Originals

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General boundary element method for non-linear heat transfer problems governed by hyperbolic heat conduction equation

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Abstract The general Boundary Element Method (BEM) for strongly non-linear problems proposed by Liao (1995) is further applied to solve a two-dimensional unsteady non-linear heat transfer problem in the time domain, governed by the hyperbolic heat conduction equation (HHCE) with the temperature-dependent thermal conductivity coefficients which are different in the x and y directions. This paper confirms that the general BEM can be used to solve even those non-linear unsteady heat transfer problems whose governing equations do not contain any linear terms in spatial domain.

Nomenclature

A	non-linear operator
c	thermal propagation speed
k_1	thermal conductivity coefficient in the x direction
k_2	thermal conductivity coefficient in the y direction
L	auxiliary linear operator
L_0	linear operator
N_0	non-linear operator
p	imbedding parameter
$\vec{r}$	point vector
S	heat-source term
t	time
T	temperature
x, y	spatial coordinates

Greek letters

α	thermal diffusivity
β	$=1/\Delta t$
Γ	boundary of spatial domain
λ	iterative parameter
θ	non-dimensional temperature
θ_0	initial approximation of θ
Θ	homotopy
ω	fundamental solution
Ω	spatial domain

1 Introduction

Although the Boundary Element Method (BEM) is essentially based on the linear superposition of fundamental solutions, many researchers such as Baumeister and Hamil (1969), Partridge (1995) and so on, have successfully applied it to solve non-linear boundary-value problems governed by an equation with a non-linear differential operator A such as

$$A(u) = f(\vec{r}) \ . \tag{1.1}$$

If the non-linear differential operator A can be divided into two parts L_0 and N_0, i.e. $A = L_0 + N_0$, where L_0 is a linear operator and N_0 is a non-linear one, Eq. (1.1) can be rewritten as

$$L_0(u) = f(\vec{r}) - N_0(u) \tag{1.2}$$

from which we can obtain an equation of integral operators

$$c(\vec{r})u(\vec{r}) = \int_\Gamma [uB_0(\omega_0) - \omega_0 B_0(u)]\,\mathrm{d}\Gamma + \int_\Omega [f - N_0(u)]\omega_0\,\mathrm{d}\Omega \ , \tag{1.3}$$

where ω_0 is the fundamental solution of the adjoint operator of the linear differential operator L_0, B_0 is the corresponding operator on the boundary Γ.

The basic idea of the above traditional BEM for non-linear problems is to move all non-linear terms to the right-hand side of the equation and then find the fundamental solution ω_0 of the linear operator L_0 remaining on the left-hand side, although it seems not easy in most cases to find such a fundamental solution ω_0 of the linear operator L_0, especially when L_0 is unfamiliar to us. It means that both the linear operator and the corresponding fundamental solution are very important and absolutely necessary for the traditional BEM. However, there exists obviously such a possibility that there is NOTHING left after moving all non-linear terms to the right-hand side of the equation. In this special case, the above-mentioned traditional BEM does not work at all.

This situation indeed occurs for many non-linear problems in engineering. For instance, consider a 2D steady-state non-linear heat transfer problem of inhomogeneous materials, governed by

$$\frac{\partial}{\partial x}\left[k_1(\theta)\frac{\partial\theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_2(\theta)\frac{\partial\theta}{\partial y}\right] + S(x,y) = 0, \quad (x,y) \in \Omega \ , \tag{1.4}$$

where θ denotes the temperature, $S(x,y)$ is the heat-source term, Ω denotes the domain of the temperature distribution, $k_1(\theta)$ and $k_2(\theta)$ are thermal conductivity coefficients

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in the x and y directions, respectively. If the thermal conductivity coefficients $k_1(\theta)$ and $k_2(\theta)$ are not only dependent on temperature θ but also different in the x and y directions, for instance,

$$k_1(\theta) = \exp(\gamma_1\theta), \quad k_2(\theta) = \exp(\gamma_2\theta) \ (\gamma_1 \neq \gamma_2) \ , \tag{1.5}$$

then nothing is left on the left-hand side of Eq. (1.1) after moving all non-linear terms and the heat-source term to the right-hand side, so that the traditional BEM is invalid at all to solve this kind of steady-state non-linear heat transfer problems.

Liao (1995) proposed a general BEM for strongly non-linear problems which is valid even for those non-linear problems whose governing equations do not contain any linear terms. Liao and Chwang (1996) successfully applied the general BEM to solve the above-mentioned steady-state non-linear heat transfer problem of inhomogeneous materials.

In this paper, we further apply the general BEM to solve 2D unsteady non-linear heat transfer problems of imhomogeneous materials, but now governed by the so-called hyperbolic heat conduction equation (HHCE)

$$\frac{1}{c^2}\frac{\partial^2 T}{\partial \bar{t}^2} + \frac{1}{\alpha}\frac{\partial T}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}}\left[\bar{k}_1(T)\frac{\partial T}{\partial \bar{x}}\right] + \frac{\partial}{\partial \bar{y}}\left[\bar{k}_2(T)\frac{\partial T}{\partial \bar{y}}\right] + \bar{S}(\bar{x},\bar{y},T) \ , \tag{1.6}$$

where T denotes temperature, α is the thermal diffusivity and c is the thermal propagation speed. Note that every variable in (1.6) has unit. Currently, many researchers are attracted by HHCE. The second derivative in time renders Eq. (1.6) hyperbolic in nature, and thus models the thermal wave propagation. The first derivative in time which accounts for the diffusive nature of the thermal propagation introduces the damping term. Therefore, Eq. (1.6) can model the propagation of thermal waves with a sharp front damped in time. In case $\bar{k}_1(T) = \bar{k}_2(T)$, Eq. (1.6) can be solved by the traditional BEM. However, in case $\bar{k}_1(T) \neq \bar{k}_2(T)$ which occurs when imhomogeneous materials are under consideration, the traditional BEM is invalid at all to solve Eq. (1.6). In this paper, we will show that the general BEM is still valid to solve the HHCE even in the case $\bar{k}_1(T) \neq \bar{k}_2(T)$.

2
Formulations of the general BEM

One can rewrite Eq. (1.6) in such a non-dimensional form:

$$\frac{\partial^2\theta}{\partial t^2} + \frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left[k_1(\theta)\frac{\partial\theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_2(\theta)\frac{\partial\theta}{\partial y}\right] + S(x,y,\theta) \ , \tag{2.1}$$

with the boundary conditions

$$\theta = g_1(x,y,t), \quad (x,y) \in \Gamma_1, \quad t \geq 0 \ , \tag{2.2}$$

$$\frac{\partial\theta}{\partial n} = g_2(x,y,t), \quad (x,y) \in \Gamma_2, \quad t \geq 0 \ , \tag{2.3}$$

and the two initial conditions

$$\theta = f_0(x,y), \quad (x,y) \in \Omega, \quad t = 0 \ , \tag{2.4}$$

$$\frac{\partial\theta}{\partial t} = f_1(x,y), \quad (x,y) \in \Omega, \quad t = 0 \ , \tag{2.5}$$

where θ denotes the non-dimensional temperature, $S(x,y,\theta)$ is the heat-source term, $k_1(\theta)$ and $k_2(\theta)$ are thermal conductivity coefficients in the x and y directions, Ω denotes the domain of the temperature distribution, $\Gamma_1 \cup \Gamma_2 = \Gamma$ denotes the boundary of the domain Ω, and $f_0(x,y), f_1(x,y), g_1(x,y,t), g_2(x,y,t)$ are known real functions, t and x,y are coordinates in time and space, respectively. Note that all of these variables are non-dimensional.

Let Δt denotes the time step and write $\theta^n = \theta(x,y,n\Delta t)$. Moreover, write $\beta = 1/\Delta t$. Then, at the nth time step $t = n\Delta t \ (n \geq 2)$, Eq. (2.1) can be approximately rewritten as

$$(\beta^2+\beta)\theta^n - (2\beta^2+\beta)\theta^{n-1} + \beta^2\theta^{n-2} = \frac{\partial}{\partial x}\left[k_1(\theta^n)\frac{\partial\theta^n}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_2(\theta^n)\frac{\partial\theta^n}{\partial y}\right] + S(x,y,\theta^n), \quad (x,y) \in \Omega \ , \tag{2.6}$$

with the boundary conditions

$$\theta^n = g_1(x,y,n\Delta t) \quad (x,y) \in \Gamma_1 \ , \tag{2.7}$$

$$\frac{\partial\theta^n}{\partial n} = g_2(x,y,n\Delta t) \quad (x,y) \in \Gamma_2 \ , \tag{2.8}$$

This is a non-linear boundary value problem about the non-dimensional temperature distribution $\theta^n = \theta(x,y,n\Delta t)$. Here, we use the fully implicit expression of Eq. (2.1) in time domain. Note that, when we solve Eqs. (2.6) to (2.8) at the nth time step $t = n\Delta t (n \geq 2)$, the temperature distribution $\theta^{n-1} = \theta(x,y,n\Delta t - \Delta t)$ at $(n-1)$th time step and $\theta^{n-2} = \theta(x,y,n\Delta t - 2\Delta t)$ at $(n-2)$th time step are known. For simplicity in expression, we define such a non-linear operator:

$$A(\theta^n) = \frac{\partial}{\partial x}\left[k_1(\theta^n)\frac{\partial\theta^n}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_2(\theta^n)\frac{\partial\theta^n}{\partial y}\right] + S(x,y,\theta^n) - (\beta^2+\beta)\theta^n + (2\beta^2+\beta)\theta^{n-1} - \beta^2\theta^{n-2} \ . \tag{2.9}$$

Obviously, if we consider

$$S(x,y,\theta^n) - (\beta^2+\beta)\theta^n + (2\beta^2+\beta)\theta^{n-1} - \beta^2\theta^{n-2} \tag{2.10}$$

as a new "heat-source term", then at each time step $t = n\Delta t \ (n \geq 2)$, the boundary value problem governed by Eqs. (2.6) and (2.8) can be seen as a steady-state heat transfer problem governed by (1.4) which has been successfully solved by Liao and Chwang (1996) by means of the general BEM.

First of all, we select an auxiliary 2D linear operator L, whose fundamental solution ω is known, to construct such a family of equations as follows:

$$(1-p)L[\Theta(x,y;p) - \theta_0(x,y)] = -pA[\Theta(x,y;p)], \ (x,y) \in \Omega, \quad p \in [0,1] \ , \tag{2.11}$$

with boundary conditions

$$\Theta(x,y;p) = pg_1(x,y,n\Delta t) + (1-p)\theta_0(x,y), \quad (x,y) \in \Gamma_1, \quad p \in [0,1] \ , \tag{2.12}$$

$$\frac{\partial\Theta(x,y;p)}{\partial n} = pg_2(x,y,n\Delta t) + (1-p)\frac{\partial\theta_0}{\partial n}, \quad (x,y) \in \Gamma_2, \quad p \in [0,1] \ , \tag{2.13}$$

where p is an embedding parameter, $A(\Theta)$ is defined by (2.9), $\theta_0(x,y)$ is an initial approximation of the temperature distribution θ^n.

Note that $\Theta(x,y;p)$ is also a function of the embedding parameter p. At $p=0$, we obtain from Eqs. (2.11) to (2.13) that

$$L[\Theta(x,y;0)] = L[\theta_0(x,y)], \quad (x,y) \in \Omega \ , \tag{2.14}$$

with boundary conditions

$$\Theta(x,y;0) = \theta_0(x,y) \quad (x,y) \in \Gamma_1 \ , \tag{2.15}$$

$$\frac{\partial\Theta(x,y;0)}{\partial n} = \frac{\partial\theta_0(x,y)}{\partial n} \quad (x,y) \in \Gamma_2 \ , \tag{2.16}$$

whose solution is obviously

$$\Theta(x,y;0) = \theta_0(x,y) \ . \tag{2.17}$$

Moreover, at $p=1$, we obtain from Eqs. (2.11) and (2.13) the governing equation $A[\Theta(x,y;1)] = 0$, i.e.

$$\frac{\partial}{\partial x}\left[k_1(\Theta)\frac{\partial\Theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_2(\Theta)\frac{\partial\Theta}{\partial y}\right] + S(x,y,\Theta) - (\beta^2+\beta)\Theta + (2\beta^2+\beta)\theta^{n-1} - \beta^2\theta^{n-2} = 0 \ , \quad (x,y) \in \Omega, \quad p = 1 \ , \tag{2.18}$$

with the two boundary conditions

$$\Theta(x,y;1) = g_1(x,y,n\Delta t), \quad (x,y) \in \Gamma_1 \ , \tag{2.19}$$

$$\frac{\partial\Theta(x,y;1)}{\partial n} = g_2(x,y,n\Delta t), \quad (x,y) \in \Gamma_2 \ . \tag{2.20}$$

Comparing Eqs. (2.18) to (2.20) with the governing Eq. (2.6) and the two boundary conditions (2.7) and (2.8), it is clear that $\Theta(x,y,1)$ is just the solution of Eqs. (2.6) to (2.8). Thus, we have

$$\Theta(x,y;1) = \theta^n = \theta(x,y,n\Delta t) \ . \tag{2.21}$$

Therefore, Eqs. (2.11) to (2.13) construct a family of equations in parameter $p \in [0,1]$, whose solution at $p=0$ is simply the known initial approximation $\theta_0(x,y)$ but at $p=1$ is the unknown temperature distribution $\theta^n = \theta(x,y,n\Delta t)$ itself. The process of the continuous change of the imbedding parameter p from 0 to 1 is just the process of the continuous variation of solution $\Theta(x,y;p)$, governed by Eqs. (2.11) to (2.13), from $\theta_0(x,y)$ to $\theta^n = \theta(x,y,n\Delta t)$. This kind of continuous variation is called *deformation* in topology, meanwhile, $\Theta(x,y;p)$ is called *homotopy*, $\theta_0(x,y)$ and $\theta^n = \theta(x,y,n\Delta t)$ are called *homotopic*. Notice that this kind of continuous deformation is completely governed by Eqs. (2.11) to (2.13). So, we call them *the zeroth-order deformation equations.*

Thus, expanding at first $\Theta(x,y;p)$ at $p=0$ by the Taylor formula and then using (2.17), we have

$$\Theta(x,y;p) = \Theta(x,y,0) + \sum_{m=1}^{\infty}\left[\frac{\theta_0^{[m]}(x,y)}{m!}\right]p^m = \theta_0(x,y) + \sum_{m=1}^{\infty}\left[\frac{\theta_0^{[m]}(x,y)}{m!}\right]p^m \ , \tag{2.22}$$

where $\theta_0^{[m]}(x,y)(m \geq 1)$ is called the mth-order deformation derivative at $p=0$, defined by

$$\theta_0^{[m]}(x,y) = \left.\frac{\partial^m\Theta(x,y;p)}{\partial p^m}\right|_{p=0} \quad (m \geq 1) \ . \tag{2.23}$$

Assume that the convergence radius of (2.22) is not less than one. Then, in case $p=1$, we have by (2.21) and (2.22) that

$$\theta(x,y,n\Delta t) = \theta_0(x,y) + \sum_{m=1}^{\infty}\frac{\theta_0^{[m]}(x,y)}{m!} \ , \tag{2.24}$$

which gives a relationship between the initial approximation $\theta_0(x,y)$ and the unknown temperature distribution $\theta^n = \theta(x,y,n\Delta t)$ at the nth time step.

By (2.24), the key of the problem becomes how to solve the so-called mth-order derivatives $\theta_0^{[m]}(x,y)(m \geq 1)$. For this purpose, we had to first of all give the equations governing $\theta_0^{[m]}(x,y)(m \geq 1)$. Differentiating the zeroth-order deformation equations (2.11) to (2.13) m times with respect to p and then setting $p=0$, we obtain the following *mth-order deformation equation* at $p=0$:

$$L[\theta_0^{[m]}(x,y)] = R_m(x,y), \quad m \geq 1, \quad (x,y) \in \Omega \ , \tag{2.25}$$

with the boundary conditions $(m \geq 1)$

$$\theta_0^{[m]} = [g_1(x,y) - \theta_0(x,y)]\delta_{1,m} \quad (x,y) \in \Gamma_1 \ , \tag{2.26}$$

$$\frac{\partial\theta_0^{[m]}}{\partial n} = \left[g_2(x,y) - \frac{\partial\theta_0(x,y)}{\partial n}\right]\delta_{1,m} \quad (x,y) \in \Gamma_2 \ , \tag{2.27}$$

where $\delta_{l,m}$ is the Kronecker delta and

$$R_1(x,y) = -A(\theta_0), \tag{2.28}$$

$$R_m(x,y) = m\left[\nabla^2\theta_0^{[m-1]} - \left.\frac{d^{m-1}A[\Theta(x,y;p)]}{dp^{m-1}}\right|_{p=0}\right] \quad (m \geq 2) \ . \tag{2.29}$$

Moreover, we have by (2.9) that

$$\left.\frac{dA[\Theta(x,y;p)]}{dp}\right|_{p=0} = \left[k_1'(\theta_0)\frac{\partial^2\theta_0}{\partial x^2} + k_2'(\theta_0)\frac{\partial^2\theta_0}{\partial y^2} + k_1''(\theta_0)\left(\frac{\partial\theta_0}{\partial x}\right)^2 + k_2''(\theta_0)\left(\frac{\partial\theta_0}{\partial y}\right)^2\right]\theta_0^{[1]} + 2k_1'(\theta_0)\left(\frac{\partial\theta_0}{\partial x}\right)\frac{\partial\theta_0^{[1]}}{\partial x}$$

$$
\begin{aligned}
&+2k_2'(\theta_0)\left(\frac{\partial\theta_0}{\partial y}\right)\frac{\partial\theta_0^{[1]}}{\partial y}\\
&+k_1(\theta_0)\frac{\partial^2\theta_0^{[1]}}{\partial x^2}+k_2(\theta_0)\frac{\partial^2\theta_0^{[1]}}{\partial y^2}\\
&+\left(\frac{\partial S(x,y,\theta_0)}{\partial\theta}-\beta^2-\beta\right)\theta_0^{[1]}\ ,
\end{aligned}
\tag{2.30}
$$

and so on.

We emphasize that the mth-order deformation Eqs. (2.25) to (2.27) are linear about the mth-order deformation derivative $\theta_0^{[m]}(x,y)(m \geq 1)$. The linear Eq. (2.25) contains the free selected, familiar auxiliary linear operator L so that it can be easily solved by the traditional BEM. For instance, we can solve the following integral equation

$$
\begin{aligned}
&\oint_\Gamma\left(\omega\frac{\partial\theta_0^{[m]}}{\partial n}-\frac{\partial\omega}{\partial n}\theta_0^{[m]}\right)\mathrm{d}\Gamma\\
&\quad=\iint_\Omega \omega(\vec{r}',\vec{r})R_m(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad (m\geq 1)\\
&\vec{r}'=(\xi,\eta)\in\Omega^c,\quad \vec{r}=(x,y)\in\Omega\ ,
\end{aligned}
\tag{2.31}
$$

to determine the unknown values of $\theta_0^{[m]}(x,y)$ on Γ_2 and $\partial\theta_0^{[m]}(x,y)/\partial n$ on Γ_1, where $\omega(\vec{r}',\vec{r})$ is the fundamental solution of the auxiliary linear operator L, and Ω^c denotes the exterior of domain Ω, excluding its boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. After obtaining the unknown values of $\theta_0^{[m]}(x,y)$ on Γ_2 and $\partial\theta_0^{[m]}(x,y)/\partial n$ $(m \geq 1)$ on Γ_1, we have at point $(\xi,\eta) \in \Omega\cup\Gamma$ such an integral equation

$$
\begin{aligned}
a(\xi,\eta)\theta_0^{[m]}(\xi,\eta)&=\oint_\Gamma\left(\omega\frac{\partial\theta_0^{[m]}}{\partial n}-\frac{\partial\omega}{\partial n}\theta_0^{[m]}\right)\mathrm{d}\Gamma\\
&\quad-\iint_\Omega \omega(\vec{r}',\vec{r})R_m(x,y)\,\mathrm{d}x\,\mathrm{d}y,\\
\vec{r}'=(\xi,\eta)&\in\Omega\cup\Gamma,\quad \vec{r}=(x,y)\in\Omega,\ m\geq 1\ ,
\end{aligned}
\tag{2.32}
$$

where

$$
a(\xi,\eta)=\begin{cases}1 & \text{if } (\xi,\eta)\in\Omega\ ,\\ 1/2 & \text{if } (\xi,\eta)\in\Gamma\ .\end{cases}
\tag{2.33}
$$

The fundamental solution $\omega(\vec{r}',\vec{r})$ depends on the selected auxiliary linear operator L. For the 2D problem under consideration, we use two kinds of auxiliary linear operators. One is the 2D Laplace operator

$$
L(u)=\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\ ,
\tag{2.34}
$$

whose fundamental solution is

$$
\omega(\vec{r}',\vec{r})=-\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-\eta)^2}\ .
\tag{2.35}
$$

The other is the 2D modified Helmholtz operator

$$
L(u)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-(\beta^2+\beta)u\ ,
\tag{2.36}
$$

whose fundamental solution is

$$
\omega(\vec{r}',\vec{r})=-\frac{1}{2\pi}K_0\left(\Lambda\sqrt{(x-\xi)^2+(y-\eta)^2}\right)\ ,
\tag{2.37}
$$

where $\Lambda = \sqrt{\beta^2+\beta}$ $(\beta = 1/\Delta t)$, and $K_0(t)$ is the modified Bessel function of the second kind of order zero, $\vec{r} = (x,y)$ and $\vec{r}' = (\xi,\eta)$ are field point and source point, respectively.

Note that only limited numbers of deformation derivatives $\theta_0^{[m]}(x,y)(m \geq 1)$ can be obtained. If the convergence radius ρ of the Taylor series (2.24) is greater than or equal to one, we can use

$$
\theta(x,y,n\Delta t)\approx\theta_0(x,y)+\sum_{m=1}^{M}\frac{\theta_0^{[m]}(x,y)}{m!}
\tag{2.38}
$$

to obtain a new approximation better than $\theta_0(x,y)$, where M denotes the order of the approximation. However, the convergence radius ρ of the Taylor series (2.22) may be less than one so that (2.24) does not hold. Even in this case, $\Theta(x,y;\lambda)(0 < \lambda < \rho)$ is in most cases still better than the initial approximation $\theta_0(x,y)$ so that we can use the following iterative formula

$$
\theta_{k+1}(x,y,n\Delta t)=\theta_k(x,y,n\Delta t)+\sum_{m=1}^{M}\frac{\lambda^m\theta_0^{[m]}(x,y)}{m!}\qquad (k=0,1,2,3,\ldots)\ ,
\tag{2.39}
$$

where λ $(0 < \lambda < \rho)$ is treated as an iterative parameter. We call (2.39) the Mth-order iterative formula. At the beginning of each iteration process related to $\theta^n = \theta(x,y,n\Delta t)$, we simply use the known temperature distribution $\theta^{n-1} = \theta(x,y,n\Delta t - \Delta t)$ as the initial approximation $\theta_0(x,y)$. However, we should keep in mind that $\theta_0(x,y)$ appearing in all of the above expressions should be set new values before each iteration.

The temperature distribution $\theta(x,y,0)$ is determined directly by the initial condition (2.2). We use the initial condition $\frac{\partial\theta(x,y,0)}{\partial t} = f_1(x,y)$ to obtain the temperature distribution $\theta(x,y,\Delta t)$ at the first time step, i.e.

$$
\theta(x,y,\Delta t)=\theta(x,y,0)+f_1(x,y)\Delta t\ .
$$

Then, by means of the above-mentioned BEM formulations, we can obtain the temperature distribution $\theta(x,y,2\Delta t)$ at the second time step. Similarly, we can further use $\theta(x,y,\Delta t), \theta(x,y,2\Delta t)$ to obtain the temperature distribution $\theta(x,y,3\Delta t)$ at the third time step, and so on. In this way, we obtain the solution of the original unsteady Eqs. (2.1), (2.2) and (2.3) in the whole time domain.

3
Numerical example

In order to show the validity of the above-mentioned formulations of the general BEM for unsteady non-linear heat transfer problems of inhomogeneous materials governed by HHCE, we consider here such an unsteady heat transfer problem of a 2D unit plate made of inhomogeneous materials, governed by

$$
\frac{\partial^2\theta}{\partial t^2}+\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial x}\left[k_1(\theta)\frac{\partial\theta}{\partial x}\right]+\frac{\partial}{\partial y}\left[k_2(\theta)\frac{\partial\theta}{\partial y}\right]\ ,
\tag{3.1}
$$

with the initial conditions $\theta = 0, \frac{\partial \theta}{\partial t} = 0$ and the boundary conditions

$$\theta = 1 \quad (x,y) \in \Gamma_1 \ , \tag{3.2}$$

$$\frac{\partial \theta}{\partial t} = 0 \quad (x,y) \in \Gamma_2 \ , \tag{3.3}$$

where Γ_1 is the half center part of the left side of the unit plate, Γ_2 denotes the other sides of heat insulation, as shown in Fig. 1.

According to Eqs. (2.11) to (2.13), the corresponding zeroth-order deformation equation for the temperature distribution $\theta^n = \theta(x, y, n\Delta t)(n \geq 2)$ is

$$(1-p)L[\Theta(x,y;p) - \theta_0(x,y)] = -p\tilde{A}[\Theta(x,y;p)], \quad (x,y) \in [0,1]\times[0,1], \quad p \in [0,1] \ , \tag{3.4}$$

with the two boundary conditions

$$\Theta(x,y;p) = p + (1-p)\theta_0(x,y) \quad (x,y) \in \Gamma_1, \quad p \in [0,1] \ , \tag{3.5}$$

$$\frac{\partial \Theta(x,y;p)}{\partial n} = (1-p)\frac{\partial \theta_0(x,y)}{\partial n} \quad (x,y) \in \Gamma_2, \quad p \in [0,1] \ , \tag{3.6}$$

where L is either the 2D Laplace operator (2.34) or the 2D modified Helmholtz operator (2.36), and the non-linear operator $\tilde{A}(\Theta)$ is now defined by

$$\tilde{A}[\Theta(x,y;p)] = k_1(\Theta)\frac{\partial^2 \Theta}{\partial x^2} + k_2(\Theta)\frac{\partial^2 \Theta}{\partial y^2} + k_1'(\Theta)\left[\frac{\partial \Theta}{\partial x}\right]^2 + k_2'(\Theta)\left[\frac{\partial \Theta}{\partial y}\right]^2 - (\beta^2+\beta)\Theta + (2\beta^2+\beta)\theta^{n-1} - \beta^2\theta^{n-2}, \quad (x,y) \in [0,1]\times[0,1], \quad p \in [0,1] \ , \tag{3.7}$$

where $\beta = \frac{1}{\Delta t}$.

According to Eqs. (2.25) to (2.27), $\theta_0^{[m]}(x,y)(m \geq 1)$ are governed by

$$L[\theta_0^{[m]}(x,y)] = \tilde{R}_m(x,y), \quad (x,y) \in [0,1]\times[0,1], \quad m \geq 1 \tag{3.8}$$

with boundary conditions

$$\theta_0^{[m]}(x,y) = \delta_{1,m}[1 - \theta_0(x,y)], \quad (x,y) \in \Gamma_1 \ , \tag{3.9}$$

$$\frac{\partial \theta_0^{[m]}(x,y)}{\partial n} = -\delta_{1,m}\frac{\partial \theta_0(x,y)}{\partial n}, \quad (x,y) \in \Gamma_2 \ , \tag{3.10}$$

where $\delta_{1,m}$ is the Kronecker delta and

$$\tilde{R}_1(x,y) = -\tilde{A}(\theta_0) \ , \tag{3.11}$$

$$\tilde{R}_m(x,y) = m\left[\nabla^2\theta_0^{[m-1]} - \left.\frac{d^{m-1}A[\Theta(x,y;p)]}{dp^{m-1}}\right|_{p=0}\right] \quad (m \geq 2) \ . \tag{3.12}$$

The linear Eq. (3.8) with the linear boundary conditions (3.9) and (3.10) can be easily solved by the traditional BEM. Note that the boundary has now four sides, each of which is divided into N_Γ equal parts. At each corner, two very close points, each respectively belongs to a different boundary, are used to deal with the discontinuation. Within every boundary element, the unknowns $\theta_0^{[m]}(x,y)$ (on Γ_2) and $\frac{\partial \theta_0^{[m]}(x,y)}{\partial n}$ (on Γ_1) are linearly distributed. Therefore, we have all together $4(N_\Gamma + 1)$ unknowns on the four sides of the unit plate. For the purpose of domain integral, the domain $[0,1]\times[0,1]$ is divided into equal $N_\Omega \times N_\Omega$ subdomains and four-point Gauss-integral formula is used.

Our convergence criterion is

$$\Delta = \sqrt{\frac{\sum_{i=0}^{N_\Omega}\sum_{j=0}^{N_\Omega}\left|\tilde{A}\left[\theta(x_i,y_j,n\Delta t)\right]\right|^2}{(N_\Omega+1)^2}} \leq 10^{-4} \ , \tag{3.13}$$

where the non-linear operator $\tilde{A}$ is defined by (3.7).

We find that, if we use the 2D Laplace operator (2.34) as the auxiliary linear operator, the iteration diverges in most cases, even if a very small value of λ is used. However, if we use the 2D modified Helmholtz operator (2.36), whose fundamental solution (2.37) is related to the modified Bessel function of the second kind of order zero, as our auxiliary linear operator, the corresponding iteration process converges and we always get accurate enough approximations of the temperature distribution $\theta^n = \theta(x, y, n\Delta t)$ at the nth time step. This is quite inter-

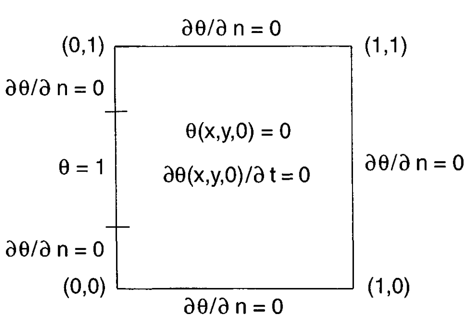


Fig. 1. Boundary and initial conditions for microwave heating of a 2D unit plate

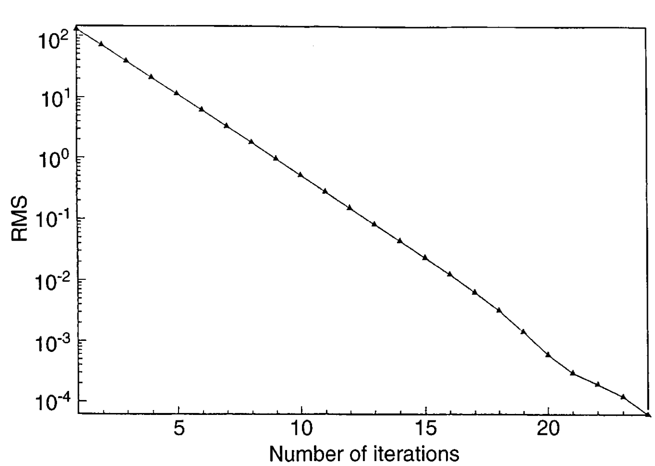


Fig. 2. Error versus number of iterations at the first time step in case $k_1(\theta) = \exp(\theta), k_2(\theta) = 1, \Delta t = 0.01, \lambda = 0.5, N_\Gamma = N_\Omega = 40$

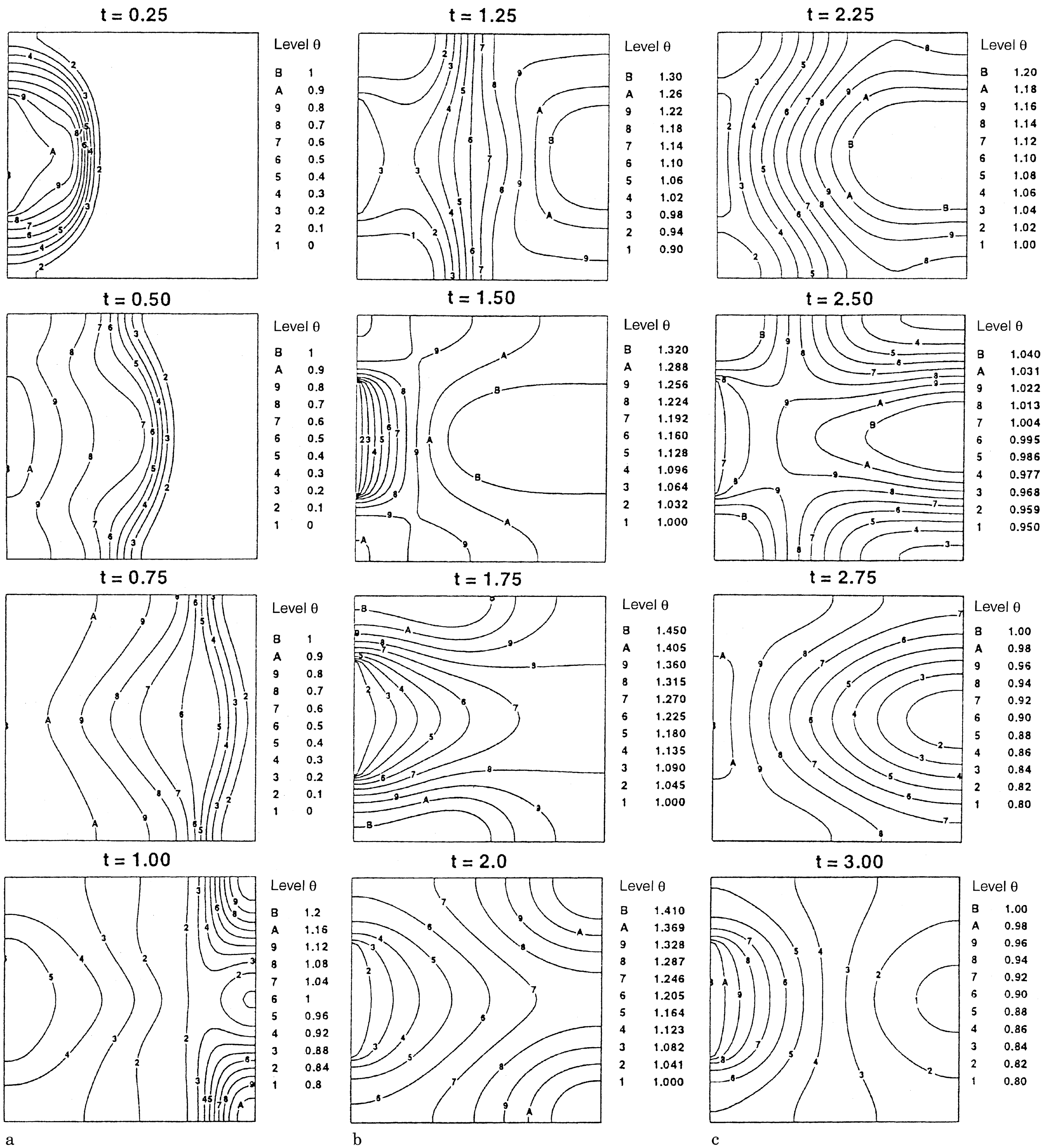
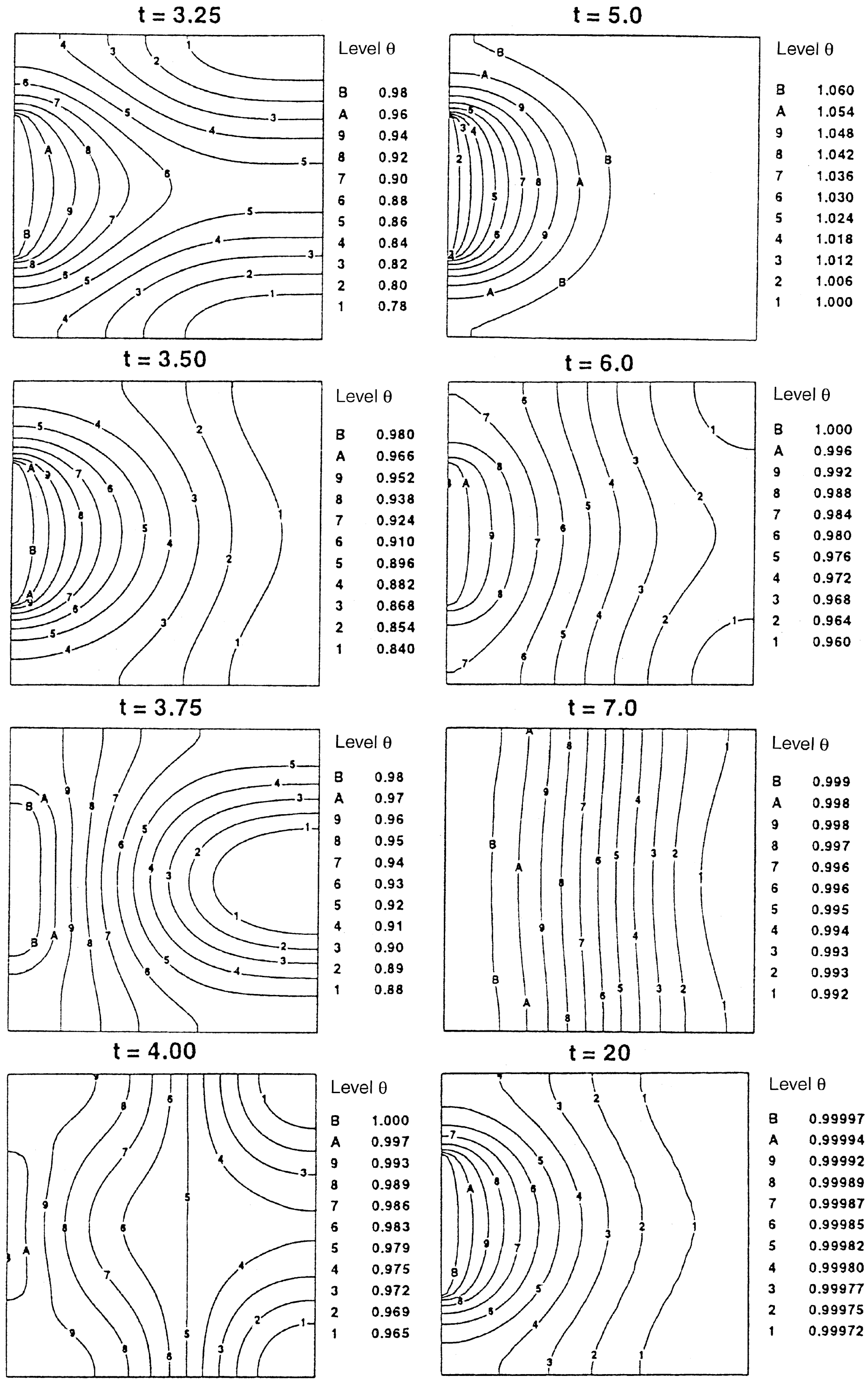


Fig. 3. Continuation and legend on page 403



d e

Fig. 3. a Propagation of thermal wave of a 2D unit plate in the case $k_1(\theta) = \exp(\theta), k_2(\theta) = 1$ at different non-dimensional time: $t = 0.25, 0.50, 0.75, 1.00$ ($N_\Gamma = N_\Omega = 40$, $\lambda = 0.5, \Delta t = 0.01$). **b** Propagation of thermal wave of a 2D unit plate in the case $k_1(\theta) = \exp(\theta)$, $k_2(\theta) = 1$ at different non-dimensional time: $t = 1.25, 1.50, 1.75, 2.00$ ($N_\Gamma = N_\Omega = 40$, $\lambda = 0.5$, $\Delta t = 0.01$). **c** Propagation of thermal wave of a 2D unit plate in the case $k_1(\theta) = \exp(\theta), k_2(\theta) = 1$ at different non-dimensional time: $t = 2.25, 2.50, 2.75, 3.00$ ($N_\Gamma = N_\Omega = 40$, $\lambda = 0.5, \Delta t = 0.01$). **d** Propagation of thermal wave of a 2D unit plate in the case $k_1(\theta) = \exp(\theta)$, $k_2(\theta) = 1$ at different non-dimensional time: $t = 3.25, 3.50, 3.75, 4.00$ ($N_\Gamma = N_\Omega = 40$, $\lambda = 0.5$, $\Delta t = 0.01$). **e** Propagation of thermal wave of a 2D unit plate in the case $k_1(\theta) = \exp(\theta), k_2(\theta) = 1$ at different non-dimensional time: $t = 5, 6, 7, 10$ ($N_\Gamma = N_\Omega = 40$, $\lambda = 0.5$, $\Delta t = 0.01$)

esting and deserves further research. It seems that the auxiliary linear operator should be appropriately selected. It means that, although the general BEM gives us great freedom to select an auxiliary linear operator, this kind of freedom is not absolute but restricted by something such as the form of governing equations under consideration and so on.

In the following parts of this paper, the 2D modified Helmholtz operator (2.36) is used as the auxiliary linear operator, if we do not explicitly point out. Using different values of M, we can obtain by (2.39) a family of iterative formulae at different order of approximation. However, we find that even the first-order iterative formula

$$\theta_{k+1}(x,y,n\Delta t) = \theta_k(x,y,n\Delta t) + \lambda\theta_0^{[1]}(x,y) \qquad (3.14)$$

is good enough to obtain convergent results. Mostly, the iteration converges quickly, as shown in Fig. 2 for the history of the root-mean-square errors *RMS* at the second time step ($t = 2\Delta t$) in case $k_1(\theta) = \exp(\theta)$ and $k_2(\theta) = 1$ under the numerical parameters $N_\Gamma = 40, N_\Omega = 40$, $\lambda = 0.5, \Delta t = 0.01$.

Without loss of generality, we consider in this paper only such a case where $k_1(\theta) = \exp(\mu\theta)$ and $k_2(\theta) = 1$, which can not be solved by the traditional BEM unless $\mu = 0$. We use the following numerical parameters such as $\lambda = 0.5, \Delta t = 0.01, N_\Omega = 40, N_\Gamma = 40$, and the first-order iterative formula (3.14) ($M = 1$). In case $\mu = 0$, our computer program gives the propagation of thermal waves in the time domain which agrees very well with that given by the traditional BEM. In case $\mu = 1$ where the traditional BEM is invalid, the above-mentioned formulations of the general BEM are still valid and the iteration process converges quickly at each time step with the iteration number less than 25. Our numerical results clearly show the propagation of the thermal shock front at the beginning of heat transfer and later the multiple reflections and interactions of the thermal shocks off the insulated walls, as shown in Fig. 3(a) to (e). Moreover, if time is long enough, the temperature distribution in the whole inner field tends to a steady-state, i.e., $\theta = 1$. Our numerical results show that the thermal shock front in case $k_1(\theta) = \exp(\theta)$, $k_2(\theta) = 1$ propagates faster than that in case $k_1(\theta) = k_2(\theta) = 1$, for instance as shown in Fig. 4 ($t = 0.75$). This is reasonable, because in the former case, it holds $k_1(\theta) = \exp(\theta) > 1$ for positive non-dimensional temperature θ so that more heat flux goes through the half center left-side of the unit plate under the same local gradient of the non-dimensional temperature $\theta(x,y,t)$.

In the considered case $k_1(\theta) = \exp(\theta)$ and $k_2(\theta) = 1$, if $\Delta t = 0.01$ and $N_\Omega = N_\Gamma = 20$ are used as numerical parameters, the iteration process converges under $\lambda = 1$ with the number of iterations less than 10 at each time step. However, the numerical approximations are a little different from those at the same time steps obtained under $N_\Omega = N_\Gamma = 40$ and $\Delta t = 0.01$, as shown in Fig. 5 at the non-dimensional time $t = 3$. Even though, the numerical results in both cases still clearly show the propagation of the thermal shock front, and also the multiple reflections and interactions of the thermal shocks off the insulated walls.

In case $N_\Omega = N_\Gamma = 40$, we also solve the same problem under $\Delta t = 0.025$. The corresponding numerical results also clearly show the propagation of the thermal shock front and the multiple reflections and interactions of the thermal shocks off the insulated walls, although they are a little different from those obtained at the same time steps under $N_\Omega = N_\Gamma = 40$ and $\Delta t = 0.01$, as shown in Fig. 6 for $t = 3$. This small difference is acceptable, because we use difference approximations in the time domain.

Therefore, the formulations of the general BEM proposed in this paper for the unsteady non-linear heat transfer problems of inhomogeneous materials governed but the Heat Hyperbolic Conduction Eq. (2.1) are indeed valid and insensitive to the descritizations of the non-dimensional temperature $\theta(x,y,t)$ in both time and spatial domain.

4 Conclusions

This paper has some relationships with the previous work of Liao (1995, 1996) and Liao and Chwang (1996). Here, we further apply the general BEM to solve 2D unsteady non-linear heat transfer problems of inhomogeneous materials, but now governed by a 2D hyperbolic heat conduction equation (HHCE) in a rather general case: the thermal conductivity coefficients in the x and y directions may be different. As an example, we consider a 2D unit plate whose center half part of the left-side is heating ($\theta = 1$) but other boundaries are insulated. Moreover, the material of the unit plate is imhomogenous although we only

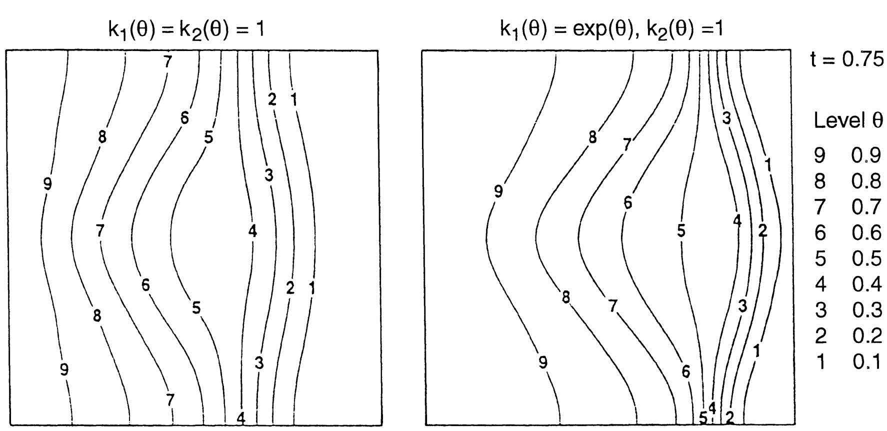


Fig. 4. Comparison of temperature distribution of inhomogeneous plate with that of homogeneous plate at $t = 0.75$

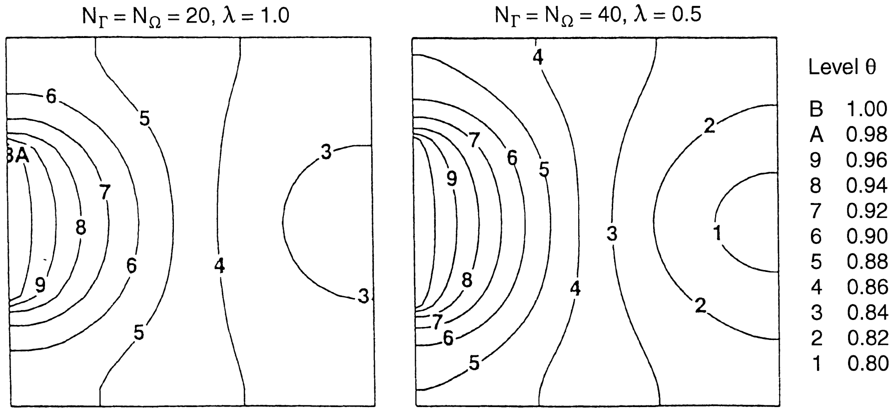


Fig. 5. Comparison of temperature distribution at $t = 3$ under different N_Γ, N_Ω in case $k_1(\theta) = \exp(\theta)$, $k_2 = 1, \Delta t = 0.01$

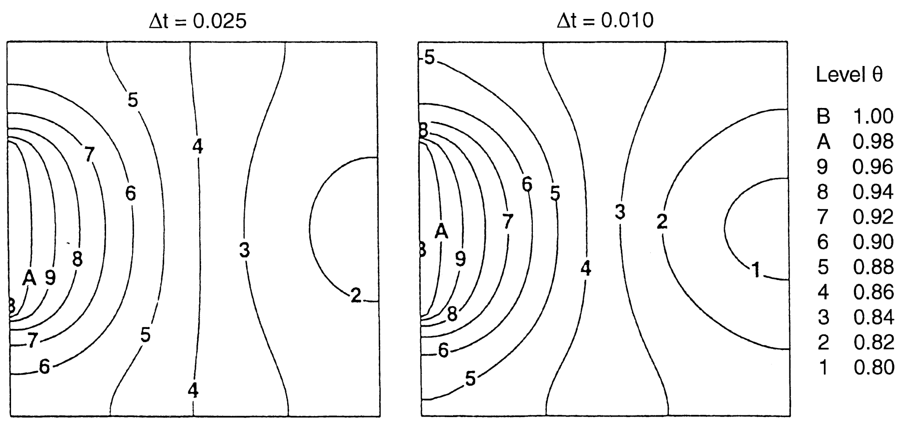


Fig. 6. Comparison of temperature distribution at $t = 3$ under different Δt in case $k_1(\theta) = \exp(\theta)$, $k_2(\theta) = 1, N_\Gamma = N_\Omega = 40$, $\lambda = 0.5$

consider the case $k_1(\theta) = \exp(\theta)$ and $k_2(\theta) = 1$. By means of the formulations of the general BEM proposed in this paper, we successfully calculate the propagation of the thermal shock front and the multiple reflections and interactions of the thermal shock off the insulated walls of the 2D unit plate. We emphasize that the traditional BEM is invalid at all for this unsteady strongly non-linear problem, but the general BEM still works quite well.

In the paper of Liao and Chwang (1996), the general BEM is successfully applied to solve 2D steady-state temperature distributions of imhomogeneous materials governed by steady-state heat transfer equation. Here, we also successfully apply the general BEM to solve unsteady heat transfer problems governed by hyperbolic heat conduction equation (HHCE). So, the general BEM is valid for both steady and unsteady equations. So, we believe that the general BEM can be applied to solve a large number of strongly non-linear problems in engineering, no matter steady or unsteady, especially when they can *not* be solved by the traditional BEM.

We note that in the above-mentioned formulations of the general BEM the domain integral appears which decreases the numerical efficiency of the method. However, a boundary element technique, called the Dual Reciprocity Boundary Element Method, has been well developed to overcome this disadvantage by transforming the domain integral to the surface integration. Thus, the general BEM for unsteady, strongly non-linear problems, which are hyperbolic in time, might become more efficient, when it is combined with the Dual Reciprocity Boundary Element Method.

Note that the mth-order deformation equations governing the mth-order deformation derivative $\theta_0^{[m]}(x, y)$ $(m \geq 1)$ are linear about $\theta_0^{[m]}(x, y)(m \geq 1)$ under a selected, familiar auxiliary linear operator L so that they can be easily solved by the traditional BEM. This is the key of the general BEM. However, we should point out that one can also apply other numerical techniques such as Finite Element Method (FEM) and Finite Difference Method (FDM) to solve these *linear* mth-order deformation equations without any difficulties.

Finally, we emphasize that the traditional BEM is invalid to solve the unsteady non-linear heat transfer problem of the inhomogeneous materials under consideration, but the general BEM is still valid for it. This confirms once again the validity and the great potential of the general BEM, although more applications in engineering are necessary.

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