ELSEVIER

J. Non-Newtonian Fluid Mech. 139 (2006) 31–43

Journal of Non-Newtonian Fluid Mechanics

www.elsevier.com/locate/jnnfm

Series solution of unsteady boundary layer flows of non-Newtonian fluids near a forward stagnation point

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Received 29 December 2005; received in revised form 8 June 2006; accepted 9 June 2006

Abstract

In this paper, the unsteady viscous flow of non-Newtonian fluids near the forward stagnation point of a two-dimensional body is studied analytically. By using the homotopy analysis method, a convergent series solution is obtained, which is uniformly valid for all dimensionless time in the whole spatial region $0 \leq \eta < \infty$. Besides, the effects of integral power-law index of the non-Newtonian fluids on the flow are investigated. To the best of our knowledge, such kind of series solutions have never been reported for this problem.
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Keywords: Non-Newtonian fluid; Stagnation point; Unsteady; Boundary-layer

1. Introduction

Many industrial fluids are non-Newtonian in their flow characteristics and are referred to as rheological fluids. Most particulate slurries (china clay and coal in water, sewage sludge, etc.), multiphase mixtures (oil–water emulsions, gas–liquid dispersions, such as froths and foams, butter), are non-Newtonian fluids. Further examples displaying a variety of non-Newtonian characteristics include pharmaceutical formulations, cosmetics and toiletries, paints, synthetic lubricants, biological fluids (blood, synovial fluid, salvia), and foodstuffs (jams, jellies, soups, marmalades). Indeed, behavior of non-Newtonian flow is so widespread that it would be no exaggeration to say that the Newtonian fluid behavior is an exception rather than the rule. Because such fluids have more complicated equations that relate the shear stresses to the velocity field than Newtonian fluids have, additional factors must be considered in examining various fluid mechanics and heat transfer phenomena (see, Irvine and Karni [1]). Newtonian fluids posses a property called viscosity and obey a relation analogous to the Hookian relation between a stress and its strain, i.e., the stress is proportional to the strain rate. Another important characteristic of rheological fluids, because of their large *apparent viscosities*, is that they have tendency toward low Reynolds and Grashof numbers. Accordingly, in engineering practice laminar flow situations are encountered more often than with Newtonian fluids. Because of the growing use of non-Newtonian fluids in various manufacturing and processing industries, considerable efforts have been directed toward understanding their friction and heat transfer characteristics. Several excellent books and review papers summarizing the state-of-the-art available in the literature testify to the maturity of the field of non-Newtonian fluids, e.g., Astarita and Marrucci [2], Darby [3], Schowalter [4], Tanner [5], Bird et al. [6], Crochet et al. [7], Shenoy and Mashelkar [8], Andersson and Irgens [9], and Ghosh et al. [10].

Several models have been proposed to explain the non-Newtonian behavior of fluids. Among these models, which are known to follow the empirical Ostwaald-de Waele model, the so-called *power-law model*, in which the shear stress varies according to a power function of the strain rate, gained much acceptance. Boundary layer assumptions were successfully applied to this model and much work has been done on it. The theoretical analysis of a steady boundary layer flow of incompressible non-Newtonian

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doi:10.1016/j.jnnfm.2006.06.003

power-law fluids was first performed by Schowalter [11] and Acrivos et al. [12]. The boundary-layer equations were formulated, and the conditions for the existence of the similarity solutions were established in [11]. A similarity solution to the boundary layer equations for a power-law fluid flowing along a horizontal flat plate including heat transfer was obtained in [12]. A detailed analysis to obtain possible similarity solutions of the steady boundary-layer equations of a non-Newtonian fluid was also performed by Kim et al. [13], and the curvature effects on the boundary layer of a non-Newtonian fluid were investigated by Yükselen and Erim [14]. The effect of wall mass injection on the flow of a non-Newtonian power-law fluid over a flat plate, near a stagnation point and past a wedge has been investigated by Thompson and Snyder [15], and Kim and Eraslan [16]. The boundary layer of a non-Newtonian fluid flow with fluid injection on a semi-infinite flat plate which moves with a constant velocity in the opposite direction to that of the uniform mainstream has been analyzed by Akçay and Yükselen [17]. The steady viscous incompressible flow of a non-Newtonian power-law fluid on a two-dimensional body in the presence of a magnetic field was studied by Sarpkaya [18], Djukić [19,20]. Finally, we mention the recent paper by Pakdemirli [21] in which the steady two-dimensional boundary layer equations of power-law fluids were derived using a special coordinate system which makes the equations independent of the body shape immersed in the flow.

Generally speaking, it is hard to analytically solve nonlinear problems. The widely applied perturbation techniques depend strongly upon small parameters, and therefore are valid in general only for weakly nonlinear problems. In 1992, using the concept of homotopy [22], which is a basic concept of topology [23], Liao [24] proposed an analytic method for nonlinear problems, namely the homotopy analysis method (HAM), and then modified it step by step [25–29]. Different from perturbation techniques, the homotopy analysis method is independent upon small parameters al all, so that it is valid for highly nonlinear problems, especially those without small/large physical parameters. Besides, it logically contains the non-perturbation techniques such as Lyapunov's artificial small parameter method [30], Adomian decomposition method [31] and the δ-expansion method [32], as proved by Liao [33]. The so-called "homotopy perturbation method" [34] proposed in 1999 is only a special case of the homotopy analysis method, as pointed out by Liao [35]. Thus, the homotopy analysis method is more general and is a unification of previous non-perturbation techniques. Besides, different from all previous analytic methods, the homotopy analysis method always gives us a family of series solutions whose convergence region can be adjusted and controlled by an auxiliary parameter. For detials, please refer to Liao [33]. The homotopy analysis method has been successfully applied to many types of nonlinear problems [33,36–45], including some unsteady nonlinear problems [42–45]. Especially, new solutions of a few nonlinear problems are even found by means of it [46,47]. All of these verify the validity of the homotopy analysis method for highly nonlinear problems.

Knowledge of non-Newtonian fluid mechanics is still in an early stage, and many aspects of the field have yet to be investigated and clarified in more details. Most of previous theoretical and experimental studies of non-Newtonian fluids have considered steady-state boundary layer flow problems, but little work has been done on unsteady boundary layer flows of a non-Newtonian fluid. The object of this paper is to study analytically the problem of unsteady boundary layer flow near the forward stagnation point of a cylindrical surface embedded in a viscous and incompressible non-Newtonian power-law fluid by using the homotopy analysis method [24,29,33].

2. Basic equations

Consider the unsteady laminar boundary-layer flow of a non-Newtonian power-law fluid in the vicinity of a two-dimensional body. It is assumed that when $t < 0$, the surface of the body and the fluid are at rest. Then at time $t = 0$, the external stream is set into an impulsive motion from rest with the velocity $u_e(x) = ax$, where x is the Cartesian coordinate measured along the surface of the body started from the stagnation point and $a > 0$ is a constant. Under these conditions the governing equations for the unsteady boundary layer flows are, see Andersson et al. [48] or Kumari and Nath [49],

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{\mathrm{d}u_e}{\mathrm{d}x} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial \tau_{xy}}{\partial y}\right), \tag{2}$$

where y is the Cartesian coordinate measured normal to the surface of the body, u and v are the velocity components along x and y axes, τ_{xy} and ρ are, respectively, the shear stress tensor, the density of the non-Newtonian fluid. Note that $u_e(\mathrm{d}u_e/\mathrm{d}x)$ represents the pressure gradient which balances the far field flow. The shear stress tensor is defined by the Ostwald-de-Wäle model, see Andersson et al. [48] or Liao [39],

$$\tau_{ij} = 2K(2D_{kl}D_{kl})^{(\kappa-1)/2}D_{ij} \tag{3}$$

where

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{4}$$

denotes the rate of stretching tensor, K is the consistency coefficient, and κ is the index in the power-law variation of the shear stress of a non-Newtonian fluid. With $\kappa = 1$, Eq.(3) represents a Newtonian fluid with dynamic coefficient of viscosity K. Therefore, the

deviation of κ from unity indicates the degree of deviation from Newtonian fluid behaviour (see e.g. Andersson and Irgens [9]). With $\kappa \neq 1$, the constitutive equation (3) represents shear-thinning or pseudoplastic fluid ($\kappa < 1$) and shear-thickening or dilatant fluid ($\kappa > 1$), respectively. The initial and boundary conditions of Eqs. (1) and (2) are

$$\begin{aligned} &t < 0: \ u = v = 0, && \text{for all points } (x, y); \\ &t \geq 0: \ u = v = 0, && \text{on } y = 0, x \geq 0; \\ &t \geq 0: \ u = u_e(x) = ax, && \text{as } y \to \infty, \ x \geq 0. \end{aligned} \tag{5}$$

Following Andersson [48], Williams and Rhyne [50] and Nazar et al. [51], we introduce the following new similarity variables and transformations

$$\psi = (\rho a^{1-2\kappa}/K)^{-1/(\kappa+1)} x^{2\kappa/(\kappa+1)} \xi^{1/(\kappa+1)} f(\eta, \xi), \quad \eta = (\rho a^{2-\kappa}/K)^{1/(\kappa+1)} x^{(1-\kappa)/(\kappa+1)} \xi^{-1/(\kappa+1)} y,$$
$$\xi = 1 - \exp(-\tau), \quad \tau = at, \tag{6}$$

where ψ is the stream function which automatically satisfies mass conservation (1). Hence, the velocity components are given by:

$$u = \frac{\partial \psi}{\partial y} = ax \frac{\partial f}{\partial \eta}, \tag{7}$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{2\kappa}{\kappa+1} \left(\frac{\rho a^{1-2\kappa}}{K} \right)^{-1/(\kappa+1)} \xi^{1/(\kappa+1)} x^{(\kappa-1)/(\kappa+1)} f + a \left(\frac{\kappa-1}{\kappa+1} \right) y \frac{\partial f}{\partial \eta}. \tag{8}$$

Note that, by means of (6), the dimensionless time ξ is defined in a finite region $0 \leq \xi \leq 1$.

Substituting (7) and (8) into Eqs. (1)–(4), the governing equations (1) and (2) can be reduced to

$$(1-\xi) \left(\frac{1}{\kappa+1} \eta \frac{\partial^2 f}{\partial \eta^2} - \xi \frac{\partial^2 f}{\partial \eta \partial \xi} \right) + \kappa \left(\frac{\partial^2 f}{\partial \eta^2} \right)^{\kappa-1} \frac{\partial^3 f}{\partial \eta^3} + \xi \left[\frac{2\kappa}{\kappa+1} f \frac{\partial^2 f}{\partial \eta^2} + 1 - \left(\frac{\partial f}{\partial \eta} \right)^2 \right] = 0, \tag{9}$$

subject to the boundary conditions

$$f(0, \xi) = 0, \ \left. \frac{\partial f(\eta, \xi)}{\partial \eta} \right|_{\eta=0} = 0, \quad \left. \frac{\partial f(\eta, \xi)}{\partial \eta} \right|_{\eta=\infty} = 1. \tag{10}$$

From Eq. (9) it is easy to obtain the following ordinary differential equations for the unsteady initial flow, corresponding to $\xi = 0$ when $t = 0$, and the final steady-state flow, corresponding to $\xi = 1$ when $t \to \infty$. When $\xi = 0$, Eq. (9) reduces to

$$\frac{1}{\kappa+1} \eta \frac{\partial^2 f}{\partial \eta^2} + \kappa \left(\frac{\partial^2 f}{\partial \eta^2} \right)^{\kappa-1} \frac{\partial^3 f}{\partial \eta^3} = 0. \tag{11}$$

When $\xi = 1$, it reduces to

$$\kappa \left(\frac{\partial^2 f}{\partial \eta^2} \right)^{\kappa-1} \frac{\partial^3 f}{\partial \eta^3} + \frac{2\kappa}{\kappa+1} f \frac{\partial^2 f}{\partial \eta^2} + 1 - \left(\frac{\partial f}{\partial \eta} \right)^2 = 0. \tag{12}$$

Note that, when $\kappa = 1$, Eq. (12) becomes the classical equation of steady flow near the forward stagnation point of an infinite plane wall, established by Hiemenz [52].

The quantity with physical interest is the local skin friction coefficient C_f, defined by

$$C_f = \frac{\tau_{xy}}{\rho [u_e(x)]^2}, \tag{13}$$

where the shear stress τ_{xy} is given by

$$\tau_{xy} = K \left(\frac{\partial u}{\partial y} \right)^{\kappa} \Bigg|_{y=0}. \tag{14}$$

Using variables (6), we have

$$C_f = \frac{\tau_w}{\rho [u_e(x)]^2} = \xi^{-\kappa/(\kappa+1)} [F_{\eta\eta}(0, \xi)]^{\kappa} Re^{-1/(1+\kappa)}, \tag{15}$$

where $Re_x = \rho x^{\kappa} [u_e(x)]^{2-\kappa}/K$ is the local Reynolds number.

3. Homotopy analysis solution

Many previous studies show that the boundary-layer flow decays exponentially at infinity (refer to [39] or [48]). Further, Liao [42] obtained accurate series solutions of a kind of unsteady boundary layer flows in a Newtonian fluid caused by an impulsively stretching flat plate. Very recently, Xu et al. [43] studied analytically the unsteady boundary-layer flow and heat transfer in an incompressible viscous electrically conducting fluid, caused by an impulsive stretching of the surface in two lateral directions. Following Liao [39,42], Andersson [48] and Xu et al. [43], we express $f(\eta, \xi)$ by a set of functions

$$\mathcal{A} = \{\xi^j\ \eta^k\ \exp(-n\eta) |\ j \geq 0,\ k \geq 0,\ n \geq 0\} \tag{16}$$

in the form

$$f(\eta, \xi) = \sum_{j=0}^{+\infty} \sum_{k=0}^{+\infty} \sum_{n=0}^{+\infty} a_{k,n}^{j} \xi^j \eta^k\ \exp(-n\eta), \tag{17}$$

where $a_{k,n}^{j}$ is the coefficient to be determined. Thus, all approximations of $f(\eta, \xi)$ must obey the above expression: this point is so important in the frame of the homotopy analysis method that it is regarded as a rule, called *the Rule of Solution Expression* for $f(\eta, \xi)$. From the initial/boundary conditions (10), and considering the *Solution Expression* denoted by (17), it is straightforward to choose

$$f_0(\eta, \xi) = \eta + \exp(-\eta) - 1 + \epsilon[1 - 2\ \exp(-\eta) + 2\ \exp(-2\eta)], \tag{18}$$

as the initial approximation of $f(\eta, \xi)$, where ϵ is an auxiliary parameter to be determined later. Note that the initial guess $f_0(\eta, \xi)$ should satisfy the *solution expression* denoted by (17) and the boundary conditions given in (10). For the reason of reducing calculation quantities, we usually choose them as simple as possible. Here, the auxiliary parameter ϵ is used to optimize the initial guess so that the accurate series solutions can be obtained in the relatively lower order approximations.

Secondly, we should choose an auxiliary linear operator $\mathcal{L}$. According to Eq. (9), and for the sake of simplicity, we choose a third-order linear operator, i.e.

$$\mathcal{L}u = u''' + b_1(\eta, \xi)\ u'' + b_2(\eta, \xi)\ u' + b_3(\eta, \xi)\ u,$$

where the prime denotes the differentiation with respect to η, and $b_1(\eta, \xi)$, $b_2(\eta, \xi)$ and $b_3(\eta, \xi)$ are functions to be determined. Let

$$u = C_1\ a_1(\eta) + C_2\ a_2(\eta) + C_3\ a_3(\eta)$$

denote the common solution of $\mathcal{L}u = 0$, where C_1, C_2, C_3 are constants independent upon η. To obey the solution expression (17) and to satisfy the boundary conditions (10), we choose the elementary solutions

$$a_1(\eta) = 1, \quad a_2(\eta) = \exp(-\eta), \quad a_3(\eta) = \exp(\eta).$$

Enforcing

$$\mathcal{L}[a_i(\eta)] = 0, \quad i = 1, 2, 3,$$

we have

$$b_1(\eta, \xi) = b_3(\eta, \xi) = 0, \quad b_2(\eta, \xi) = -1.$$

Therefore, we have the auxiliary linear operator

$$\mathcal{L}[\Phi(\xi, \eta; q)] = \frac{\partial^3 \Phi}{\partial \eta^3} - \frac{\partial \Phi}{\partial \eta}, \tag{19}$$

which has the property

$$\mathcal{L}[C_1 + C_2\ \exp(-\eta) + C_3\ \exp(\eta)] = 0, \tag{20}$$

where C_1, C_2 and C_3 are constants independent upon η. In general, the auxiliary linear operator is chosen by means of the rule of solution expression, the governing equation and boundary conditions.

From (9), we define the nonlinear operator

$$\mathcal{N}[\Phi(\eta, \xi; q)] = (1 - \xi)\left(\frac{1}{\kappa + 1}\eta \frac{\partial^2 \Phi}{\partial \eta^2} - \xi \frac{\partial^2 \Phi}{\partial \eta \partial \xi}\right) + \kappa \left(\frac{\partial^2 \Phi}{\partial \eta^2}\right)^{\kappa - 1} \frac{\partial^3 \Phi}{\partial \eta^3} + \xi \left[\frac{2\kappa}{\kappa + 1} \Phi \frac{\partial^2 \Phi}{\partial \eta^2} + 1 - \left(\frac{\partial \Phi}{\partial \eta}\right)^2\right], \tag{21}$$

where q is an embedding parameter, $\Phi(\eta, \xi; q)$ is a mapping of $f(\eta, \xi)$. Let $\hbar$ denote the auxiliary non-zero parameter. We construct the zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\Phi(\eta, \xi; q) - f_0(\eta, \xi)] = q\ \hbar\ H(\eta, \xi)\ \mathcal{N}[\Phi(\eta, \xi; q)], \tag{22}$$

subject to the boundary conditions

$$\Phi(0,\xi;q) = \left.\frac{\partial\Phi(\eta,\xi;q)}{\partial\eta}\right|_{\eta=0} = 0, \quad \left.\frac{\partial\Phi(\eta,\xi;q)}{\partial\eta}\right|_{\eta=+\infty} = 1. \tag{23}$$

where $H(\eta,\xi)$ is an auxiliary function.

Obviously, when $q=0$ and $q=1$, the above zeroth-order deformation equation (22) has the solutions

$$\Phi(\eta,\xi;0) = f_0(\eta,\xi) \tag{24}$$

and

$$\Phi(\eta,\xi;1) = f(\eta,\xi), \tag{25}$$

respectively. Thus, as q increases from 0 to 1, $\Phi(\eta,\xi;q)$ varies (or deforms) from the known initial guess $f_0(\eta,\xi)$ to the unknown solution $f(\eta,\xi)$. Expanding $\Phi(\eta,\xi;q)$ in Taylor's series with respect to q, we have

$$\Phi(\eta,\xi;q) = \Phi(\eta,\xi,0) + \sum_{m=1}^{+\infty} f_m(\eta,\xi)\, q^m, \tag{26}$$

where

$$f_m(\eta,\xi) = \left.\frac{1}{m!}\frac{\partial^m\Phi(\eta,\xi;q)}{\partial q^m}\right|_{q=0}. \tag{27}$$

Note that Eq. (22) contains the auxiliary parameter $\hbar$. Assuming that $\hbar$ is correctly chosen so that the above series is convergent at $q=1$, we have, using (25), the solution series

$$f(\eta,\xi) = f_0(\eta,\xi) + \sum_{m=1}^{+\infty} f_m(\eta,\xi). \tag{28}$$

For simplicity, define

$$\vec{f}_m = \{f_0, f_1, f_2, \ldots, f_m\}. \tag{29}$$

Differentiating the zeroth-order deformation equation (22) m times with respect to q, then setting $q=0$, and finally dividing them by $m!$, we obtain the mth-order deformation equation for $m \geq 1$ as

$$\mathcal{L}[f_m(\eta,\xi) - \chi_m f_{m-1}(\eta,\xi)] = \hbar\, H(\eta,\xi)\, R_m(\vec{f}_{m-1}), \tag{30}$$

subject to the initial/boundary conditions

$$f_m(0,\xi) = \left.\frac{\partial f_m(\eta,\xi)}{\partial\eta}\right|_{\eta=0} = 0, \quad \left.\frac{\partial f_m(\eta,\xi)}{\partial\eta}\right|_{\eta=\infty} = 0, \tag{31}$$

where

$$R_m = \left.\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\Phi(\eta,\xi;q)]}{\partial q^{m-1}}\right|_{q=0} \tag{32}$$

and

$$\chi_m = \begin{cases} 0, & m=1, \\ 1, & m>1. \end{cases} \tag{33}$$

Note that R_m is dependent on the integer power-law index κ. When $\kappa=1$, we have

$$R_m(\vec{f}_{m-1}) = (1-\xi)\left(\frac{1}{2}\eta\frac{\partial^2 f_{m-1}}{\partial\eta^2} - \xi\frac{\partial^2 f_{m-1}}{\partial\eta\partial\xi}\right) + \frac{\partial^3 f_{m-1}}{\partial\eta^3} + \xi\left[\sum_{i=0}^{m-1}\left(f_i\frac{\partial^2 f_{m-1-i}}{\partial\eta^2} - \frac{\partial f_i}{\partial\eta}\frac{\partial f_{m-1-i}}{\partial\eta}\right) + 1 - \chi_m\right]. \tag{34}$$

When $\kappa=2$, we have

$$R_m(\vec{f}_{m-1}) = (1-\xi)\left(\frac{1}{3}\eta\frac{\partial^2 f_{m-1}}{\partial\eta^2} - \xi\frac{\partial^2 f_{m-1}}{\partial\eta\partial\xi}\right) + 2A_{m-1} + \xi\left[\sum_{i=0}^{m-1}\left(\frac{4}{3}f_i\frac{\partial^2 f_{m-1-i}}{\partial\eta^2} - \frac{\partial f_i}{\partial\eta}\frac{\partial f_{m-1-i}}{\partial\eta}\right) + 1 - \chi_m\right], \tag{35}$$

where

$$A_j = \sum_{i=0}^{j} \frac{\partial^2 f_i}{\partial \eta^2} \frac{\partial^3 f_{j-i}}{\partial \eta^3}. \tag{36}$$

When $\kappa = 3$, they read

$$R_m(\vec{f}_{m-1}) = (1-\xi)\left(\frac{1}{4}\eta \frac{\partial^2 f_{m-1}}{\partial \eta^2} - \xi \frac{\partial^2 f_{m-1}}{\partial \eta \partial \xi}\right) + 3\sum_{i=0}^{m-1} A_i \frac{\partial^2 f_{m-i-1}}{\partial \eta^3}$$
$$+ \xi\left[\sum_{i=0}^{m-1}\left(\frac{3}{2} f_i \frac{\partial^2 f_{m-1-i}}{\partial \eta^2} - \frac{\partial f_i}{\partial \eta}\frac{\partial f_{m-1-i}}{\partial \eta}\right) + 1 - \chi_m\right]. \tag{37}$$

Let $f_m^*(\eta, \xi)$ denote the particular solution of Eq. (30). According to (20), its general solution reads

$$f_m(\eta, \xi) = f_m^*(\eta, \xi) + C_1^m \exp(-\eta) + C_2^m \exp(\eta) + C_3^m + \chi_m f_{m-1}(\eta, \xi), \tag{38}$$

where the constants C_1^m, C_2^m, C_3^m are determined by the boundary conditions (31), i.e.

$$C_2^m = 0,\ C_1^m = \left.\frac{\partial f_m^*(\eta, \xi)}{\partial \eta}\right|_{\eta=0},\ C_3^m = -C_1^m - f_m^*(0, \xi). \tag{39}$$

In this way, it is easy to obtain the *linear* Eq. (30) one after the other in the order $m = 1, 2, 3, \ldots$, especially by means of the symbolic computation software, such as Mathematica, Maple and so on. For more details, the readers are refer to Appendix A of this paper.

We have some freedom to choose the auxiliary function $H(\eta, \xi)$. It is found that convergent series can be obtained when $H(\eta, \xi) = 1$ or $H(\eta, \xi) = \exp(-\eta)$. However, it is found that the latter corresponds to the solution series converging faster. So, we choose $H(\eta, \xi) = \exp(-\eta)$ in the following part of this article.

4. Recursive formulae

So, by means of computer and symbolic computation, we can get high-order approximation of the considered unsteady nonlinear problem. Obviously, recursive formulas are helpful for us to recognize the structure of the series solutions and to have an insight to the nature of the considered nonlinear problem.

It is found that, when $\kappa = 1$ and $\epsilon = 0$, $f(\eta, \xi)$ is expressed by

$$f_m(\eta, \xi) = \sum_{i=0}^{m} \sum_{j=0}^{n+1-\chi_{n+1}} \sum_{k=j+\chi_{j+1}}^{2n+1-\chi_{n+1}-j+\chi_{j+1}} \alpha_{j,k}^{m,i} \xi^i \eta^j \exp(-k\eta), \tag{40}$$

where $n = [(m+i+1)/2]$ is an integer.

For brevity, write

$$f_m(\eta, \xi) = \sum_{i=0}^{m+1} \sum_{j=0}^{m+1} \sum_{k=0}^{2m+2} \alpha_{j,k}^{m,i} \lambda_{j,k}^{m,i} \xi^i \eta^j \exp(-k\eta) \tag{41}$$

where

$$\lambda_{j,k}^{m,i} = \begin{cases} 1, & 0 \le i \le m \quad \text{and } 0 \le j \le n+1-\chi_{n+1} \\ & \quad \text{and } j+\chi_{j+1} \le k \le 2n+1-\chi_{n+1}-j+\chi_{j+1} \\ 0, & \quad \text{otherelse.} \end{cases} \tag{42}$$

Substituting (41) into (30)–(34) and (38) –(39), we have the recursive formulae of $f(\eta, \xi)$ for $m \ge 1$ as

$$\alpha_{0,0}^{m,i} = \hbar \Omega_{0,0}^{m,i} - \hbar \varepsilon_{0,0}^{m,i}(1-\chi_m) + \chi_m \alpha_{0,1}^{m-1,i} \lambda_{0,0}^{m-1,i} + C_1^m, \tag{43}$$

$$\alpha_{0,0}^{m,i} = \hbar \Omega_{0,1}^{m,i} - \hbar \varepsilon_{0,1}^{m,i}(1-\chi_m) + \chi_m \alpha_{0,1}^{m-1,i} \lambda_{0,1}^{m-1,i} + C_2^m, \tag{44}$$

$$\alpha_{j,k}^{m,i} = \hbar \Omega_{j,k}^{m,i} - \hbar \varepsilon_{j,k}^{m,i}(1-\chi_m) + \chi_m \alpha_{j,k}^{m-1,i} \lambda_{j,k}^{m-1,i}, \tag{45}$$

where

$$\Omega_{j,k}^{m,i} = \sum_{q=j}^{m+1} \Lambda_{q,k}^{m,i} \mu_{k,q}^{j}, \tag{46}$$

$$C_1^m = \sum_{i=0}^{m+1}\sum_{j=0}^{m+1}\sum_{k=0}^{2m+1}[\hbar\Omega_{j,k}^{m,i} - \hbar\varepsilon_{j,k}^{m,i}(1-\chi_m)(k-j-1)\xi^i, \quad C_2^m = \sum_{i=0}^{m+1}\sum_{j=0}^{m+1}\sum_{k=0}^{2m+1}[\hbar\Omega_{j,k}^{m,i} - \hbar\varepsilon_{j,k}^{m,i}(1-\chi_m)](j-k)\xi^i, \quad (47)$$

and

$$\varepsilon_{j,k}^{m,i} = \begin{cases} 1, & m=1 \text{ and } i=j=1 \text{ and } k=0, \\ 0, & \text{otherelse.} \end{cases} \quad (48)$$

The definitions of $\Lambda_{q,k}^{m,i}$ and $\mu_{k,q}^{j}$ in (46) are

$$\Lambda_{j,k}^{m,i} = \frac{1}{2}\gamma_{j-1,k}^{m-1,i} - \frac{1}{2}\gamma_{j-1,k}^{m-1,i-1} - \omega_{j,k}^{m-1,i-1} + \omega_{j,k}^{m-1,i-2} + \vartheta_{j,k}^{m-1,i} + \Gamma_{j,k}^{m,i} - \Delta_{j,k}^{m,i} \quad (49)$$

and

$$\mu_{n,q}^{i} = \begin{cases} \displaystyle\sum_{j=i}^{q+1}\frac{q!}{j!}\frac{1}{2^{q-j+2}}, & 0 \le j \le q+1,\ n=1 \\ \displaystyle\sum_{j=i}^{q}\frac{q!}{j!}\left[\frac{1}{2(n+1)^{q-j+1}} - \frac{1}{2(n-1)^{q-j+1}}\right]\frac{1}{n^{j-i+1}}, & 0 \le j \le q,\ n>1 \end{cases} \quad (50)$$

where

$$\Gamma_{j,k}^{m,i} = \sum_{r=0}^{m-1}\sum_{i_2=\max\{0,i-r-2\}}^{\min\{m-r,i-1\}}\sum_{j_2=\max\{0,j-r-1\}}^{\min\{m-r,j\}}\sum_{k_2=\max\{0,j-2r-2\}}^{\min\{2m-2r,k\}}\lambda_{j-i_2,k-k_2}^{r,i-i_2-1}\alpha_{j-j_2,k-k_2}^{r,i-i_2-1}\gamma_{j_2,k_2}^{m-r-1,i_2}, \quad (51)$$

$$\Delta_{j,k}^{m,i} = \sum_{r=0}^{m-1}\sum_{i_2=max\{0,i-r-2\}}^{\min\{m-r,i-1\}}\sum_{j_2=\max\{0,j-r-1\}}^{\{m-r,j\}}\sum_{k_2=\max\{0,j-2r-2\}}^{\min\{2m-2r,k\}}\beta_{j-j_2,k-k_2}^{r,i-i_2-1}\beta_{j_2,k_2}^{m-r-1,i_2}, \quad (52)$$

and

$$\beta_{j,k}^{m,i} = (j+1)\lambda_{j+1,k}^{m,i}\alpha_{j+1,k}^{m,i} - k\lambda_{j,k}^{m,i}\alpha_{j,k}^{m,i}, \quad (53)$$

$$\gamma_{j,k}^{m,i} = (j+2)(j+1)\lambda_{j+2,k}^{m,i}\alpha_{j+2,k}^{m,i} - 2k(j+1)\lambda_{j+1,k}^{m,i}\alpha_{j+1,k}^{m,i} + k^2\lambda_{j,k}^{m,i}\alpha_{j,k}^{m,i}, \quad (54)$$

$$\vartheta_{j,k}^{m,i} = (j+3)(j+2)(j+1)\lambda_{j+3,k}^{m,i}\alpha_{j+3,k}^{m,i} - 3k(j+2)(j+1)\lambda_{j+2,k}^{m,i}\alpha_{j+2,k}^{m,i} + 3k^2(j+1)\lambda_{j+1,k}^{m,i}\alpha_{j+1,k}^{m,i} - k^3\lambda_{j,k}^{m,i}\alpha_{j,k}^{m,i}, \quad (55)$$

$$\delta_{j,k}^{m,i} = (i+1)\alpha_{j,k}^{m,i+1}\lambda_{j,k}^{m,i+1}, \quad (56)$$

$$\omega_{j,k}^{m,i} = (i+1)(j+1)\lambda_{j+1,k}^{m,i+1}\alpha_{j+1,k}^{m,i+1} - k(i+1)\lambda_{j,k}^{m,i+1}\alpha_{j,k}^{m,i+1}. \quad (57)$$

Thus, the constants of the series solutions can be explicitly determined from the first four coefficients: $\alpha_{0,0}^{0,0} = -1$, $\alpha_{1,0}^{0,0} = 1$, $\alpha_{0,1}^{0,0} = 1$, $\alpha_{1,1}^{0,0} = 0$. At the Mth-order of approximation, we have

$$f(\eta,\xi) = \sum_{m=0}^{M}\sum_{i=0}^{m+1}\sum_{j=0}^{m+1}\sum_{k=0}^{2m+2}\alpha_{j,k}^{m,i}\lambda_{j,k}^{m,i}\xi^i\eta^j\exp(-k\eta). \quad (58)$$

Note that, for other values of κ and ϵ, we can get high-order approximations in a similar way. For details, pleaser refer to Liao [39].

5. Result analysis

Note that the solution series (28) contains two auxiliary parameters $\hbar$ and ϵ, which can be determined in the way described below. To decide the auxiliary parameter ϵ, we define an error function

$$E(\epsilon) = \int_0^1\int_0^{+\infty}[R_1(f_0)]^2 d\eta d\xi, \quad (59)$$

Table 1
The values of parameters used in our calculation

κ	$\hbar$	ϵ
1	-1	0
2	$-1/2$	$-1/8$
3	$-1/3$	$-1/8$

which is independent upon $\hbar$. Enforcing

$$\frac{\mathrm{d}E(\epsilon)}{\mathrm{d}\epsilon} = 0, \tag{60}$$

we get the "best" value of ϵ, corresponding to the minimum residual error of the original equation. Note that, if the higher order analytic approximations are needed in the calculation, we should set $\epsilon = 0$ to reduce the calculating quantities. Then, we choose proper value of $\hbar$ by plotting the $\hbar$-curves to ensure that the solution series (28) converges, as suggested by Liao [33]. The values of parameters $\hbar$ and ϵ used in our analysis are listed in Table 1.

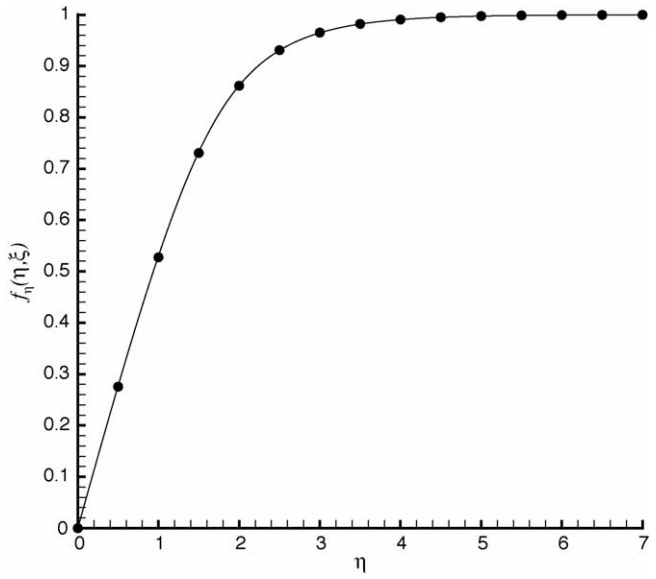

Fig. 1. The comparison of the analytic approximations with the numerical solutions at $\xi = 0$ when $\kappa = 2$. Symbol: numerical results; solid line: 20th-order HAM approximations.

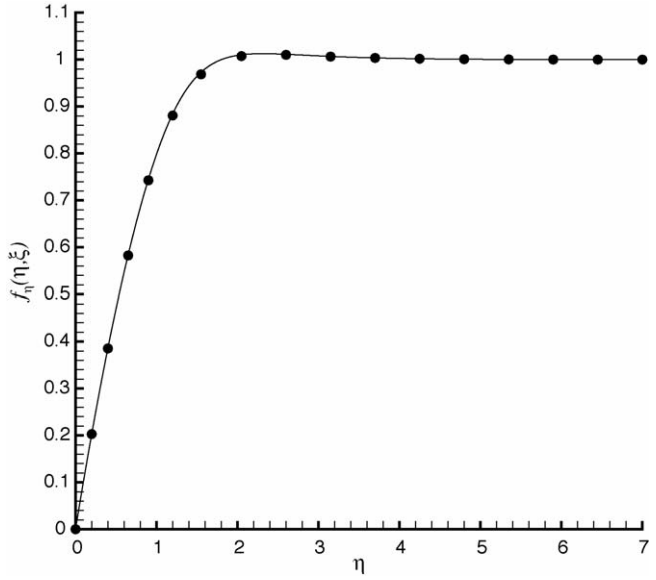

Fig. 2. The comparison of the analytic approximations with the numerical solutions at $\xi = 1$ when $\kappa = 2$. Symbol: numerical results; solid line: 20th-order HAM approximations.

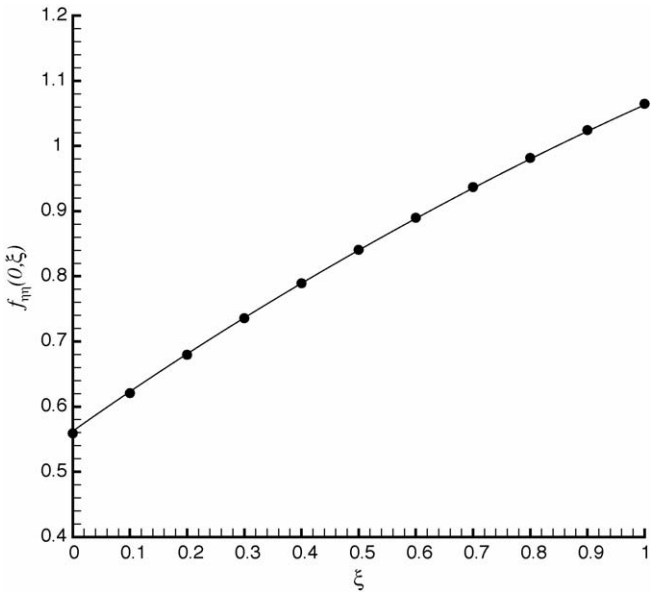

Fig. 3. The analytic approximations of $f_{\eta\eta}(0, \xi)$ for $0 \leq \xi \leq 1$ when $\kappa = 2$ given by homotopy analysis method. Solid line 17th-order approximations; symbol: 20th-order approximations.

When $\xi = 0$, corresponding to the initial state, and $\xi = 1$, corresponding to the steady state, our analytic approximations agree well with numerical ones in the whole region $0 \leq \eta < +\infty$, as shown in Figs. 1 and 2. Note that the numerical results are obtained by using the finite difference method based on Newton–Raphson technique. The initial guess of the numerical solutions is given by Eq.(18). To satisfy the boundary conditions at infinity, an integration distance $\eta_\infty = 40$, which is discretized into 10,000 intervals, is found to be adequate. The iterative integration procedure is repeated until the RMS error for each discretized Eq. (11) or (12) is less than 5×10^{-6}. It is found that, in the whole region $\xi \in [0, 1]$, our series solutions (28) are convergent, as shown in Fig. 3. Further, it is found that our series solutions (28) are convergent for all $\xi \in [0, 1]$. Thus, by means of the homotopy analysis method, we obtain convergent series solutions uniformly valid in the whole spatial region $0 \leq \eta < +\infty$ for all dimensionless time $\xi \in [0, 1]$, corresponding to $0 \leq \tau < +\infty$.

The dimensionless velocity profiles as a function of τ for $\kappa = 3$ are shown in Fig. 4 as τ increases from $\tau = 0$ (initial unsteady flow) to $\tau \to \infty$ (steady-state flow). We can see that these profiles develop rapidly from rest as τ increases from zero to ∞. Although not shown here, it is important to notice that the transition from the unsteady initial flow up to the steady-state flow is completely smooth for all values of κ and τ. The velocity profiles for the different values of κ at the same non-dimensional time τ =1 are as shown in Fig. 5.

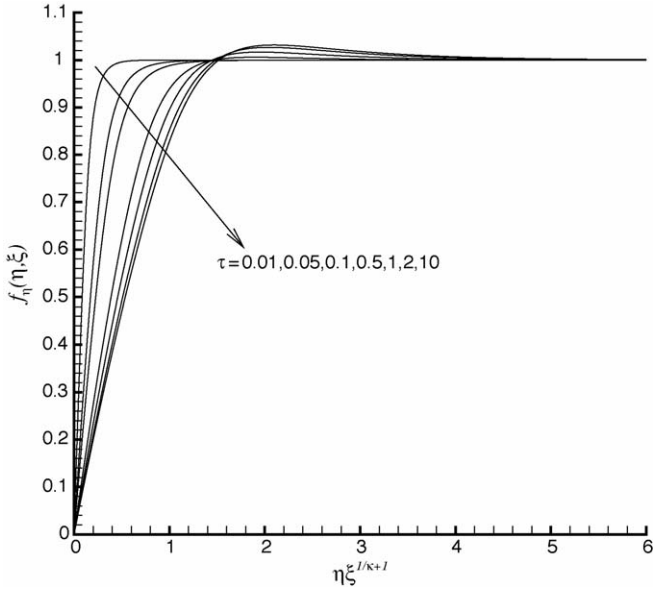

Fig. 4. The velocity profiles for the forward stagnation point when $\kappa = 3$.

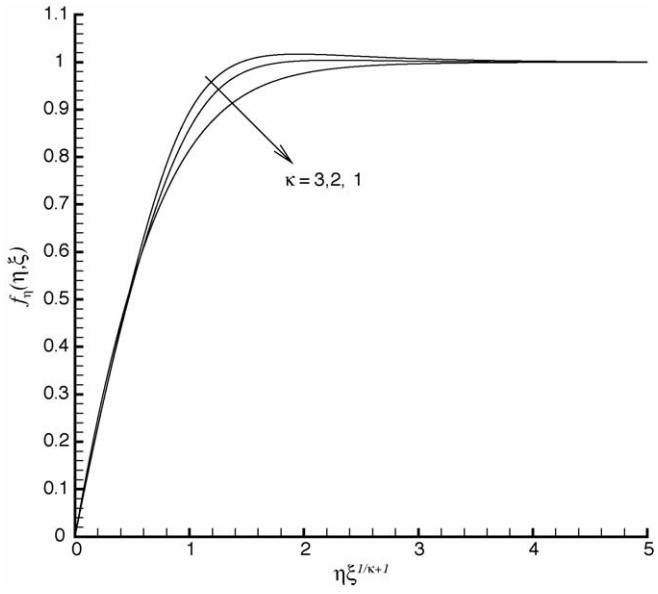

Fig. 5. The velocity profiles for the forward stagnation point at $\tau = 1$ for different κ.

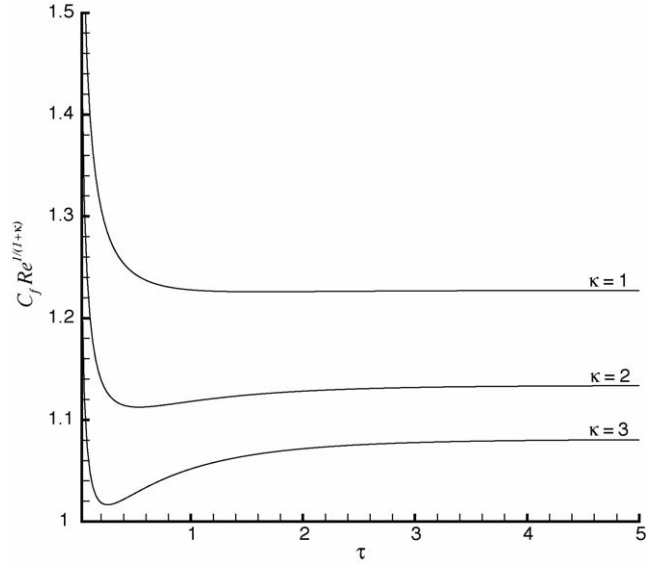

Fig. 6. The local skin friction coefficient as a function of τ for the different parameter κ.

The curves of the local skin friction coefficient C_f versus τ for some values of the power-law index κ are shown in Fig. 6. Note that, the local skin friction coefficient C_f decreases as the value of κ increases, and is asymptotic to that of steady-state flow as $\tau \to \infty$. Therefore, the transition from the initial unsteady flow to the final steady-state flow is smooth, i.e. without any singularity.

6. Conclusions

In this paper, we apply the homotopy analysis method to study the unsteady boundary layer flow of a non-Newtonian fluid near the forward stagnation point for the power-law index $\kappa = 1, 2, 3$. By this analytic technique, the convergent series solutions are obtained, which are uniformly valid for all time $0 \leq \tau < \infty$ in the whole spatial region $0 \leq \eta < \infty$. To the best of our knowledge, such kind of solutions has never been reported. Besides, the effects of integral power-law index of the non-Newtonian fluids for $\kappa = 1, 2, 3$ on the velocity are investigated. The proposed analytic approach has general meaning, and thus may be applied in the similar way to other unsteady boundary-layer flows to get accurate series solutions valid for all time in the whole spatial region.

Acknowledgements

We would like to express our sincere thanks to the reviewers for their valuable comments. This work is partly sponsored by Program of Shanghai Subject Chief Scientist (Approval No. 05XD14011) and Shanghai Committee of Science and Technology (Approval No. 05DJ14001).

Appendix A. The detailed formulas

In this part, we demonstrate the detailed approach on how to solve Eq. (9) by the homotopy analysis method. For brevity, here, we only consider the case $\epsilon = 0$ and $\kappa = 1$. It is quite similar for other ϵ and κ.

Substituting the initial guess (18) into (34), we have

$$R_1(\vec{f}_0) = -\exp(-\eta) + \xi\,\exp(-\eta) + \frac{1}{2}\eta\exp(-\eta)\frac{1}{2}\xi\eta\,\exp(-\eta). \tag{61}$$

When $m = 1$, from (30), we have the 1st-order deformation equation

$$\mathcal{L}[f_1(\eta,\xi) - \chi_1 f_0(\eta,\xi)] = \hbar R_1(\vec{f}_0), \tag{62}$$

where $\mathcal{L}$ is defined by (19) and χ_1 is defined by (33), respectively. Solving the linear equation (62) and using (38), we have

$$f_1(\eta,\xi) = f_1^*(\eta,\xi) + C_1^1\,\exp(-\eta) + C_2^1\exp(\eta) + C_3^1, \tag{63}$$

where

$$f_1^*(\eta,\xi) = -\frac{1}{72}\,\exp(-2\eta) + \frac{23}{72}\xi\exp(-2\eta) + \frac{1}{12}\eta\,\exp(-2\eta) + \frac{1}{12}\xi\eta\,\exp(-2\eta) \tag{64}$$

is a particular solution, and C_1^1, C_2^1, C_3^1 are constants. Using the boundary conditions (31), we have

$$C_1^1 = \frac{(1-5\xi)}{9},\ C_2^1 = 0,\quad C_3^1 = \frac{(-7+17\xi)}{72}. \tag{65}$$

Then, we have

$$f_1(\eta,\xi) = -\frac{1}{72}\,\exp(-2\eta) + \frac{23}{72}\xi\exp(-2\eta) + \frac{1}{12}\eta\,\exp(-2\eta) + \frac{1}{12}\xi\eta\,\exp(-2\eta) + \frac{(1-5\xi)}{9}\,\exp(-\eta) + \frac{(-7+17\xi)}{72}. \tag{66}$$

Similarly, we can obtain

$$f_2(\eta,\xi) = f_2^*(\eta,\xi) + C_1^2\,\exp(-\eta) + C_2^2\exp(\eta) + C_3^2 + \chi_2 f_1(\eta,\xi), \tag{67}$$

where

$$\begin{aligned} f_2^*(\eta,\xi) = {} & \frac{1}{54}\exp(-3\eta) - \frac{1}{648}\exp(-2\eta) - \frac{83}{43200}\xi\,\exp(-4\eta) - \frac{13}{576}\xi\,\exp(-3\eta) \\ & - \frac{85}{1296}\xi\,\exp(-2\eta) + \frac{157}{43200}\xi^2\,\exp(-4\eta) + \frac{5}{576}\xi^2\,\exp(-3\eta) - \frac{59}{1296}\xi^2\exp(-2\eta) \\ & - \frac{1}{48}\eta\,\exp(-3\eta) + \frac{1}{108}\eta\,\exp(-2\eta) + \frac{1}{720}\xi\eta\,\exp(-4\eta) + \frac{1}{48}\xi\eta\,\exp(-3\eta) \\ & - \frac{1}{27}\xi\eta\exp(-2\eta) + \frac{1}{720}\xi^2\eta\exp(-4\eta) + \frac{1}{36}\xi^2\eta\,\exp(-3\eta) - \frac{5}{108}\xi^2\eta\,\exp(-2\eta) \\ & + \frac{1}{144}\eta^2\,\exp(-3\eta) + \frac{1}{72}\xi\eta^2\exp(-3\eta) + \frac{1}{144}\xi^2\eta^2\,\exp(-3\eta) \end{aligned}$$

and

$$C_2^1 = -\frac{83}{1296} + \frac{24851}{129600}\xi + \frac{4321}{129600}\xi^2,\quad C_2^2 = 0,\quad C_3^2 = \frac{61}{1296} - \frac{13177}{129600}\xi - \frac{17}{129600}\xi^2.$$

Then, we have

$$\begin{aligned} f_2(\eta,\xi) = {} & -\frac{65}{1296} + \frac{1}{54}\exp(-3\eta) - \frac{5}{324}\exp(-2\eta) + \frac{61}{1296}\exp(-\eta) + \frac{17423}{129600}\xi \\ & - \frac{83}{43200}\xi\,\exp(-4\eta) - \frac{13}{576}\xi\,\exp(-3\eta) + \frac{329}{1296}\xi\exp(-2\eta) - \frac{47149}{129600}\xi\exp(-\eta) \end{aligned}$$

$$+\frac{17}{129600}\xi^2+\frac{157}{43200}\xi^2\exp(-4\eta)+\frac{5}{576}\xi^2\exp(-3\eta)-\frac{59}{1296}\xi^2\exp(-2\eta)$$
$$+\frac{4321}{129600}\xi^2\exp(-\eta)-\frac{1}{48}\xi\exp(-3\eta)+\frac{5}{54}\eta\exp(-2\eta)+\frac{1}{720}\xi\eta\exp(-4\eta)$$
$$+\frac{1}{48}\xi\eta\exp(-3\eta)+\frac{5}{108}\xi\eta\exp(-2\eta)+\frac{1}{720}\xi^2\eta\exp(-4\eta)+\frac{1}{36}\xi^2\eta\exp(-3\eta)$$
$$-\frac{5}{108}\xi^2\eta^2\exp(-2\eta)+\frac{1}{144}\eta^2\exp(-3\eta)+\frac{1}{72}\xi\eta^2\exp(-3\eta)+\frac{1}{144}\xi^2\eta^2\exp(-3\eta).$$

In this way, $f_m(\eta,\xi)$ can be determined one by one in the order $m=1,2,3,\ldots$.

Appendix B. Supplementary Data

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.jnnfm.2006.06.003.

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