convergent for $\varepsilon > 1.65$ as long as μ is properly selected.

All of the above illustrate that the approaching functions $\Phi_{m,n}(\hbar)$ and $\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B})$ indeed have more general meanings so that it can be applied to enlarge convergence regions of perturbation approximations which are not even in the form of power series.

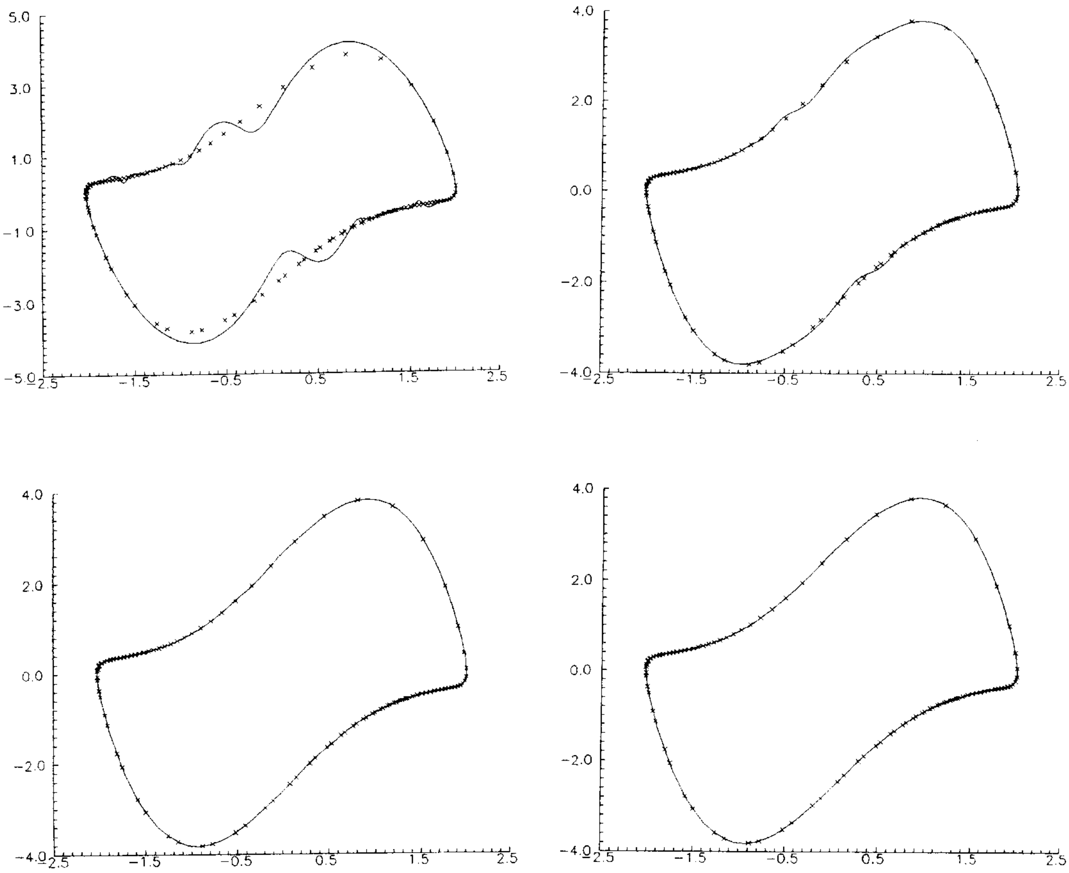

Figure 7. Limit cycle in case of $\varepsilon = 2$ given by Equation (51) when $\mu = -1/2$. Top-left: 12th-order HAM approximation, top-right: 24th-order HAM approximation, bottom-left: 36th-order HAM approximation, bottom-right: 48th-order HAM approximation, center symbols: numerical result. Horizontal-axis: $u(t)$, vertical-axis: $u'(t)$.

5. Conclusions and Discussions

In this paper, we use two nonlinear differential equations as examples to illustrate a simple approach to enlarging convergence regions of perturbation approximations. The foundation of this approach are the so-called general Taylor theorems, which provide us with a solid, rational base. The two examples illustrate that the so-called approaching functions $\Phi_{m,n}(\hbar)$ defined by Equation (6) and $\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B})$ defined by Equation (29) have indeed more general meanings which can be simply applied, in most cases, to greatly enlarge the convergence regions of the perturbation approximations. This provides us with a simple but effective approach to improving perturbation approximations.

Why is it possible that the convergence region of a general Taylor series (24) and (30) depends upon the value of $\hbar$ ($|1 + \hbar| < 1$) and may be greater than that of the corresponding classical Taylor series? To explain this, let us consider a complex power series in general, say,

$$\lim_{m\to+\infty} \sum_{k=0}^{m} c_k (z - z_0)^k. \tag{52}$$

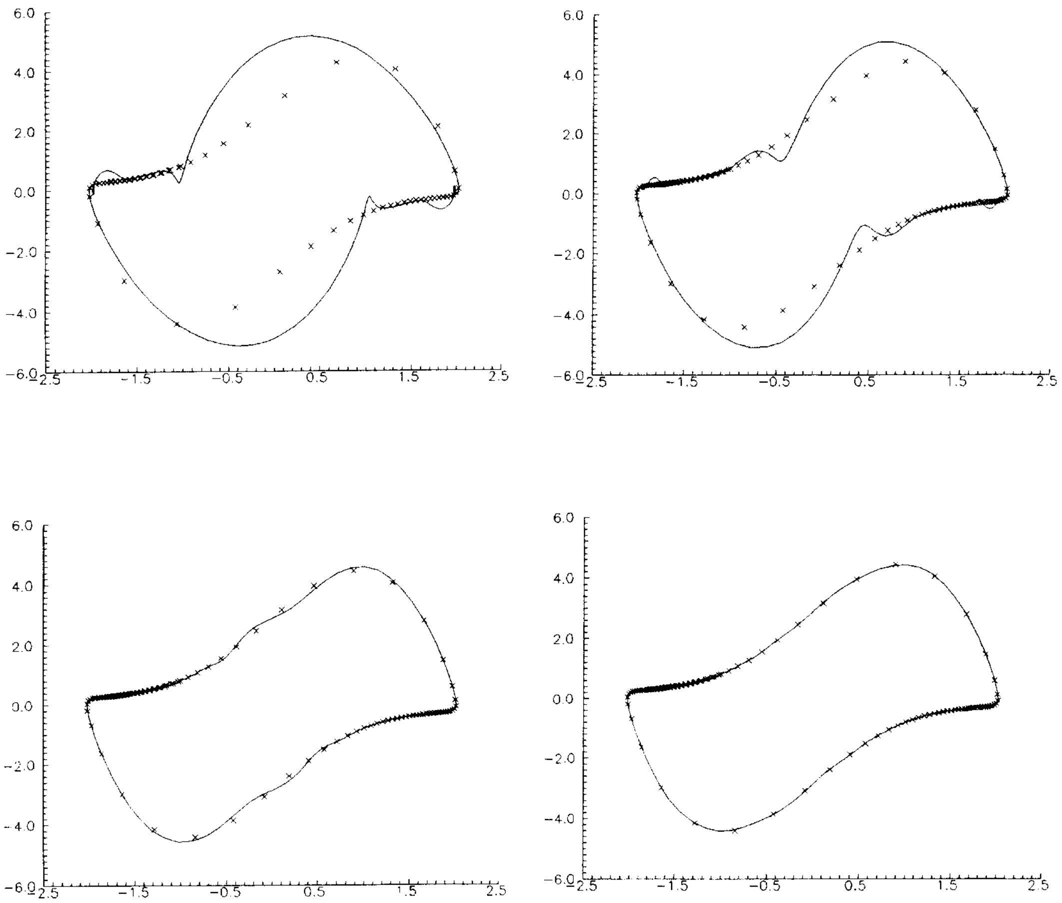

Figure 8. Limit cycle in case of $\varepsilon = 2.5$ given by Equation (51) when $\mu = -1/5$. Top-left: 12th-order HAM approximation, top-right: 24th-order HAM approximation, bottom-left: 48th-order HAM approximation, bottom-right: 72th-order HAM approximation, center symbols: numerical result. Horizontal-axis: $u(t)$, vertical-axis: $u'(t)$.

Write $\mathbf{e}_k = (z - z_0)^k$ $(k \geq 0)$. Then, $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k, \ldots$ constructs a base and

$$\mathbf{c} = (c_0, c_1, c_2, \ldots, c_k, \ldots) \in \Re^{+\infty} \tag{53}$$

can be considered as a known point in $\Re^{+\infty}$. Let $\mathbf{r} \in \Re^{+\infty}$ denote a point in general. The classical Taylor series (52) is in fact a kind of *limit* as the point $\mathbf{r} \in \Re^{+\infty}$ tends to the known point $\mathbf{c} \in \Re^{+\infty}$ along such a *unique* (traditional) *path*

$$\begin{aligned}
&(c_0, 0, 0, 0, \ldots),\\
&(c_0, c_1, 0, 0, 0, \ldots),\\
&(c_0, c_1, c_2, 0, 0, 0, \ldots),\\
&\qquad\quad\vdots\\
&(c_0, c_1, c_2, \ldots, c_k, 0, 0, 0, \ldots),\\
&\qquad\quad\vdots
\end{aligned}$$

so that the classical Taylor series (52) is convergent in the region

$$|z - z_0| < \rho_0, \tag{54}$$

where $\rho_0 = \min|\xi_k - z_0|$ and ξ_k $(k \in I)$ denotes all singularities of the complex function related to Equation (52). The corresponding general Taylor series

$$\lim_{m\to+\infty} \sum_{k=0}^{m} (c_k \mathbf{e}_k)\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B}), \tag{55}$$

where $\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B})$ is defined by Equation (29), is also a kind of *limit* as the point $\mathbf{r} \in \Re^{+\infty}$ tends to the known point $\mathbf{c} \in \Re^{+\infty}$, but along such an *infinite* number of *different* paths

$$\begin{aligned}
&(c_0\chi_{0,0}(\hbar, \mathcal{A}, \mathcal{B}), 0, 0, 0, \ldots),\\
&(c_0\chi_{1,0}(\hbar, \mathcal{A}, \mathcal{B}), c_1\chi_{1,1}(\hbar, \mathcal{A}, \mathcal{B}), 0, 0, 0, \ldots),\\
&(c_0\chi_{2,0}(\hbar, \mathcal{A}, \mathcal{B}), c_1\chi_{2,1}(\hbar, \mathcal{A}, \mathcal{B}), c_2\chi_{2,2}(\hbar, \mathcal{A}, \mathcal{B}), 0, 0, 0, \ldots),\\
&\qquad\vdots\\
&(c_0\chi_{k,0}(\hbar, \mathcal{A}, \mathcal{B}), c_1\chi_{k,1}(\hbar, \mathcal{A}, \mathcal{B}), c_2\chi_{k,2}(\hbar, \mathcal{A}, \mathcal{B}), \ldots, c_k\chi_{k,k}(\hbar, \mathcal{A}, \mathcal{B}), 0, 0, 0, \ldots),\\
&\qquad\vdots
\end{aligned}$$

so that the convergence region of the series (55) is dependent upon the parameter $\hbar$ and the two functions $\mathcal{A}(p)$, $\mathcal{B}(p)$. For example, in a special case of $\mathcal{A}(p) = \mathcal{B}(p) = p$, the general Taylor series (55) is convergent in the region

$$\bigcap_{k\in I} \left| 1 + \hbar - \hbar\left(\frac{z - z_0}{\xi_k - z_0}\right)\right| < 1, \quad |1 + \hbar| < 1, \tag{56}$$

where $\xi(k \in I)$ denotes all singularities of the complex function related to Equation (52). We emphasize that the convergence region given by Equation (56) strongly depends on the value of $\hbar$. Note that, owing to Equation (32), $\lim_{m\to+\infty} \chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B}) = 1$ holds in the case of $|1 + \hbar| < 1$ for any finite positive integer k so that, although different values of $\hbar$ and different functions $\mathcal{A}(p)$, $\mathcal{B}(p)$ correspond to different paths, all of them approach the *same* point $\mathbf{c} \in \Re^{+\infty}$. Moreover, owing to Equation (22) and Theorems 1 and 2, in the special case of $\hbar = -1$ and $\mathcal{A}(p) = \mathcal{B}(p) = p$, the path is exactly the same as that for the classical Taylor series (52) so that, in this case, Equation (56) gives the convergence region (54). Thus, the approaching functions $\Phi_{m,k}(\hbar)$ and $\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B})$ can define different paths for approaching a known point in an infinite-dimensional space.

The expression (56) is reasonable, because the limit of a function having more than two variables is mainly strongly dependent on paths approaching a given point. For example, consider a simple limit

$$\lim_{(x,y,z)\to(0,0,0)} \frac{\sqrt{x^2 + y^2 + z^2}}{|x|}. \tag{57}$$

Assume that the approaching path is defined by

$$y = \alpha x^\gamma, \quad z = \beta x^\gamma, \quad \gamma > 0. \tag{58}$$

Then, we have the limit

$$\lim_{(x,y,z)\to(0,0,0)} \frac{\sqrt{x^2+y^2+z^2}}{|x|} = \begin{cases} 1, & \gamma > 1, \\ \sqrt{1+\alpha^2+\beta^2}, & \gamma = 1, \\ +\infty, & 0 < \gamma < 1, \end{cases} \tag{59}$$

which is strongly dependent on the path. That is to say, although the point (0,0,0) is singular, the limit (57) is so strongly dependent on the way of approaching this point, that the limit itself may be quite different. This simple example can well explain why the convergence region (56) of the general Taylor series (55) may be dependent upon $\hbar$ and the two embedding functions $\mathcal{A}(p)$, $\mathcal{B}(p)$. Although, by the general Taylor series (55), all paths tend to the same known point $\mathbf{c} \in \Re^{+\infty}$, they are however different for different values of $\hbar$ and different embedding functions $\mathcal{A}(p)$, $\mathcal{B}(p)$. Therefore, the general Taylor series (55), as a kind of limit, is strongly dependent on $\hbar$ and two embedding functions $\mathcal{A}(p)$, $\mathcal{B}(p)$. Certainly, some of them may be better than others and can give series which are convergent in larger regions. This is the reason why we call $\Phi_{m,k}(\hbar)$ the general approach function, respectively, and also the essential reason why $\Phi_{m,k}(\hbar)$, $\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B})$ can enlarge the convergence region of a classical Taylor series. Similarly, if we write $\bar{\mathbf{e}}_k = \varepsilon^{2k+1}\sin[(2k+1)\tau]$ $(k \geq 0)$ and $\hat{\mathbf{e}}_k = \varepsilon^{2k}\cos[(2k+1)\tau]$ $(k \geq 0)$ and consider them as a base, we can understand why the convergence region of Equation (51) is also dependent on $\hbar$ and embedding functions $\mathcal{A}(p)$, $\mathcal{B}(p)$. Therefore, for any an infinite sequence $\sum_0^{+\infty} c_k\tilde{\mathbf{e}}_k$ within the classical meaning, where $\tilde{\mathbf{e}}_k$ is a base in general, one can write such a related general sequence

$$\lim_{m\to+\infty} \sum_{k=0}^{m} (c_k\tilde{\mathbf{e}}_k)\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B}), \quad |1+\hbar| < 1, \tag{60}$$

whose convergence region is a function of $\hbar$ and two embedding functions $\mathcal{A}(p)$, $\mathcal{B}(p)$. This might explain why the approaching function $\Phi_{m,k}(\hbar)$ and the general approach function $\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B})$ can enlarge the convergence region of an infinite sequence not in the form of power series. Therefore, the approaching function $\chi_{m,k}(\hbar, \mathcal{A}, \mathcal{B})$ has indeed some rather general meanings.

The proposed approach is based on the so-called general Taylor theorems cited in Section 3 so that it has a solid mathematical base. In fact, it can be proved that the Euler transformation is only a special case of Theorem 2. The detailed mathematical proofs will be given elsewhere in the near-future, if possible.

For the proposed approach, perturbation sequence at a rather high order of approximations had to be given. This is nearly impossible to calculate by hand. However, we are now in the time when both computer hardware and software have developed rather quickly so that the computer has become an indispensable, powerful tool. By means of high-performance computers and some well-developed software such as MATHEMATICA, MAPLE, and so on, it is now not very difficult for us to get very high-order perturbation approximations. So, the proposed approach has practical meanings.

The advantage of the proposed approach is its simplicity in application: we do not need to rewrite a perturbation power series in a new variable and, besides, we do not need to know the exact positions and properties of all singularities. In many cases, the proposed approach is a kind of simplification of the homotopy analysis method [6–10], because, by means of the homotopy analysis method, we can obtain the same corresponding results and even some better approximations if we use proper auxiliary linear operators, as pointed out by Liao in [9, 10]. The difficulty in applying the proposed approach is that, for perturbation approximations not in power series, we currently have no idea of how to exactly determine the related convergence regions, although, as shown in this paper, the proposed simple approach would indeed seem valid in this case. Even so, the considered two examples illustrate well the validity of the proposed simple approach to enlarge convergence regions of perturbation approximations.

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