

Analytic Solutions of Von Kármán Plate under Arbitrary Uniform Pressure Part (II): Equations in Integral Form

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Abstract *In this paper, the homotopy analysis method (HAM) is successfully applied to solve the Von Kármán's plate equations in the integral form for a circular plate with the clamped boundary under an arbitrary uniform external pressure. Two HAM-based approaches are proposed. One is for a given external load Q , the other for a given central deflection. Both of them are valid for an arbitrary uniform external pressure by means of choosing a proper value of the so-called convergence-control parameters c_1 and c_2 in the frame of the HAM. Besides, it is found that iteration can greatly accelerate the convergence of solution series. In addition, we prove that the interpolation iterative method [1, 2] is a special case of the HAM-based 1st-order iteration approach for a given external load Q when $c_1 = -\theta$ and $c_2 = -1$, where θ denotes the interpolation parameter of the interpolation iterative method. Therefore, like Zheng and Zhou [3], one can similarly prove that the HAM-based approaches are valid for an arbitrary uniform external pressure, at least in some special cases such as $c_1 = -\theta$ and $c_2 = -1$. Furthermore, it is found that the HAM-based iteration approaches converge much faster than the interpolation iterative method [1, 2]. All of these illustrate the validity and potential of the HAM for the famous Von Kármán's plate equations, and show the superiority of the HAM over perturbation methods.*

Key Words circular plate, uniform external pressure, homotopy analysis method

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1. Introduction

In Part (I), we solved the Von Kármán's plate equations [4] in the differential form for a circular plate under an arbitrary uniform external pressure by means of the homotopy analysis method (HAM) [5–11], an analytical approximation method for highly nonlinear problems. By means of the so-called convergence-control parameter c_0 , convergent analytic approximations for four kinds of boundaries were obtained, with large enough ratio of central deflection to thickness $w(0)/h > 20$. It is found that the convergence-control parameter c_0 plays an important role: it is the convergence-control parameter c_0 that can guarantee the convergence of solution series and iteration, and thus distinguishes the HAM from other analytic methods. Besides, it is found that the iteration technique can greatly increase the computational efficiency. In addition, it was proved in Part (I) that the perturbation methods for *any* a perturbation quantity (including Vincent's [12] and Chien's [13] perturbation methods) and the modified iteration method [14] are only the special cases of the HAM when $c_0 = -1$.

The Von Kármán's plate equations [4] in the integral form describing the large deflection of a circular thin plate under a uniform external pressure read

$$\varphi(y) = - \int_0^1 K(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot S(\varepsilon) \varphi(\varepsilon) d\varepsilon - \int_0^1 K(y, \varepsilon) \cdot Q d\varepsilon, \quad (1)$$

$$S(y) = \frac{1}{2} \int_0^1 G(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \varphi^2(\varepsilon) d\varepsilon, \quad (2)$$

in which

$$K(y, \varepsilon) = \begin{cases} (\lambda - 1)y\varepsilon + y, & y \leq \varepsilon, \\ (\lambda - 1)y\varepsilon + \varepsilon, & y > \varepsilon, \end{cases} \quad (3)$$

$$G(y, \varepsilon) = \begin{cases} (\mu - 1)y\varepsilon + y, & y \leq \varepsilon, \\ (\mu - 1)y\varepsilon + \varepsilon, & y > \varepsilon, \end{cases} \quad (4)$$

with the definitions

$$y = \frac{r^2}{R_a^2}, \quad W(y) = \sqrt{3(1 - \nu^2)} \frac{w(y)}{h}, \quad \varphi(y) = y \frac{dW(y)}{dy}, \quad (5)$$

$$S(y) = 3(1 - \nu^2) \frac{R_a^2 N_r}{Eh^3} y, \quad Q = \frac{3(1 - \nu^2) \sqrt{3(1 - \nu^2)} R_a^4}{4Eh^4} p, \quad (6)$$

where r is the radial coordinate whose origin locates at the center of the plate, E , ν , R_a , h , $w(y)$, N_r and p are Young's modulus of elasticity, the Poisson's ratio, radius, thickness, deflection, the radial membrane force of the plate and the external uniform load, respectively, λ and μ are parameters related to the boundary conditions at $y = 1$, Q is a constant related to the uniform external load, respectively. The dimensionless central deflection

$$W(y) = - \int_y^1 \frac{1}{z} \varphi(z) dz \quad (7)$$

can be derived from Eq.(5). Four kinds of boundaries are considered:

- (a) Clamped: $\lambda = 0$ and $\mu = 2/(1 - \nu)$;
- (b) Moveable clamped: $\lambda = 0$ and $\mu = 0$;
- (c) Simple support: $\lambda = 2/(1 + \nu)$ and $\mu = 0$;
- (d) Simple hinged support: $\lambda = 2/(1 + \nu)$ and $\mu = 2/(1 - \nu)$.

Keller and Reiss [1] proposed the interpolation iterative method to solve the Von Kármán's plate equations in the integral form by introducing an interpolation parameter to the iteration procedure, and they successfully obtained convergent solutions for loads as high as $Q = 7000$. Further, Zheng and Zhou [2, 3] proved that convergent solutions can be obtained by the interpolation iterative method for an arbitrary uniform external pressure. Their excellent work is a milestone in this field.

Iterative procedures of the interpolation iterative method [1, 2] for the Von Kármán's plate equations in the integral form are:

$$\psi_n(y) = \frac{1}{2} \int_0^1 G(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \vartheta_n^2(\varepsilon) d\varepsilon, \quad (8)$$

$$\begin{aligned} \vartheta_{n+1}(y) &= (1 - \theta)\vartheta_n(y) - \theta \int_0^1 K(y, \varepsilon) Q d\varepsilon \\ &- \theta \int_0^1 K(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \vartheta_n(\varepsilon) \psi_n(\varepsilon) d\varepsilon. \end{aligned} \quad (9)$$

with the definition of the initial guess

$$\vartheta_1(y) = -\frac{Q\theta}{2}[(\lambda + 1)y - y^2]. \quad (10)$$

The homotopy analysis method (HAM) [5–11] was proposed by Liao [5] in 1992. Unlike perturbation technique, the HAM is independent of any

small/large physical parameters. Besides, unlike other analytic techniques, the HAM provides a simple way to guarantee the convergence of solution series by means of introducing the so-called “convergence-control parameter”. In addition, the HAM provides us great freedom to choose equation-type and solution expression of the high-order linear equations. As a powerful technique to solve highly nonlinear equations, the HAM was successfully employed to solve various types of nonlinear problems over the past two decades [15–25]. Especially, as shown in [26–30], the HAM can bring us something completely new/different: the steady-state resonant waves were first predicted by the HAM in theory and then confirmed experimentally in a lab [28].

In this paper, we propose two approaches in the frame of the HAM to solve the Von Kármán’s plate equations in the integral form. Besides, we proof that the interpolation iterative method [1, 2] is only a special case of the HAM.

2. HAM approach for given external load Q

2.1. Mathematical formulas

Following Zheng [2], we express $\varphi(y)$ and $S(y)$ as

$$\varphi(y) = \sum_{k=1}^{+\infty} a_k \cdot y^k, \quad S(y) = \sum_{k=1}^{+\infty} b_k \cdot y^k, \quad (11)$$

where a_k and b_k are constant coefficients to be determined. They provide us the so-called “solution expression” of $\varphi(y)$ and $S(y)$ in the frame of the HAM.

Let $\varphi_0(y)$ and $S_0(y)$ be initial guesses of $\varphi(y)$ and $S(y)$. Moreover, let c_1 and c_2 denote the non-zero auxiliary parameters, called the convergence-control parameters, and $q \in [0, 1]$ the embedding parameter, respectively. Then we construct a family of differential equations in $q \in [0, 1]$, namely the zeroth-order deformation equation:

$$(1 - q)[\Phi(y; q) - \varphi_0(y)] = c_1 q \mathcal{N}_1(y; q), \quad (12)$$

$$(1 - q)[\Xi(y; q) - S_0(y)] = c_2 q \mathcal{N}_2(y; q), \quad (13)$$

where

$$\begin{aligned}\mathcal{N}_1(y; q) &= \Phi(y; q) + \int_0^1 K(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \Phi(\varepsilon; q) \Xi(\varepsilon; q) d\varepsilon \\ &+ \int_0^1 K(y, \varepsilon) \cdot Q d\varepsilon,\end{aligned}\tag{14}$$

$$\mathcal{N}_2(y; q) = \Xi(y; q) - \frac{1}{2} \int_0^1 G(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \Phi^2(\varepsilon; q) d\varepsilon,\tag{15}$$

are two nonlinear operators.

When $q = 0$, Eqs. (12) and (13) have the solution

$$\Phi(y; 0) = \varphi_0(y), \quad \Xi(y; 0) = S_0(y).\tag{16}$$

When $q = 1$, they are equivalent to the original equations (1)-(2), provided

$$\Phi(y; 1) = \varphi(y), \quad \Xi(y; 1) = S(y).\tag{17}$$

Therefore, as q increases from 0 to 1, $\Phi(y; q)$ varies continuously from the initial guess $\varphi_0(y)$ to the solution $\varphi(y)$, so does $\Xi(y; q)$ from the initial guess $S_0(y)$ to the solution $S(y)$. In topology, these kinds of continuous variation are called deformation. That is why Eqs. (12)-(13) constructing the homotopies $\Phi(y; q)$ and $\Xi(y; q)$ are called the zeroth-order deformation equations.

Expanding $\Phi(y; q)$ and $\Xi(y; q; a)$ into Taylor series with respect to the embedding parameter q , we have the so-called homotopy-series:

$$\Phi(y; q) = \varphi_0(y) + \sum_{k=1}^{+\infty} \varphi_k(y) q^k,\tag{18}$$

$$\Xi(y; q) = S_0(y) + \sum_{k=1}^{+\infty} S_k(y) q^k,\tag{19}$$

where

$$\varphi_k(y) = \mathcal{D}_k[\Phi(y; q)], \quad S_k(y) = \mathcal{D}_k[\Xi(y; q)],\tag{20}$$

in which

$$\mathcal{D}_k[f] = \left. \frac{1}{k!} \frac{\partial^k f}{\partial q^k} \right|_{q=0}\tag{21}$$

is called the k th-order homotopy-derivative of f .

Note that there are two convergence-control parameters c_1 and c_2 in the homotopy-series (18) and (19), respectively. Assume that c_1 and c_2 are properly chosen so that the homotopy-series (18) and (19) converge at $q = 1$. Then according to Eq. (17), the so-called homotopy-series solutions read:

$$\varphi(y) = \varphi_0(y) + \sum_{k=1}^{+\infty} \varphi_k(y), \quad (22)$$

$$S(y) = S_0(y) + \sum_{k=1}^{+\infty} S_k(y). \quad (23)$$

The n th-order approximation of $\varphi(y)$ and $S(y)$ read

$$\tilde{\Phi}_n(y) = \sum_{k=0}^n \varphi_k(y), \quad (24)$$

$$\tilde{\Xi}_n(y) = \sum_{k=0}^n S_k(y). \quad (25)$$

Substituting the power series (18) and (19) into the zeroth-order deformation equations (12) and (13), and then equating the like-power of the embedding parameter q , we have the so-called k th-order deformation equations

$$\varphi_k(y) - \chi_k \varphi_{k-1}(y) = c_1 \delta_{1,k-1}(y), \quad (26)$$

$$S_k(y) - \chi_k S_{k-1}(y) = c_2 \delta_{2,k-1}(y), \quad (27)$$

where

$$\begin{aligned} \delta_{1,k-1}(y) &= \mathcal{D}_{k-1}[\mathcal{N}_1(y; q)] \\ &= \varphi_{k-1}(y) + \int_0^1 K(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \sum_{i=0}^{k-1} \varphi_i(\varepsilon) S_{k-1-i}(\varepsilon) d\varepsilon \\ &+ (1 - \chi_k) \int_0^1 K(y, \varepsilon) \cdot Q d\varepsilon, \end{aligned} \quad (28)$$

$$\begin{aligned} \delta_{2,k-1}(y) &= \mathcal{D}_{k-1}[\mathcal{N}_2(y; q)] \\ &= S_{k-1}(y) - \frac{1}{2} \int_0^1 G(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \sum_{i=0}^{k-1} \varphi_i(\varepsilon) \varphi_{k-1-i}(\varepsilon) d\varepsilon, \end{aligned} \quad (29)$$

with the definition

$$\chi_m = \begin{cases} 1 & \text{when } m = 1, \\ 0 & \text{when } m \geq 2. \end{cases} \quad (30)$$

Note that the k th-order deformation equations (26)-(27) are linear.

Note that we have great freedom to choose the initial guesses $\varphi_0(y)$ and $S_0(y)$. According to [6], $\varphi_0(y)$ and $S_0(y)$ should obey the solution expression (11), thus, we choose the initial guesses:

$$\varphi_0(y) = \frac{Qc_0}{2}[(\lambda + 1)y - y^2], \quad (31)$$

$$S_0(y) = 0, \quad (32)$$

which satisfy all boundary conditions. Besides, for the sake of simplicity, set

$$c_1 = c_2 = c_0. \quad (33)$$

In order to measure the accuracy of approximations, we define the sum of the two discrete squared residuals

$$Err = \frac{1}{K+1} \sum_{i=0}^K \left\{ \left[\mathcal{N}_1 \left(\frac{i}{K} \right) \right]^2 + \left[\mathcal{N}_2 \left(\frac{i}{K} \right) \right]^2 \right\}, \quad (34)$$

where \mathcal{N}_1 and \mathcal{N}_2 are two nonlinear operators defined by (14) and (15), respectively, and $K = 100$ is used in the whole of this paper. Obviously, the smaller the Err , the more accurate the approximation.

2.2. Convergent results given by the HAM approach without iteration

Without loss of generality, the Von Kármán's plate equations with the clamped boundary is studied at first, and the Poisson's ratio ν is taken to be 0.3 in all cases considered in this paper.

First of all, the HAM-based approach (without iteration) is used to solve the Von Kármán's plate equations for a given external load Q with the clamped boundary. Take the case of $Q = 5$ as an example. The optimal value of c_0 is determined by the minimum of Err defined by (34), i.e. the sum of the squared residual errors of the two governing equations. As shown in Fig. 1, the sum of the squared residual errors arrives its minimum at $c_0 \approx -0.35$, which suggests that the optimal value of c_0 is about -0.35 . As shown in Table 1, the sum of the squared residual errors quickly decreases to 1.7×10^{-7} by means of $c_0 \approx -0.35$ in the case of $Q = 5$, with

Table 1: The sum of the squared residual errors Err and the central deflection $w(0)/h$ versus the order of approximation in the case of $Q = 5$ for a circular plate with the clamped boundary, given by the HAM approach (see § 2.1) without iteration using $c_0 = -0.35$.

m , order of approx.	Err	$w(0)/h$
10	3.3×10^{-4}	0.64
20	6.5×10^{-5}	0.62
30	1.1×10^{-5}	0.62
40	1.6×10^{-6}	0.62
50	1.7×10^{-7}	0.62

the convergent ratio of the maximum central deflection to plate thickness $w(0)/h = 0.62$. Note that Vincent's perturbation result [12] (using Q as the perturbation quantity) for a circular plate with the clamped boundary is only valid for $w(0)/h < 0.52$, corresponding to $Q < 3.9$, and thus fails in the case of $Q = 5$. So, the convergence control parameter c_0 indeed provides us a simple way to guarantee the convergence of solution series. This illustrates that our HAM-based approach (mentioned in § 2.1) has advantages over the perturbation method [12].

For given values of Q , we can always find its optimal convergence control parameter c_0 in a similar way, which can be expressed by the empirical formula:

$$c_0 = -\frac{13}{13 + Q^2}. \quad (35)$$

Besides, the convergent homotopy-approximations of central deflection $w(0)/h$ in case of different values of Q for a circular plate with the clamped boundary are given in Table 2.

2.3. Convergence acceleration by iteration

According to Liao [6], the convergence of the homotopy-series solutions can be greatly accelerated by means of iteration technique, which uses the

Table 2: The convergent results of the central deflection $w(0)/h$ in the case of the different values of Q for a circular plate with the clamped boundary, given by the HAM-based approach (see § 2.1) without iteration using the optimal convergence-control parameter c_0 given by (35).

Q	c_0	$w(0)/h$
1	-0.93	0.15
2	-0.76	0.29
3	-0.59	0.41
4	-0.45	0.53
5	-0.34	0.62

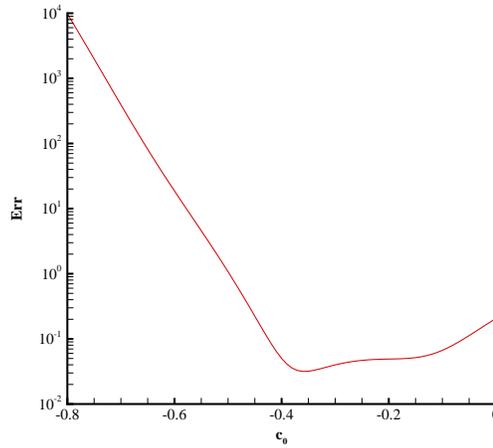


Figure 1: The sum of the squared residual errors Err versus c_0 in the case of the given external load $Q = 5$ for a circular plate with the clamped boundary, given by the HAM-based approach (without iteration).

M th-order homotopy-approximations

$$\varphi^*(y) \approx \varphi_0(y) + \sum_{k=1}^M \varphi_k(y), \quad (36)$$

$$S^*(y) \approx S_0(y) + \sum_{k=1}^M S_k(y), \quad (37)$$

as the new initial guesses of $\varphi_0(y)$ and $S_0(y)$ for the next iteration. This provides us the M th-order iteration approach of the HAM.

Note that the length of the solution expressions increases exponentially in iteration. To avoid this, we truncate the right-hand sides of Eqs.(26) and (27) in the following way

$$\delta_{1,k}(y) \approx \sum_{m=0}^N A_{k,m} \cdot y^k, \quad \delta_{2,k}(y; a) \approx \sum_{m=0}^N B_{k,m} \cdot y^k, \quad (38)$$

where $A_{k,m}$ and $B_{k,m}$ are constant coefficients, and N is called the truncation order, respectively.

Without loss of generality, let us consider the case of $Q = 132.2$, corresponding to $w(0)/h = 3.0$. As shown in Fig. 2, the higher the order M of iteration, the less times of iteration is required for a given accuracy-level of approximation. Besides, the sum of the squared residual errors Err versus the CPU time for different order of iteration (M) are given in Fig. 3. Note that the higher the order of iteration, the faster the approximation converges. Thus, it is natural for us to choose the 5th-order iteration approach (i.e. $M = 5$) from the view-point of computational efficiency.

For a given Q , we use the truncation order $N = 100$. It is found that the corresponding optimal convergence-control parameter c_0 can be expressed by the empirical formula:

$$c_0 = -\frac{23}{Q + 23}. \quad (39)$$

As shown in Table 3, convergent analytic solutions can be obtained even in the case of $Q = 1000$, corresponding to $w(0)/h = 6.1$. Here, it should be emphasized that Vincent's perturbation result [12] (using Q as the perturbation quantity) for a circular plate with the clamped boundary is only valid for $w(0)/h < 0.52$, corresponding to $Q < 3.9$ only. This illustrates again the

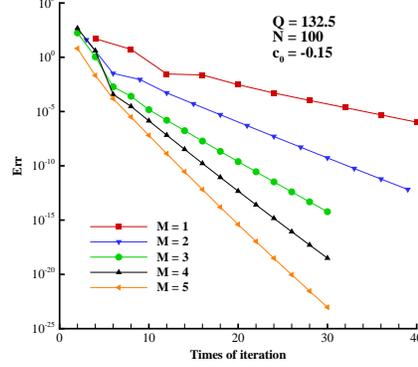


Figure 2: The sum of the squared residual errors Err versus the times of iteration in the case of $Q = 132.2$ for a circular plate with the clamped boundary, given by the HAM-based iteration approach (with different orders of iteration) using the convergence-control parameter $c_0 = -0.15$ and the truncation order $N = 100$. Square: 1st-order; Triangle down: 2nd-order; Circle: 3rd-order; Triangle up: 4th-order; Triangle left: 5th-order.

superiority of the HAM over perturbation methods. The convergent approximations of central deflection $w(0)/h$ in case of different values of Q for a circular plate with the clamped boundary are listed in Table 4. In addition, several deflection curves under different loads Q are shown in Fig. 4.

2.4. Relations between the HAM and the interpolation iterative method

Here we prove that the interpolation iterative method [1, 2] is a special case of the HAM-based 1st-order iteration approach.

Let θ denote the interpolation parameter in the interpolation iterative method [1, 2]. Set $c_1 = -\theta$ and $c_2 = -1$, since we have the great freedom to choose their values in the frame of the HAM. According to (26)-(30) and (36)-(37), we have the 1st-order homotopy-approximations of $\varphi(y)$ and $S(y)$:

$$\begin{aligned}
\varphi^*(y) &= \varphi_0(y) + \varphi_1(y) \\
&= \varphi_0(y) - \theta \delta_{1,1}(y) \\
&= (1 - \theta)\varphi_0(y) - \theta \int_0^1 K(y, \varepsilon) Q d\varepsilon \\
&- \theta \int_0^1 K(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \varphi_0(\varepsilon) S_0(\varepsilon) d\varepsilon, \tag{40}
\end{aligned}$$

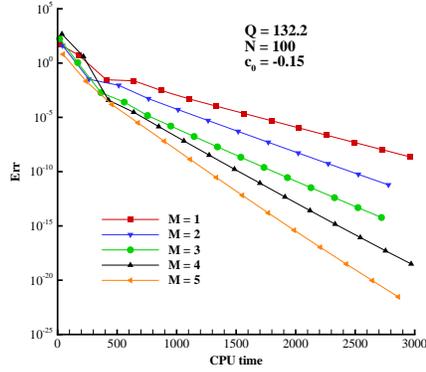


Figure 3: The sum of the squared residual errors Err versus the CPU time in the case of $Q = 132.2$ for a circular plate with the clamped boundary, given by the HAM-based iteration approach (with different orders of iteration) using the convergence-control parameter $c_0 = -0.15$ and the truncation order $N = 100$. Square: 1st-order; Triangle down: 2nd-order; Circle: 3rd-order; Triangle up: 4th-order; Triangle left: 5th-order.

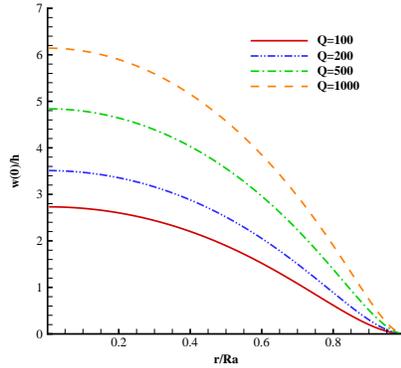


Figure 4: The deflection curves of a circular plate with the clamped boundary given by the HAM-based 5th-order iteration approach in the case of $Q = 100, 200, 500, 1000$. Solid line: $Q = 100$; Dash-double-dotted line: $Q = 200$; Dash-dotted line: $Q = 500$; Dashed line: $Q = 1000$.

Table 3: The sum of the squared residual errors Err and the homotopy-approximations of $w(0)/h$ at the m th iteration in the case of $Q = 1000$ for a circular plate with the clamped boundary, given by the HAM-based 5th-order iteration approach with the optimal convergence-control parameter $c_0 = -0.02$ and the truncation order $N = 100$.

m , times of iteration	Err	$w(0)/h$
20	2.0×10^{-1}	6.1
40	1.4×10^{-3}	6.1
60	1.4×10^{-5}	6.1
80	1.4×10^{-7}	6.1
100	1.5×10^{-9}	6.1

Table 4: The convergent homotopy-approximation of the central deflection $w(0)/h$ in case of different values of Q for a circular plate with the clamped boundary, given by the HAM-based 5th-order iteration approach using the optimal convergence-control parameter c_0 given in (39) and the truncation order $N = 100$.

Q	c_0	$w(0)/h$
200	-0.10	3.5
400	-0.05	4.5
600	-0.04	5.2
800	-0.03	5.7
1000	-0.02	6.1

$$\begin{aligned}
 S^*(y) &= S_0(y) + S_1(y) \\
 &= S_0(y) - \delta_{2,1}(y) \\
 &= \frac{1}{2} \int_0^1 G(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \varphi_0^2(\varepsilon) d\varepsilon.
 \end{aligned} \tag{41}$$

Assume that $\varphi_0(y)$ and $S_0(y)$ are known. We use the following iterative procedure:

- (A) Calculate $S^*(y)$ according to Eq. (41);
- (B) $S_0(y)$ is replaced by $S^*(y)$ that is used as the new initial guess;
- (C) Calculate $\varphi^*(y)$ according to Eq. (40);
- (D) $\varphi_0(y)$ is replaced by $\varphi^*(y)$ that is used as the new initial guess.

At the n th times of iteration, write

$$\hat{\Phi}_n(y) = \varphi^*(y), \quad \hat{\Xi}_{n-1}(y) = S^*(y).$$

Then, the procedure of the HAM-based 1st-order iteration approach is expressed by

$$\hat{\Xi}_{n-1}(y) = \frac{1}{2} \int_0^1 G(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \hat{\Phi}_{n-1}^2(\varepsilon) d\varepsilon, \quad (42)$$

$$\begin{aligned} \hat{\Phi}_n(y) &= (1 - \theta) \hat{\Phi}_{n-1}(y) - \theta \int_0^1 K(y, \varepsilon) Q d\varepsilon \\ &- \theta \int_0^1 K(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \hat{\Phi}_{n-1}(\varepsilon) \hat{\Xi}_{n-1}(\varepsilon) d\varepsilon. \end{aligned} \quad (43)$$

We choose the initial guess

$$\hat{\Phi}_0(y) = -\frac{Q\theta}{2} [(\lambda + 1)y - y^2]. \quad (44)$$

Note that, the iterative procedures (8), (10) and the initial solution (10) of the interpolation iterative method [1, 2] are exactly the same as those of the HAM-based iteration approach (42)-(44). Thus, the interpolation iterative method [1, 2] is a special case of the HAM-based 1st-order iteration approach when $c_1 = -\theta$ and $c_2 = -1$. It should be emphasized that, as shown in Figs. 2 and 3, the higher the order of the iteration, the faster the approximations converge. So, the interpolation iterative method [1, 2] should correspond to the slowest one among the HAM-based M th-order iteration approaches (up to $M = 5$). Finally, it should be emphasized that the interpolation iterative method [1, 2] is valid for an *arbitrary* uniform external pressure, as proved by Zheng and Zhou [3]. So, following Zheng and Zhou [3], one could prove that the HAM-based approach mentioned in § 2.1 is valid for an arbitrary uniform pressure at least in some special cases, such as $c_1 = -\theta$ and $c_2 = -1$. This reveals the important role of the convergence-control parameters c_1 and c_2 in the frame of the HAM.

3. HAM approach for given central deflection

According to Chien [13], it makes sense to introduce the central deflection into the Von Kármán's plate equations for a circular plate so as to enlarge the convergent region. Based on this knowledge, the HAM-based approach for given central deflection is employed to solve the Von Kármán's plate equations in the integral form with the clamped boundary.

3.1. Mathematical formulas

Given

$$W(0) = a,$$

we have due to Eq. (7) an additional equation for the corresponding unknown value of Q :

$$\int_0^1 \frac{1}{\varepsilon} \varphi(\varepsilon) d\varepsilon = -a. \quad (45)$$

Let $\varphi_0(y)$ and $S_0(y)$ denote initial guesses of $\varphi(y)$ and $S(y)$, which satisfy the restriction condition (45), c_1 and c_2 the non-zero auxiliary parameters, called the convergence-control parameters, $q \in [0, 1]$ the embedding parameter, respectively. Then, we construct the so-called zeroth-order deformation equations

$$(1 - q)[\tilde{\Phi}(y; q) - \varphi_0(y)] = c_1 q \tilde{\mathcal{N}}_1(y; q), \quad (46)$$

$$(1 - q)[\tilde{\Xi}(y; q) - S_0(y)] = c_2 q \tilde{\mathcal{N}}_2(y; q), \quad (47)$$

subject to the restricted condition

$$\int_0^1 \frac{1}{\varepsilon} \tilde{\Phi}(\varepsilon; q) d\varepsilon = -a. \quad (48)$$

where

$$\begin{aligned} \tilde{\mathcal{N}}_1(y; q) &= \tilde{\Phi}(y; q) + \int_0^1 K(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \tilde{\Phi}(\varepsilon; q) \tilde{\Xi}(\varepsilon; q) d\varepsilon \\ &+ \int_0^1 K(y, \varepsilon) \cdot \tilde{\Theta}(q) d\varepsilon, \end{aligned} \quad (49)$$

$$\tilde{\mathcal{N}}_2(y; q) = \tilde{\Xi}(y; q) - \frac{1}{2} \int_0^1 G(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \tilde{\Phi}^2(\varepsilon; q) d\varepsilon, \quad (50)$$

are the two nonlinear operators, corresponding to Eqs. (1) and (2), respectively. Note that we introduce here a continuous variation $\tilde{\Theta}(q)$ from the initial guess Q_0 to Q , since Q is unknown for a given central deflection a .

When $q = 0$, it holds

$$\tilde{\Phi}(y; 0) = \varphi_0(y), \quad \tilde{\Xi}(y; 0) = S_0(y), \quad (51)$$

since $\varphi_0(y)$ satisfies the restriction condition. When $q = 1$, they are equivalent to the original equations (1), (2) and (45), provided

$$\tilde{\Phi}(y; 1) = \varphi(y), \quad \tilde{\Xi}(y; 1) = S(y), \quad \tilde{\Theta}(1) = Q. \quad (52)$$

Then, as q increases from 0 to 1, $\tilde{\Phi}(y; q)$ varies continuously from the initial guess $\varphi_0(y)$ to the solution $\varphi(y)$, so do $\tilde{\Xi}(y; q)$ from the initial guess $S_0(y)$ to the solution $S(y)$, $\tilde{\Theta}(q)$ from the initial guess Q_0 to the unknown load Q for the given central deflection a , respectively.

Expanding $\tilde{\Phi}(y; q)$, $\tilde{\Xi}(y; q)$ and $\tilde{\Theta}(q)$ into Taylor series with respect to the embedding parameter q , we have the so-called homotopy-series:

$$\tilde{\Phi}(y; q) = \varphi_0(y) + \sum_{k=1}^{+\infty} \varphi_k(y) q^k, \quad (53)$$

$$\tilde{\Xi}(y; q) = S_0(y) + \sum_{k=1}^{+\infty} S_k(y) q^k, \quad (54)$$

$$\tilde{\Theta}(q) = Q_0 + \sum_{k=1}^{+\infty} Q_k q^k, \quad (55)$$

in which

$$\varphi_k(y) = D_k[\Phi(y; q)], \quad S_k(y) = D_k[\Xi(y; q)], \quad Q_k = D_k[\Theta(q)], \quad (56)$$

with the definition \mathcal{D}_k by (21). Assume that the homotopy-series (53)-(55) are convergent at $q = 1$. According to (52), we have the so-called homotopy-series solutions:

$$\varphi(y) = \varphi_0(y) + \sum_{k=1}^{+\infty} \varphi_k(y), \quad (57)$$

$$S(y) = S_0(y) + \sum_{k=1}^{+\infty} S_k(y), \quad (58)$$

$$Q = Q_0 + \sum_{k=1}^{+\infty} Q_k. \quad (59)$$

Substituting the power series (53)-(55) into the zeroth-order deformation equations (46)-(48), and then equating the like-power of q , we have the so-called k th-order deformation equations

$$\varphi_k(y) - \chi_k \varphi_{k-1}(y) = c_1 \delta_{1,k-1}(y), \quad (60)$$

$$S_k(y) - \chi_k S_{k-1}(y) = c_2 \delta_{2,k-1}(y), \quad (61)$$

subject to the restriction condition

$$\int_0^1 \frac{1}{\varepsilon} \varphi_k(\varepsilon) d\varepsilon = 0, \quad (62)$$

where χ_k is defined by (30), and

$$\begin{aligned} \delta_{1,k-1}(y) &= \mathcal{D}_{k-1}[\mathcal{N}_1(y; q)] \\ &= \varphi_{k-1}(y) + \sum_{i=0}^{k-1} \int_0^1 K(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \varphi_i(\varepsilon) S_{k-1-i}(\varepsilon) d\varepsilon \\ &\quad + \int_0^1 K(y, \varepsilon) \cdot Q_{k-1} d\varepsilon, \end{aligned} \quad (63)$$

$$\begin{aligned} \delta_{2,k-1}(y) &= \mathcal{D}_{k-1}[\mathcal{N}_2(y; q)] \\ &= S_{k-1}(y) - \sum_{i=0}^{k-1} \frac{1}{2} \int_0^1 G(y, \varepsilon) \cdot \frac{1}{\varepsilon^2} \cdot \varphi_i(\varepsilon) \varphi_{k-1-i}(\varepsilon) d\varepsilon, \end{aligned} \quad (64)$$

in which Q_{k-1} is determined by the restriction condition (62). Consequently, $\varphi_k(y)$ and $S_k(y)$ of Eqs. (60) and (61) are obtained. Then, we have the n th-order approximation:

$$\varphi(y) = \sum_{k=0}^n \varphi_k(y), \quad (65)$$

$$S(y) = \sum_{k=0}^n S_k(y), \quad (66)$$

$$Q = \sum_{k=0}^n Q_k. \quad (67)$$

According to [6], the initial guesses $\varphi_0(y)$ and $S_0(y)$ should obey the solution expression (11) and satisfy the boundary conditions. Thus, we choose

$$\varphi_0(y) = \frac{-2a}{2\lambda + 1} [(\lambda + 1)y - y^2], \quad (68)$$

$$S_0(y) = 0, \quad (69)$$

as the initial guesses of $\varphi(y)$ and $S(y)$, respectively.

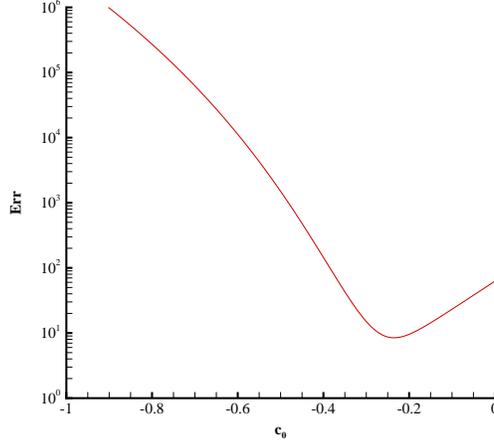


Figure 5: The sum of the squared residual errors Err versus c_0 in the case of $a = 5$ for a circular plate with the clamped boundary, gained by the HAM-based approach (without iteration).

3.2. Results given by the HAM-approach without iteration

First, the HAM-based approach (without iteration) for given central deflection is used to solve the Von Kármán's plate equations in the integral form with the clamped boundary. Without loss of generality, let us consider the same case of $a = 5$, equivalent to the central deflection $w(0)/h = 3$. As shown in Fig. 5, the sum of the squared residual errors Err arrives its minimum at $c_0 \approx -0.25$. As shown in Table 5, the sum of the squared residual errors quickly decreases to 3.6×10^{-7} by means of $c_0 \approx -0.25$ in the case of $a = 5$. Note that, the Chien's perturbation method [13] (using a as the perturbation quantity) is only valid for $w(0)/h < 2.44$, equivalent to $a < 4$, for a plate with the clamped boundary. This again illustrates the superiority of the HAM-based approach over the perturbation method.

Similarly, for given value of a , we can always find its optimal value of c_0 , which can be expressed by the empirical formula:

$$c_0 = -\frac{11}{11 + a^2} \quad (a \leq 5). \quad (70)$$

In addition, the convergent results of the external load Q in case of different values of a for a circular plate with the clamped boundary are given in

Table 5: The sum of the squared residual errors Err and the corresponding load Q versus the order of approximation in the case of $a = 5$ for a circular plate with the clamped boundary, given by the HAM without iteration using $c_0 = -0.25$.

Order of approx.	Err	Q
20	3.4×10^{-2}	132.3
40	2.2×10^{-3}	132.5
60	8.8×10^{-5}	132.3
80	9.7×10^{-7}	132.2
100	3.6×10^{-7}	132.2

Table 6: The convergent results of the external load Q in case of different values of a for a circular plate with the clamped boundary, given by the HAM-based approach without iteration using the optimal convergence-control parameter c_0 given by (70).

a	c_0	Q
1	-0.92	4.8
2	-0.73	14.6
3	-0.55	35.2
4	-0.41	72.4
5	-0.31	132.2

Table 6.

3.3. Convergence acceleration by means of iteration

As shown in § 2.3, iteration can greatly accelerate the convergence of the homotopy-series solutions. Therefore, we used the HAM-based iteration approach for given central deflection to solve the Von Kármán's plate equations with the clamped boundary in this subsection. The iteration order M and the truncation order N are defined in a similar way to Eqs. (36) and (38).

The iteration order $M = 5$ and the truncation order $N = 100$ are used in all cases described below. Without loss of generality, let us first consider the same case of $a = 5$. Note that the sum of the squared residual errors quickly decreases to 1.4×10^{-28} in only 10 iteration times, as shown in Table 7. In addition, as shown in Figs. 6 and 7, the convergence is much faster

Table 7: The sum of the squared residual Err and the load Q versus the iteration times in the case of $a = 5$ for a circular plate with the clamped boundary, given by the HAM-based 5th-order iteration approach using $c_0 = -0.5$ and the truncation order $N = 100$.

m , times of iteration.	Err	Q
2	2.0×10^{-3}	132.0
4	8.4×10^{-9}	132.2
6	1.7×10^{-16}	132.2
8	5.3×10^{-22}	132.2
10	1.4×10^{-28}	132.2

by introducing $W(0)$ into the Von Kármán's plate equations. Therefore, using the central deflection indeed makes sense. It should be emphasized that the HAM-based iteration approaches converges much faster than the interpolation iterative method [1, 2], as shown in Figs. 6 and 7.

Similarly, for any a given value of a , we can always find the optimal value of the convergence-control parameter c_0 , which can be expressed by the empirical formula:

$$c_0 = -\frac{25}{25 + a^2}. \quad (71)$$

As shown in Table 8, convergent solutions can be obtained even in the case of $a = 30$, equivalent to $w(0)/h = 18.2$. As shown in Fig. 8, as a increases, the interpolation parameter θ of the interpolation iterative method [1, 2] approaches to 0 much faster than the optimal convergence-control parameter c_0 mentioned-above. This explains why the HAM-based iteration approaches converge much faster than the interpolation iterative method [1, 2].

4. Concluding remarks

In this paper, the homotopy analysis method (HAM) is successfully applied to solve the Von Kármán's plate equations in the integral form for a circular plate with the clamped boundary under an arbitrary uniform external pressure. Two HAM-based approaches are proposed. One is for given load Q , the other for given central deflection. Both of them are valid for arbitrary uniform external pressure, by means of choosing a proper value of the so-called convergence-control parameters c_1 and c_2 in the frame of the HAM.

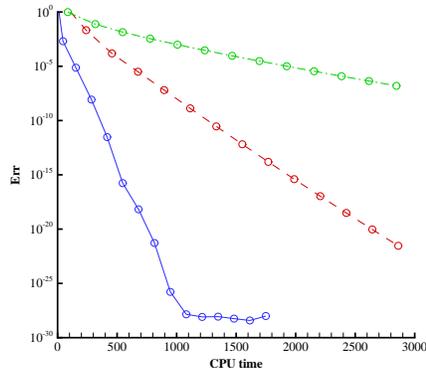


Figure 6: The sum of the squared residual errors Err versus the CPU time in the case of $a = 5$ for a circular plate with the clamped boundary, gained by the interpolation iterative method [1, 2], and the HAM-based approach for given external load Q and central deflection a , respectively. Dash-dotted line: results given by the interpolation iterative method [1, 2] using the interpolation parameter $\theta = 0.1$; Dashed line: results given by the HAM-based 5th-order iteration approach for given external load Q using the optimal convergence-control parameter $c_0 = -0.15$ and the truncation order $N = 100$; Solid line: results given by the HAM-based 5th-order iteration approach for given central deflection a using the optimal convergence-control parameter $c_0 = -0.5$ and the truncation order $N = 100$.

Besides, it is found that iteration can greatly accelerate the convergence of solution series. In addition, it is proved that the interpolation iterative method [1, 2] is a special case of the HAM-based 1st-order iteration approach for given external load Q when $c_1 = -\theta$ and $c_2 = -1$, where θ denotes the interpolation parameter of the interpolation iterative method[†]. Therefore, like Zheng and Zhou [3], one can similarly prove that the HAM-based approaches for the Von Kármán's plate equations in the integral form are valid for an arbitrary uniform external pressure, at least in some special cases such as $c_1 = -\theta$ and $c_2 = -1$. Finally, it should be emphasized that the HAM-based

[†]Note that, it was proved in Part (I) that the perturbation methods for *any* a perturbation quantity (including Vincent's [12] and Chien's [13] perturbation methods) and the modified iteration method [14] are only the special cases of the HAM-based approaches when $c_0 = -1$, for the Von Kármán's plate equations in the differential form for a circular plate under a uniform external pressure.

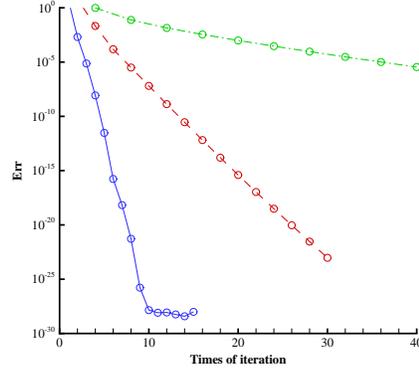


Figure 7: The sum of the squared residual errors Err versus iteration times in the case of $a = 5$ for a circular plate with the clamped boundary, gained by the interpolation iterative method [1, 2], and the HAM-based approach for given external load Q and central deflection a , respectively. Dash-dotted line: results given by the interpolation iterative method [1, 2] using the interpolation parameter $\theta = 0.1$; Dashed line: results given by the HAM-based 5th-order iteration approach for given external load Q using the optimal convergence-control parameter $c_0 = -0.15$ and the truncation order $N = 100$; Solid line: results given by the HAM-based 5th-order iteration approach for given central deflection a using the optimal convergence-control parameter $c_0 = -0.5$ and the truncation order $N = 100$.

Table 8: The convergent homotopy-approximation of Q in case of different values of a for a circular plate with the clamped boundary, given by the HAM-based 5th-order iteration approach with the optimal convergence-control parameter c_0 given by (71) and the truncation order $N = 100$.

a	c_0	Q
5	-0.50	132.2
10	-0.20	957.7
15	-0.10	3152.1
20	-0.06	7386.9
25	-0.04	14334.1
30	-0.03	24665.7

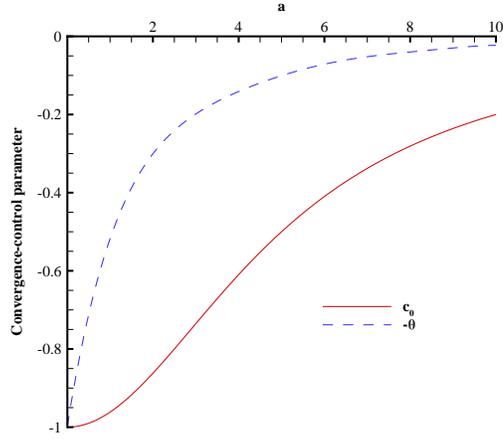


Figure 8: The convergence-control parameter c_0 and the interpolation parameter θ versus the central deflection a for a circular plate with the clamped boundary, gained by the interpolation iterative method [2] for given external load Q and the HAM-based 5th-order iteration approach for given central deflection a using the truncation order $N = 100$. Dashed line: $-\theta$; Solid line: c_0 .

iteration approaches converge much faster than the interpolation iterative method [1, 2], as shown in Figs. 6 and 7. All of these illustrate the validity and potential of the HAM for the famous Von Kármán's plate equations, and show the superiority of the HAM over perturbation methods. Without doubt, the HAM can be applied to solve some challenging nonlinear problems in solid mechanics.

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