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Series solutions of non-linear Riccati differential equations with fractional order

Jie Cang, Yue Tan, Hang Xu, Shi-Jun Liao *

School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai 200030, China

Abstract

In this paper, based on the homotopy analysis method (HAM), a new analytic technique is proposed to solve non-linear Riccati differential equation with fractional order. Different from all other analytic methods, it provides us with a simple way to adjust and control the convergence region of solution series by introducing an auxiliary parameter $\hbar$. Besides, it is proved that well-known Adomian's decomposition method is a special case of the homotopy analysis method when $\hbar = -1$. This work illustrates the validity and great potential of the homotopy analysis method for the non-linear differential equations with fractional order. The basic ideas of this approach can be widely employed to solve other strongly non-linear problems in fractional calculus.
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1. Introduction

Fractional differential equations have been found to be effective to describe some physical phenomena such as damping laws, rheology, diffusion processes, and so on. Several methods have been used to solve Fractional differential equations, such as Laplace transform method [1,2], Fourier transform method [3], Adomian's decomposition method (ADM) [4–7] and so on.

There are many different types of definitions of fractional calculus. For example, the Riemann–Liouville integral operator [1] of order μ is defined by

$$(J^{\mu}f)(x) = \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1} f(t)\,\mathrm{d}t, \quad (\mu > 0) \tag{1}$$

$$(J^{0}f)(x) = f(x) \tag{2}$$

and its fractional derivative of order μ ($\mu \geqslant 0$) is normally used:

$$(D_l^{\mu}f)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (J^{n-\mu}f)(x), \quad (\mu > 0,\ n-1 < \mu < n), \tag{3}$$

where n is an integer. For Riemann–Liouville's definition, one has

* Corresponding author. Tel.: +86 21 6293 2676; fax: +86 21 6293 3156.
E-mail address: sjliao@sjtu.edu.cn (S.-J. Liao).

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$$J^{\mu}x^{v} = \frac{\Gamma(v+1)}{\Gamma(v+1+\mu)}x^{v+\mu}. \tag{4}$$

The Riemann–Liouville integral operator plays an important role in the development of the theory of fractional derivatives and integrals. However, it has some disadvantages for fractional differential equations with initial and boundary conditions. Therefore, we adopt here Caputo's definition [8,9], which is a modification of Riemann–Liouville definition:

$$(D^{\mu}f)(x) = \frac{1}{\Gamma(n-\mu)}\int_0^x (x-t)^{n-\mu-1}f^{(n)}(t)\,\mathrm{d}t, \quad (\mu > 0,\ n-1 < \mu < n), \tag{5}$$

where n is an integer. Caputo's integral operator has an useful property [8,9]:

$$(J^{\mu}D^{\mu}f)(x) = f(x) - \sum_{\kappa=0}^{n-1} f^{(\kappa)}(0^+)\frac{x^{\kappa}}{\kappa!}, \quad (n-1 < \mu < n), \tag{6}$$

where n is an integer.

We consider here the following non-linear fractional Riccati differential equation

$$D^{\mu}y = A(t) + B(t)y + C(t)y^2, \quad n-1 < \mu < n, \quad t > 0, \tag{7}$$

subject to the initial conditions

$$y^{(k)}(0) = c_k, \quad k = 0, 1, \ldots, n-1, \tag{8}$$

where μ is fractional derivative order, n is an integer, $A(t)$, $B(t)$ and $C(t)$ are known real functions, and c_k is a constant. When μ is a positive integer, the fractional equation becomes the classical Riccati differential equation. The importance of this equation usually arises in the optimal control problems.

Momani and Shawagfeh [7] solved the fractional Riccati differential equations (7) and (8) by means of the Adomian's decomposition method [10]. However, the convergence region of the corresponding results is rather small, as shown later in this paper. Currently, the homotopy analysis method (HAM) [11–16] is developed to solve lots of non-linear problems. Different from perturbation techniques, the homotopy analysis method does not depend upon any small or large parameters. Besides, it logically contains other non-perturbation techniques, such as Adomian's decomposition method [10], Lyapunov's artificial small parameter method [17], and the δ-expansion method [18], as proved by Liao [14]. Currently, Hayat and Sajid [19], Sajid et al. [20] and Abbasbandy [21,22] pointed out that the so-called "homotopy perturbation method" proposed by He [23] in 1999 is only a special case of the homotopy analysis method propounded by Liao [11,14,16] in 1992. Thus, the homotopy analysis method is valid for much more non-linear problems in science and engineering, especially for those without small/large parameters. Furthermore, different from all other analytic techniques, it provides us with a simple way to control and adjust the convergence of solution series, therefore one can always get accurate enough approximations. The homotopy analysis method has been successfully applied to many non-linear problems, such as non-linear vibration [24], non-linear water waves [25], viscous flows of non-Newtonian fluids [19,20,26,27], unsteady viscous flows [28], the generalized Hirota–Satsuma coupled KdV equation [22], Thomas–Fermi's equation [29], non-linear heat transfer [21,30], a third grade fluid past a porous plate [31], the flow of an Oldroyd 6-constant fluid [32], MHD flows over a stretching surface [33], and so on. Currently, Zhu [34,35] applied the HAM to give, for the first time, an explicit series solution of the famous Black–Scholes type equation in finance for American put option, which is a system of non-linear PDEs with an unknown moving boundary. All of these previous works show the validity of the HAM and provide us a good background to the problem mentioned above.

All of our previous applications of the HAM are related with differential/integral equations with integer order. In this paper, we further apply the HAM, for the first time, to solve the non-linear fractional Riccati differential equation (7). By means of introducing an auxiliary parameter $\hbar$, we can adjust and control the convergence region of solution series. Besides, we prove that Adomian's decomposition method is a special case of the HAM when $\hbar = -1$. Much better approximations are obtained by the HAM, which agree well with Podlubny's numerical solutions [1]. This work illustrates the validity of the HAM for the non-linear fractional differential equations.

2. Homotopy analysis method

2.1. Zeroth-order deformation equation

Let $\mathscr{L}$ denote an auxiliary linear operator, $y_0(t)$ is an initial approximation that satisfies the initial conditions (8). Here, we emphasize that we have freedom to choose the auxiliary linear operator $\mathscr{L}$ and the initial guess $y_0(t)$. From (7), one has

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$$J^{\mu}D^{\mu}y = J^{\mu}A(t) + J^{\mu}[B(t)y + C(t)y^2], \quad n-1 < \mu < n, \quad t > 0,$$

which gives, according to (6) and (8), that

$$y(t) = \sum_{k=0}^{n-1} c_k \frac{t^k}{k!} + J^{\mu}A(t) + J^{\mu}[B(t)y(t) + C(t)y^2(t)], \quad n-1 < \mu < n, \quad t > 0.$$

Neglecting the unknown terms on the right-hand side, we have the initial guess

$$y_0(t) = \sum_{k=0}^{n-1} c_k \frac{t^k}{k!} + J^{\mu}A(t). \tag{9}$$

Note that the original equation contains the linear operator D^{μ}. So, it is straightforward for us to choose the auxiliary linear operator

$$\mathscr{L}\phi = D^{\mu}\phi. \tag{10}$$

For simplicity, we define, according to Eq. (7), the non-linear operator

$$\mathscr{N}\phi = D^{\mu}\phi - A(t) - B(t)\phi - C(t)\phi^2. \tag{11}$$

Then, we construct the so-called zeroth-order deformation equation

$$(1-q)\mathscr{L}[G(t;q) - y_0(t)] = q\hbar\mathscr{N}[G(t;q)], \tag{12}$$

subject to the initial conditions

$$G^{(\kappa)}(0;q) = c_{\kappa}, \quad \kappa = 0, 1, \ldots, n-1, \tag{13}$$

where $\hbar \neq 0$ denote an auxiliary parameter, $q \in [0,1]$ is an embedding parameter, $G(t;q)$ is a kind of mapping of $y(t)$, as described later. Note that the above equations define a *family* of solutions $G(t;q)$ with respect to the embedding parameter $q \in [0,1]$. When $q = 0$, since $y_0(t)$ satisfies all initial conditions (8) and besides $\phi = 0$ is a solution of $\mathscr{L}\phi = 0$, we have obviously

$$G(t;0) = y_0(t). \tag{14}$$

When $q = 1$, since $\hbar \neq 0$, the zero-order deformation equations (12) and (13) are equivalent to the original ones, provides

$$G(t;1) = y(t). \tag{15}$$

Thus, as q increases from 0 to 1, $G(t;q)$ continuously varies (or *deforms*) from the initial approximation $y_0(t)$ to the exact solution $y(t)$ of Eqs. (7) and (8). Using (14), we expand $G(t;q)$ in the Taylor series

$$G(t;q) = y_0(t) + \sum_{m=1}^{+\infty} y_m(t)q^m, \tag{16}$$

where

$$y_m(t) = \frac{1}{m!}\left.\frac{\partial^m G(t;q)}{\partial q^m}\right|_{q=0}. \tag{17}$$

Assume that $\mathscr{L}$, $\hbar$ and $y_0(t)$ are properly chosen so that the series (16) converges at $q = 1$. Then, using (15), we have the solution series

$$y(t) = y_0(t) + \sum_{m=1}^{+\infty} y_m(t). \tag{18}$$

2.2. The mth-order deformation equation

For the sake of simplicity, define the vector

$$\vec{y}_m = \{y_0(t), y_1(t), y_2(t), \ldots, y_m(t)\}.$$

According to the fundamental theorem in calculus, the Taylor series (16) of $G(t;q)$ is *unique* for a given $G(t;q)$. Since $G(t;q)$ is completely determined by the zero-order deformation equations (12) and (13), $y_m(t)$ is therefore *uniquely*

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determined by (12) and (13). Thus, the governing equations and initial conditions of $y_m(t)$ should be *unique*. Following Liao [11–15], differentiating m times the zero-order deformation equations (12) and (13) with respect to q, then setting $q = 0$, and finally dividing them by $m!$, we have the mth-order deformation equation

$$\mathscr{L}[y_m(t) - \chi_m y_{m-1}(t)] = \hbar R_m(\vec{y}_{m-1}), \tag{19}$$

subject to the initial conditions

$$y_m^{(\kappa)}(0) = 0, \quad \kappa = 0, \ldots, n-1, \tag{20}$$

where

$$R_m(\vec{y}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathscr{N}[G(t;q)]}{\partial q^{m-1}}\bigg|_{q=0} = D^\mu y_{m-1}(t) - A(t)(1-\chi_m) - B(t)y_{m-1}(t) - C(t)\sum_{k=0}^{m-1} y_k(t)y_{m-1-k}(t) \tag{21}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{22}$$

Directly substituting the series (16) into the zero-order deformation equations (12) and (13), and then equating the coefficients of the like power of q, one can obtain exactly the *same* high-order deformation equations as (19) and (20), as proved by Sajid et al. [20] and Hayat and Sajid [19]. Therefore, for the same zero-order deformation equation, the corresponding high-order deformation equations are *unique*, no matter how one obtains them. This agrees with the fundamental theorem in calculus, as mentioned above.

According to (19) and (10), one has

$$J^\mu D^\mu[y_m(t) - \chi_m y_{m-1}(t)] = \hbar J^\mu[R_m(\vec{y}_{m-1})].$$

Using the property (6) and the initial conditions (20), we further have

$$y_m(t) = \chi_m y_{m-1}(t) + \hbar J^\mu[R_m(\vec{y}_{m-1})]. \tag{23}$$

Substituting (21) into the above expression, and using (9) and the property (6), we have the solution

$$y_m(t) = (\chi_m + \hbar)y_{m-1}(t) - \hbar(1-\chi_m)y_0(t) - \hbar J^\mu\left[B(t)y_{m-1}(t) + C(t)\sum_{i=0}^{m-1} y_i(t)y_{m-1-i}(t)\right]. \tag{24}$$

When $\hbar = -1$, the above expression gives

$$y_m(t) = J^\mu\left[B(t)y_{m-1}(t) + C(t)\sum_{i=0}^{m-1} y_i(t)y_{m-1-i}(t)\right], \quad m \geqslant 1. \tag{25}$$

2.3. Relation to Adomian's decomposition method

Liao [14] proved that Adomian's decomposition method [10] is just a special case of the HAM for non-linear differential equations with integer order. Here, we prove that Liao's conclusion is true for non-linear fractional Riccati differential equation.

Momani and Shawagfeh [7] solved Eq. (7) by mans of the Adomian's decomposition method. Applying the operator J^μ to both sides of Eq. (7) and using the initial conditions, Momani and Shawagfeh [7] gave

$$y(t) = \sum_{\kappa=0}^{n-1} c_\kappa \frac{t^\kappa}{\kappa!} + J^\mu[A(t) + B(t)y + C(t)y^2]. \tag{26}$$

The Adomain's decomposition method suggests that the solution $y(t)$ be decomposed by an infinite series of components

$$y(t) = y_0(t) + \sum_{m=1}^{+\infty} y_m(t). \tag{27}$$

Note that in form the above expression is exactly the *same* as the HAM series (18). The non-linear term in Eq. (26) is decomposed in the form

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$$y^2(t) = \sum_{m=0}^{+\infty} A_m(t), \tag{28}$$

where

$$A_m(t) = \sum_{i=0}^{m} y_i(t) y_{m-i}(t) \tag{29}$$

is the so-called the Adomian polynomials. Substituting (28) and (27) into (26) gives

$$\sum_{m=0}^{+\infty} y_m(t) = \sum_{\kappa=0}^{n-1} c_\kappa \frac{t^\kappa}{\kappa!} + J^\mu[A(t) + B(t)\sum_{m=0}^{+\infty} y_m(t) + C(t)\sum_{m=0}^{+\infty} A_m(t)]. \tag{30}$$

By means of the Adomian's decomposition method, one has from the above expression that

$$\begin{aligned} y_0(t) &= \sum_{\kappa=0}^{n-1} c_\kappa \frac{t^\kappa}{\kappa!} + J^\mu(A(t)), \\ y_m(t) &= J^\mu[B(t)y_{m-1} + C(t)A_{m-1}(t)] = J^\mu[B(t)y_{m-1} + C(t)\sum_{i=0}^{m-1} y_i(t)y_{m-1-i}(t)], \quad m \geqslant 1. \end{aligned} \tag{31}$$

Note that the expression (31) is exactly the same as (25) given by the homotopy analysis method when $\hbar = -1$. Thus, Adomian's decomposition method is only a special case of the homotopy analysis method in the special case of $\hbar = -1$. In other words, the homotopy analysis method logically contains the Adiomian's decomposition method, and thus is more general.

Currently, Hayat and Sajid [19] and Sajid et al. [20] generally proved that the so-called "homotopy perturbation method" (HPM) [23] is only a special case of the previous "homotopy analysis method" (HAM) [11] when $\hbar = -1$. Thus, using the so-called "homotopy perturbation method" [23], one should obtain exactly the *same* results as those given by the Adomian's decomposition method for the non-linear fractional differential equation at hand, whose convergent regions are small, as shown in Figs. 1 and 2. This is mainly because, like Adomian's decomposition method, the so-called "homotopy perturbation method" [23] is *sensitive* to physical parameters, and thus is invalid for strongly non-linear problems without small/large physical parameters, as shown by Abbasbandy [21,22].

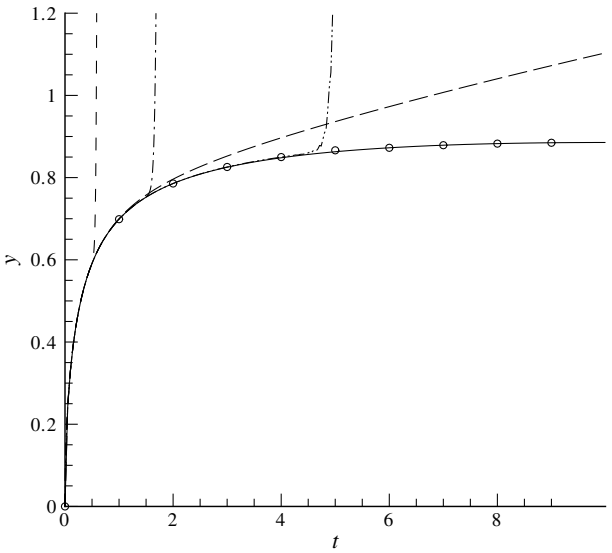

Fig. 1. Comparison of the numerical result with the analytic approximations of Eq. (32) when $\mu = 1/2$. Dashed line: ADM solution; dash-dotted line: HAM solution when $\hbar = -1/2$; Dash-dot-dotted line: HAM solution when $\hbar = -1/5$; Long-dashed line: the traditional [5,5] Padé approximate; solid line: [5,5] homotopy-Padé approximate; symbols: numerical results.

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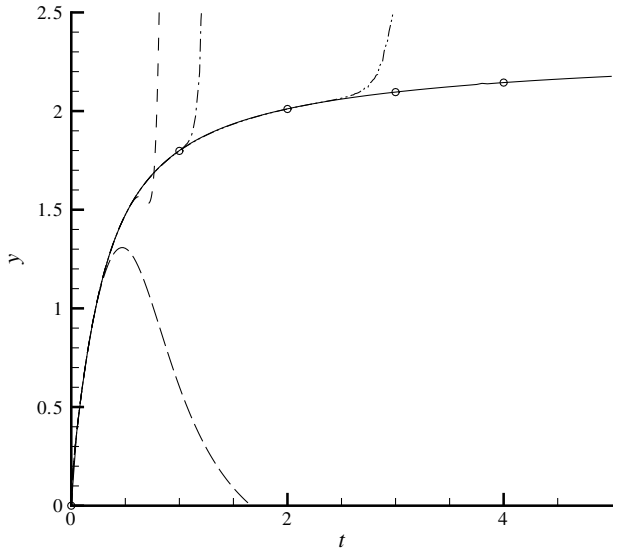

Fig. 2. Comparison of the numerical result with the analytic approximations of Eq. (40) when $\mu = 1/2$. Dashed line: ADM solution; dash-dotted line: HAM solution when $\hbar = -1/2$; dash-dot-dotted line: HAM solution when $\hbar = -1/5$; long-dashed line: the traditional [10,10] Padé approximate; solid line: [5,5] homotopy-Padé approximate; symbols: numerical results.

3. Some examples

3.1. Example A

Consider the fractional Riccati equation:

$$D^{\mu}y + y^2 = 1, \quad n-1 < \mu < n, \quad t > 0, \tag{32}$$

subject to the initial conditions

$$y(0) = 0. \tag{33}$$

In this case, $A(t) = 1, B(t) = 0, C(t) = -1$. According to (9), we have the initial guess

$$y_0(t) = \frac{t^{\mu}}{\Gamma(\mu+1)}. \tag{34}$$

According to (24), we have

$$y_m(t) = (\chi_m + \hbar)y_{m-1}(t) - \hbar(1-\chi_m)y_0(t) + \hbar J^{\mu}\left[\sum_{i=0}^{m-1-i} y_i(t)y_{m-1-i}(t)\right], \tag{35}$$

which gives

$$\begin{aligned}
y_1(t) &= \frac{\hbar t^{3\mu}\Gamma(1+2\mu)}{\Gamma(1+\mu)^2\Gamma(1+3\mu)},\\
y_2(t) &= \frac{\hbar t^{3\mu}\Gamma(1+2\mu)}{\Gamma(1+\mu)^2\Gamma(1+3\mu)} + \frac{\hbar^2 t^{3\mu}\Gamma(1+2\mu)}{\Gamma(1+\mu)^2\Gamma(1+3\mu)} + \frac{2\hbar^2 t^{5\mu}\Gamma(1+2\mu)\Gamma(1+4\mu)}{\Gamma(1+\mu)^3\Gamma(1+3\mu)\Gamma(1+5\mu)},\\
y_3(t) &= \frac{\hbar t^{3\mu}\Gamma(1+2\mu)}{\Gamma(1+\mu)^2\Gamma(1+3\mu)} + \frac{2\hbar^2 t^{3\mu}\Gamma(1+2\mu)}{\Gamma(1+\mu)^2\Gamma(1+3\mu)} + \frac{\hbar^3 t^{3\mu}\Gamma(1+2\mu)}{\Gamma(1+\mu)^2\Gamma(1+3\mu)} + \frac{4\hbar^2 t^{5\mu}\Gamma(1+2\mu)\Gamma(1+4\mu)}{\Gamma(1+\mu)^3\Gamma(1+3\mu)\Gamma(1+5\mu)}\\
&\quad + \frac{4\hbar^3 t^{5\mu}\Gamma(1+2\mu)\Gamma(1+4\mu)}{\Gamma(1+\mu)^3\Gamma(1+3\mu)\Gamma(1+5\mu)} + \frac{\hbar^3 t^{7\mu}\Gamma(1+2\mu)^2\Gamma(1+5\mu)\Gamma(1+6\mu)}{\Gamma(1+2\mu)^4\Gamma(1+3\mu)^2\Gamma(1+7\mu)}\\
&\quad + \frac{4\hbar^3 t^{7\mu}\Gamma(1+2\mu)\Gamma(1+4\mu)\Gamma(1+6\mu)}{\Gamma(1+\mu)^4\Gamma(1+3\mu)\Gamma(1+5\mu)\Gamma(1+7\mu)}.
\end{aligned} \tag{36}$$

When $\hbar = -1$, the above expressions are exactly the same as those given by the Adomian's decomposition method:

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$$
\begin{aligned}
y_1(t) &= -\frac{t^{3\mu}\Gamma(1+2\mu)}{\Gamma(1+\mu)^2\Gamma(1+3\mu)},\\
y_2(t) &= \frac{2t^{5\mu}\Gamma(1+2\mu)\Gamma(1+4\mu)}{\Gamma(1+\mu)^3\Gamma(1+3\mu)\Gamma(1+5\mu)},\\
y_3(t) &= -\frac{t^{7\mu}\Gamma(1+2\mu)^2\Gamma(1+5\mu)\Gamma(1+6\mu)}{\Gamma(1+2\mu)^4\Gamma(1+3\mu)^2\Gamma(1+7\mu)} - \frac{4t^{7\mu}\Gamma(1+2\mu)\Gamma(1+4\mu)\Gamma(1+6\mu)}{\Gamma(1+\mu)^4\Gamma(1+3\mu)\Gamma(1+5\mu)\Gamma(1+7\mu)},
\end{aligned}
\tag{37}
$$

respectively. This illustrates that the Adomian's decomposition method is indeed a special case of the homotopy analysis method. However, mostly, the results given by the Adomian's decomposition method converge to the corresponding numerical solutions in a rather small region, as shown in Fig. 1 in case of $\mu = 0.5$. But, different from Adomian's decomposition method, the homotopy analysis method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter $\hbar$. As shown in Fig. 1, the convergence region of the HAM solutions is enlarged when $\hbar = -1/2$ and $\hbar = -1/5$. It seems that the convergence region of the HAM series tends to infinity as $\hbar$ tends to zero. This is an obvious advantage of the homotopy analysis method over Adomian's decomposition method.

Besides, using the so-called homotopy-Páde acceleration technique [14], one can greatly accelerate the convergence of the HAM solutions. In general, the homotopy-Páde method is more effective than the traditional Páde technique [36] used by Momani [7] for the fractional differential equations. For example, when $\mu = 1/2$, the traditional [2,2] and [4,4] Páde approximations read

$$
\begin{aligned}
\mathrm{TP}[2,2] &= \frac{1.12838t^{0.5}}{1+0.848826t},\\
\mathrm{TP}[4,4] &= \frac{1.12838t^{0.5}+1.0855t^{1.5}}{1+1.81083t+0.38427t^2}.
\end{aligned}
\tag{38}
$$

However, the corresponding [2,2] and [4,4] homotopy-Páde approximations are

$$
\begin{aligned}
\mathrm{HP}[2,2] &= \frac{1.12838t^{0.5}+1.59633t^{1.5}+0.260161t^{2.5}}{1+2.26354t+0.999102t^2},\\
\mathrm{HP}[4,4] &= \frac{1.12838t^{0.5}+3.67419t^{1.5}+3.68162t^{2.5}+1.15758t^{3.5}+0.0551944t^{4.5}}{1+4.10499t+5.59436t^2+2.80361t^3+0.374705t^4},
\end{aligned}
\tag{39}
$$

respectively, which are much more accurate than the traditional ones, as shown in Fig. 1.

3.1.1. Example B

Let us consider the fractional Riccati equation:

$$
D^{\mu}y = 2y(t) - y^2 + 1, \quad n-1 < \mu < n, \quad t > 0, \tag{40}
$$

subject to the initial condition

$$
y(0) = 0. \tag{41}
$$

Now, $A(t) = 1,\ B(t) = 2,\ C(t) = -1$. Similarly, according to (9), we have the initial guess

$$
y_0(t) = \frac{t^{\mu}}{\Gamma(\mu+1)}. \tag{42}
$$

Using (24), we have

$$
y_m(t) = (\chi_m + \hbar)y_{m-1}(t) - \hbar(1-\chi_m)y_0(t) + \hbar J^{\mu}\left[\sum_{i=0}^{m-1-i} y_i(t)y_{m-1-i}(t) - 2y_{m-1}(t)\right]. \tag{43}
$$

Similarly, the result of Adomian's decomposition method, which corresponds to the HAM solution in the special case of $\hbar = -1$, is valid in a rather small region, as shown in Fig. 2. However, the convergence region of the HAM series solution can be enlarged simply by choosing a proper value of the auxiliary parameter $\hbar$, as shown in Fig. 2. It seems that our HAM series solution converges to the exact solution in the whole region $0 \le t < +\infty$ as $\hbar$ tends to zero. Besides, the homotopy-Páde technique is more effective than the traditional Páde method to accelerate the convergence, as shown in Fig. 2.

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4. Conclusion and discussion

In this paper, based on the homotopy analysis method (HAM) [11–16], a new analytic technique is proposed to solve non-linear fractional differential equations. Different from all other analytic methods, it provides us with a simple way to adjust and control the convergence region of solution series by introducing an auxiliary parameter $\hbar$. This is an obvious advantage of the homotopy analysis method. Besides, it is proved that Adomian's decomposition method is only a special case of the homotopy analysis method. This work illustrates the validity and great potential of the homotopy analysis method for non-linear fractional differential equations. The basic ideas of this approach can be further employed to solve other strongly non-linear problems in fractional calculus.

In this paper, we choose the same initial guess $y_0(t)$ as that used in Adomian's decomposition method. Besides, we simply choose the fractional operator D^μ as the auxiliary linear operator. In this way, we obtain solutions in power series. However, it is well-known that a power series often has a small convergence radius. It should be emphasized that, in the frame of the homotopy analysis method, we have great freedom to choose the initial guess and the auxiliary linear operator $\mathscr{L}$, as shown in [16]. So, it might be completely *unnecessary* for us to choose the auxiliary linear operator $\mathscr{L} = D^\mu$. It would be valuable to give series solution of non-linear fractional differential equations by means of other basis functions better than power series.

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