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GENERAL BOUNDARY ELEMENT METHOD FOR NON-LINEAR PROBLEMS

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SUMMARY

In this paper the well-known non-linear equation $f''' + \frac{1}{2}ff'' = 0$ with boundary conditions $f(0) = 0, f'(0) = 0$ and $f(\infty) = 1$ is used as an example to describe the basic ideas of a kind of general boundary element method for non-linear problems whose governing equations and boundary conditions may not contain any linear terms at all.

KEY WORDS: general BEM; non-linear operators; homotopy

1. INTRODUCTION

Many researchers are interested in applying the boundary element method[1–3] (BEM) to solve non-linear boundary value problems governed by a non-linear differential operator A:

$$A(u) = g(\vec{r}) \tag{1}$$

where u is a dependent variable and g is a function of the position vector $\vec{r}$. If this non-linear operator A can be divided into two parts L_0 and N_0, where L_0 is linear, N_0 is non-linear and $A = L_0 + N_0$ holds, then, traditionally, writing the original equation (1) as $L_0(u) = g(\vec{r}) - N_0(u)$, we can obtain the following equation involving integral operators:

$$c(\vec{r})u(\vec{r}) = \int_\Gamma [uB_0(\omega_0) - \omega_0 B_0(u)]\mathrm{d}\Gamma + \int_\Omega [g(\vec{r}) - N_0(u)]\omega_0 \mathrm{d}\Omega, \tag{2}$$

where ω_0 is the fundamental solution of the adjoint operator of the linear differential operator L_0, B_0 is the corresponding boundary operator and Γ denotes the boundary of domain Ω. The parameter $c(\vec{r})$ in (2) is a geometric factor with the following values, depending on the location of $\vec{r}$:

$$c(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \in \Omega, \\ 0 & \text{if } \vec{r} \in \Omega^c \\ \dfrac{\theta}{2\pi} & \text{if } \vec{r} \in \Gamma, \end{cases}$$

where Ω^c denotes the exterior of domain Ω, excluding its boundary Γ, and θ is the angle formed between the tangents to the boundary at point $\vec{r}$, approaching it from each side. For points at which the boundary is differentiable, $\theta = \pi$.

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The basic idea of the above-mentioned traditional BEM is to move all non-linear terms of the original equation to the right-hand side of the equation and then to find the fundamental solution of the linear operator on the left-hand side. Therefore both the linear operator and the corresponding fundamental solution are very important and absolutely necessary for the traditional BEM. However, there obviously exists a special case in which nothing is left after moving all non-linear terms to the right-hand side of the equation. In this special case the traditional non-linear BEM does not work at all. Moreover, even if there exists a linear operator L_0, it may be either too simple so that not all boundary conditions can be satisfied (for example, L_0 is a first-order differential operator but the original problem is a second-order non-linear differential equation) or too complex so that it is difficult to find the corresponding fundamental solution—the traditional BEM is of no use in the former case and is very difficult to apply in the latter case. Hence the traditional BEM seems to have the following restrictions.

1. Many non-linear differential operators do *not* contain any linear terms at all, i.e. $A = L_0 + N_0$ does *not* hold, so that the traditional BEM is of no use in this case.
2. Even if the non-linear differential operator A contains such a linear operator L_0, L_0 may be either so complex that its fundamental solution is unknown or so simple that it cannot satisfy all given boundary conditions.

We can give such an example in fluid mechanics. Let us consider a well-known non-linear ordinary differential equation describing 2D laminar viscous flow over a flat plate:[4]

$$f''' + \tfrac{1}{2}ff'' = 0 \tag{3}$$

with boundary conditions

$$f = f' = 0 \quad \text{when} \quad \eta = 0, \tag{4}$$

$$f' = 1 \quad \text{when} \quad \eta \to \infty, \tag{5}$$

where $f(\eta)$ is related to the streamfunction ψ by

$$f(\eta) = \frac{\psi}{\surd(\nu U x)}$$

and η is given by

$$\eta = y\surd\left(\frac{U}{\nu x}\right).$$

Here U is the velocity at infinity, ν is the kinematic viscosity coefficient and x and y are the two independent co-ordinates.

Defining $h = f' = \mathrm{d}f/\mathrm{d}\eta$, we can transform equation (3) into

$$f\left(\frac{\mathrm{d}h}{\mathrm{d}f}\right) + 2\left(\frac{\mathrm{d}h}{\mathrm{d}f}\right)^2 + 2h\left(\frac{\mathrm{d}^2h}{\mathrm{d}f^2}\right) = 0 \tag{6}$$

with two boundary conditions

$$h = 0 \quad \text{when} \quad f = 0, \tag{7}$$

$$h = 1 \quad \text{when} \quad f \to \infty. \tag{8}$$

Defining also

$$\tau = \frac{\sqrt{f}}{1+\sqrt{f}}, \tag{9}$$

equation (6) becomes

$$\tau(1-\tau)^5\left[h\left(\frac{d^2h}{d\tau^2}\right)+\left(\frac{dh}{d\tau}\right)^2\right]+[\tau^4-(1+2\tau)(1-\tau)^4h]\frac{dh}{d\tau}=0 \tag{10}$$

with two corresponding boundary conditions

$$h = 0 \quad \text{when} \quad \tau = 0 \tag{11}$$

$$h = 1 \quad \text{when} \quad \tau = 1. \tag{12}$$

If $h(\tau)$ can be obtained by solving the non-linear second-order differential equation (10) with boundary conditions (11) and (12), the solution of the original equation (3) with boundary conditions (4) and (5) can be expressed as a function of the variable $\tau \in [0, 1]$:

$$f(\tau) = \left(\frac{\tau}{1-\tau}\right)^2, \tag{13}$$

$$f'(\tau) = h(\tau), \tag{14}$$

$$\eta(\tau) = \int_0^f \frac{df}{h(\tau)} = \int_0^\tau \frac{2t\,dt}{(1-t)^3 h(t)}. \tag{15}$$

Clearly

$$f''(\tau) = \frac{dh}{d\eta} = \frac{dh/d\tau}{d\eta/d\tau} = \frac{(1-\tau)^3 h(\tau)}{2\tau}\left(\frac{dh}{d\tau}\right), \tag{16}$$

so we have

$$f''(0) = \frac{1}{2}\left(\frac{dh}{d\tau}\right)^2\Bigg|_{\tau=0}. \tag{17}$$

However, it seems difficult to solve the non-linear equation (10) whose linear and non-linear differential operators are respectively

$$L_0(h) = \tau^4\left(\frac{dh}{d\tau}\right), \tag{18}$$

$$N_0(h) = \tau(1-\tau)^5\left[h\left(\frac{d^2h}{d\tau^2}\right)+\left(\frac{dh}{d\tau}\right)^2\right]-(1+2\tau)(1-\tau)^4h\left(\frac{dh}{d\tau}\right). \tag{19}$$

Note that L_0 is now a first-order differential operator which needs only one boundary condition. However, we have unfortunately two boundary conditions (11) and (12). Therefore the traditional BEM is certainly useless in this case. Thus it seems necessary to develop a kind of quite general BEM for non-linear problems which should give us great freedom to select a proper linear operator whose fundamental solution is familiar to us whether the linear operator L_0 exists or not.

In this paper we will use the non-linear equation (10) as an example to describe the basic ideas of a new, quite general boundary element method which can overcome the difficulties mentioned in the previous paragraph.

2. BASIC IDEAS OF THE PROPOSED BEM

We use two kinds of linear operators,

$$\text{mode 1:}\quad L_1(h) = \frac{\mathrm{d}^2 h}{\mathrm{d}\tau^2} - \beta^2 h, \quad \beta > 0,$$

$$\text{mode 2:}\quad L_2(h) = \frac{\mathrm{d}^2 h}{\mathrm{d}\tau^2} + \beta^2 h, \quad \beta > 0,$$

to construct a homotopy $\hbar(\tau, p)$: $[0, 1] \times [0, 1] \to \mathbb{R}$ which satisfies the equation

$$L_\alpha(\hbar) = (1-p)L_\alpha(h_0) + p[L_\alpha(\hbar) - \tilde{A}(\hbar)], \quad \tau \in [0, 1], \quad p \in [0, 1], \quad \alpha = \{1, 2\} \tag{20}$$

with two boundary conditions

$$\hbar(\tau, p) = 0, \quad \tau = 0, \quad p \in [0, 1], \tag{21}$$

$$\hbar(\tau, p) = 1, \quad \tau = 1, \quad p \in [0, 1], \tag{22}$$

where

$$\tilde{A}(\hbar) = \tau(1-\tau)^5\left[\hbar\left(\frac{\mathrm{d}^2\hbar}{\mathrm{d}\tau^2}\right) + \left(\frac{\mathrm{d}\hbar}{\mathrm{d}\tau}\right)^2\right] + [\tau^4 - (1+2\tau)(1-\tau)^4\hbar]\frac{\mathrm{d}\hbar}{\mathrm{d}\tau},$$

and $h_0(\tau)$ is a freely selected initial solution that satisfies (11) and (12), $p \in [0, 1]$ is an imbedding parameter and $\hbar(\tau, p)$ is a real function of both $p \in [0, 1]$ and $\tau \in [0, 1]$. We call equation (20) the *zeroth-order deformation equation*. Obviously the two expressions

$$\hbar(\tau, 0) = h_0(\tau), \tag{23}$$

$$\hbar(\tau, 1) = h(\tau) \tag{24}$$

hold, where $h(\tau)$ is the solution of (10)–(12). This means that $h_0(\tau)$ and $h(\tau)$ are homotopic.

Assume that the 'continuous deformation' $\hbar(\tau, p)$ is smooth enough about p so that the *mth-order deformation derivatives*

$$\hbar^{[m]}(\tau, p) = \frac{\partial^m \hbar(\tau, p)}{\partial p^m}, \quad m = 1, 2, 3, \ldots, \tag{25}$$

exist. Then, according to the Taylor formula, we have from (23) that

$$\hbar(\tau, p) = \hbar(\tau, 0) + \sum_{m=1}^{\infty} \left.\frac{\partial^m \hbar(\tau, p)}{\partial p^m}\right|_{p=0} \left(\frac{p^m}{m!}\right) = h_0(\tau) + \sum_{m=1}^{\infty} h_0^{[m]}(\tau)\left(\frac{p^m}{m!}\right), \tag{26}$$

where $h_0^{[m]}(\tau)$ is the value of $\hbar^{[m]}(\tau, p)$ at $p = 0$, which can be obtained in the way described later. For simplicity we call the above expression the *Taylor homotopy series*. The value of the convergence radius ρ of the above series is generally finite. Thus in the case $\rho \leqslant 1$ we have

$$\hbar(\tau, \lambda) = h_0(\tau) + \sum_{m=1}^{\infty} h_0^{[m]}(\tau)\left(\frac{\lambda^m}{m!}\right), \tag{27}$$

where $0 < \lambda \leqslant \rho \leqslant 1$. Note that the above-obtained $\hbar(\tau, \lambda)$ is mostly a better approximation than the initial solution $h_0(\tau)$, so that expression (27) gives in fact a family of high-order iterative formulae

$$h_{k+1}(\tau) = h_k(\tau) + \sum_{m=1}^{M} h_k^{[m]}(\tau)\left(\frac{\lambda^m}{m!}\right), \quad k = 0, 1, 2, 3, \ldots, \tag{28}$$

where M $(M = 1, 2, 3, \ldots)$ denotes the order of the iterative formula.

Differentiating the zeroth-order deformation equation (20) m times with respect to the imbedding parameter p and then setting $p=0$, we obtain the *mth-order deformation equations* at $p=0$ as

$$L_\alpha(h_k^{[m]}) = g_m(\tau), \quad \tau \in [0,1], \quad m \geq 1, \quad \alpha = \{1,2\}, \quad k = 0,1,2,\ldots, \tag{29}$$

with two corresponding boundary conditions

$$h_k^{[m]}(0) = 0, \quad m \geq 1, \quad k = 0,1,2,3,\ldots, \tag{30}$$

$$h_k^{[m]}(1) = 0, \quad m \geq 1, \quad k = 0,1,2,3,\ldots, \tag{31}$$

where

$$g_1(\tau) = -\tilde{A}[h_k(\tau)], \quad \tau \in [0,1], \quad k = 0,1,2,\ldots, \tag{32}$$

$$g_m(\tau) = m\left(L_\alpha[h_k^{[m-1]}(\tau)] - \left.\frac{\mathrm{d}^{m-1}A[\hbar(\tau,p)]}{\mathrm{d}p^{m-1}}\right|_{p=0}\right),$$
$$\tau \in [0,1], \quad m > 1, \quad \alpha = \{1,2\}, \quad k = 0,1,2,\ldots. \tag{33}$$

Note that $g_1(\tau)$ is the minus residual of the original equation (10) and is the same for *any* proper linear operators L_0. Substituting the definition of $\tilde{A}$ into (33), we have

$$g_2(\tau) = 2\left\{L_\alpha(h_k^{[1]}) - \tau^4\left(\frac{\mathrm{d}h_k^{[1]}}{\mathrm{d}\tau}\right) + (1+2\tau)(1-\tau)^4\left[h_k^{[1]}\left(\frac{\mathrm{d}h_k}{\mathrm{d}\tau}\right) + h_k\left(\frac{\mathrm{d}h_k^{[1]}}{\mathrm{d}\tau}\right)\right]\right.$$
$$\left. - \tau(1-\tau)^5\left[h_k^{[1]}\left(\frac{\mathrm{d}^2h_k}{\mathrm{d}\tau^2}\right) + h_k\left(\frac{\mathrm{d}^2h_k^{[1]}}{\mathrm{d}\tau^2}\right) + 2\left(\frac{\mathrm{d}h_k}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}h_k^{[1]}}{\mathrm{d}\tau}\right)\right]\right\}, \tag{34}$$

$$g_3(\tau) = 3\left\{L_\alpha(h_k^{[2]}) - \tau^4\left(\frac{\mathrm{d}h_k^{[2]}}{\mathrm{d}\tau}\right) + (1+2\tau)(1-\tau)^4\left[h_k^{[2]}\left(\frac{\mathrm{d}h_k}{\mathrm{d}\tau}\right) + 2h_k^{[1]}\left(\frac{\mathrm{d}h_k^{[1]}}{\mathrm{d}\tau}\right) + h_k\left(\frac{\mathrm{d}h_k^{[2]}}{\mathrm{d}\tau}\right)\right]\right.$$
$$\left. - \tau(1-\tau)^5\left[h_k^{[2]}\left(\frac{\mathrm{d}^2h_k}{\mathrm{d}\tau^2}\right) + h_k\left(\frac{\mathrm{d}^2h_k^{[2]}}{\mathrm{d}\tau^2}\right) + 2\left(\frac{\mathrm{d}h_k}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}h_k^{[2]}}{\mathrm{d}\tau}\right) + 2h_k^{[1]}\left(\frac{\mathrm{d}^2h_k^{[1]}}{\mathrm{d}\tau^2}\right) + 2\left(\frac{\mathrm{d}h_k^{[1]}}{\mathrm{d}\tau}\right)^2\right]\right\}. \tag{35}$$

Hence, according to (2), we obtain the corresponding boundary integral equation

$$h_k^{[m]}(\tau) = C_m\omega_\alpha(\tau,0) - D_m\omega_\alpha(\tau,1) + \int_0^1 \omega_\alpha(\tau,t)g_m(t)\mathrm{d}t, \quad \tau \in [0,1], \quad m \geq 1,$$
$$\alpha = \{1,2\}, \quad k = 0,1,2,\ldots, \tag{36}$$

where

$$\omega_1(\tau,t) = -\frac{\exp(-\beta|\tau-t|)}{2\beta}, \tag{37}$$

$$\omega_2(\tau,t) = \frac{\sin(\beta|\tau-t|)}{2\beta}, \tag{38}$$

are fundamental solutions of mode 1 and mode 2 respectively. The two unknown coefficients C_m and D_m are determined by two boundary conditions $h_k^{[m]}(0) = 0$ and $h_k^{[m]}(1) = 0$, which means that a set of two linear algebraic equations

$$\begin{bmatrix} \omega_\alpha(0,0) & \omega_\alpha(0,1) \\ \omega_\alpha(1,0) & \omega_\alpha(1,1) \end{bmatrix} \begin{bmatrix} C_m \\ D_m \end{bmatrix} = -\begin{bmatrix} \int_0^1 \omega_\alpha(0,t) g_m(t) \mathrm{d}t \\ \int_0^1 \omega_\alpha(1,t) g_m(t) \mathrm{d}t \end{bmatrix}, \quad \alpha = \{1,2\}, \quad m \geqslant 1, \tag{39}$$

must be solved. Certainly a proper value of β should be selected so that

$$E_\alpha(\beta) = \omega_\alpha(0,0)\omega_\alpha(1,1) - \omega_\alpha(0,1)\omega_\alpha(1,0) \neq 0, \quad a = \{1,2\}, \tag{40}$$

holds in order to avoid the singularity of the corresponding matrix.

3. NUMERICAL RESULTS

We select $h_0(\tau) = \tau$ as the initial approximation for all subsequent discussion. For the domain integral we divide the domain [0, 1] into N equal subdomains, i.e. $\tau_k = k/N$. Second-order accurate central difference approximations are applied to the first- and second-order derivatives appearing in the expressions of $g_m(\tau)(m = 1, 2, 3, \ldots)$. For the computation of $\mathrm{d}h/\mathrm{d}\tau$ at $\tau = 0$ we use respectively the second-order finite difference approximation

$$\left.\frac{\mathrm{d}h}{\mathrm{d}\tau}\right|_{\tau=0} = \frac{4h(\tau_1) - h(\tau_2)}{2\Delta\tau} \tag{41}$$

and the third-order finite difference approximation

$$\left.\frac{\mathrm{d}h}{\mathrm{d}\tau}\right|_{\tau=0} = \frac{48h(\tau_1) - 36h(\tau_2) + 16h(\tau_3) - 3h(\tau_4)}{12\Delta\tau}. \tag{42}$$

In this paper the root-mean-square error is defined as

$$RMS = \sqrt{\left(\frac{\sum_{k=0}^{N}\{\tilde{A}[h(\tau_k)]\}^2}{N+1}\right)}. \tag{43}$$

3.1. *Selection of numerical parameters*

The numerical parameters that we need to select are M ($M \geqslant 1$), which denotes the order of iterative formula (28), α ($\alpha = 1$ or 2), which is the subscript of the two linear operators L_1 and L_2, λ ($\lambda > 0$), which acts as a kind of iterative factor, and β ($\beta > 0$), which is a parameter of the linear operators L_1 and L_2. In this subsection $N = 100$ is used.

At first we simply select $\beta = 1$, $\lambda = 0{\cdot}75$ and observe the corresponding iteration procedures under various orders of iterative formula (28) ($M = 1$, 2, 3). The histories of errors during the iterative process using different formulae ($M = 1$, 2, 3) for two different linear operators ($\alpha = 1$ or 2) are shown in Figures 1 and 2 respectively. We note from Figures 1 and 2 that higher-order iterative formulae accelerate the iteration. In fact, this is true for all other values of $\lambda \leqslant 1$ for both cases $\alpha = 1$ and 2. This is mainly because the higher-order formulae, which are based on the Taylor series, can give a more accurate approximation than the lower-order ones if λ is less than the convergence radius ρ.

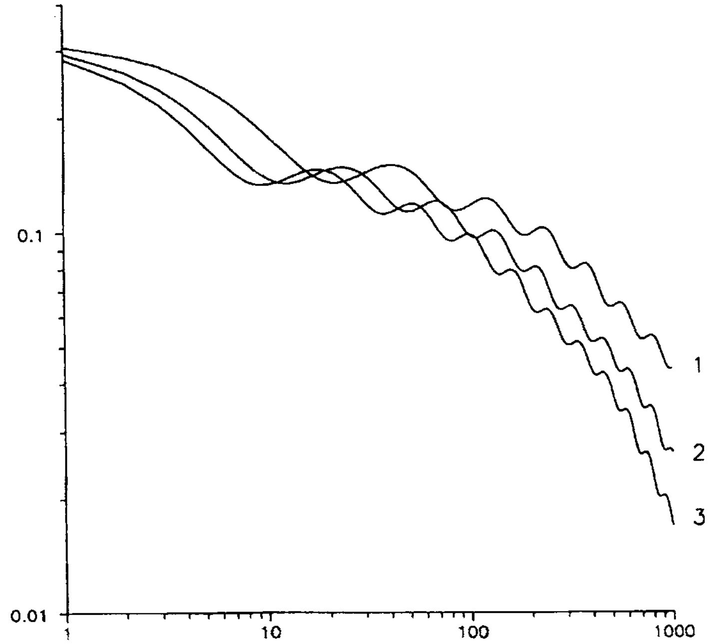

Figure 1. History of errors during iterative process for different order formulae ($\alpha = 1$): horizontal axis, iterative times; vertical axis, *RMS*; numerical parameters, $\beta = 1$, $\lambda = 0{\cdot}75$, $N = 100$; curve 1, first-order formula; 2, second-order formula; 3, third-order formula

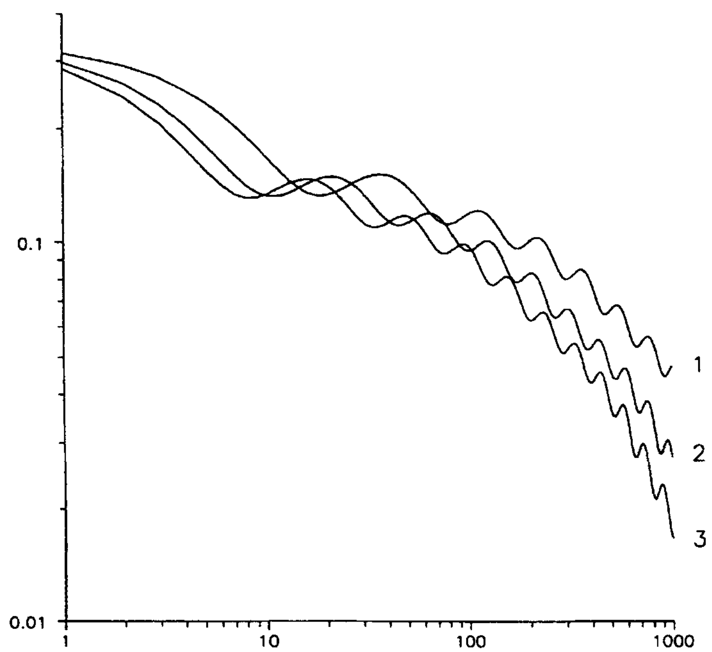

Figure 2. History of errors during iterative process for different order formulae ($\alpha = 2$): horizontal axis, iterative times; vertical axis, *RMS*; numerical parameters, $\beta = 1$, $\lambda = 0{\cdot}75$, $N = 100$; curve 1, first-order formula; 2, second-order formula; 3, third-order formula

Secondly, in the case $\beta=1$, $M=3$ we observe the iteration procedure under various values of λ ($\lambda=0{\cdot}25$, $0{\cdot}5$, $0{\cdot}75$, 1) for both $\alpha=1$ and 2. The corresponding histories of errors during the iterative process are shown in Figures 3 and 4 respectively, which obviously indicate that larger values of λ ($0 \leqslant \lambda \leqslant 1$) accelerate the iteration. This can be easily understood, because $\hbar(\tau, p)$ at $p=1$ is equal to $h(\tau)$. It implies that the convergence radius ρ of the corresponding *Taylor homotopy series* (26) must be equal to or greater than unity. According to our experience, this seems to be the case.

According to (40), we have

$$E_1(\beta)=\frac{1-\exp(-2\beta)}{4\beta^2}\neq 0, \tag{44}$$

$$E_2(\beta)=\frac{\sin^2(\beta)}{4\beta^2}\neq 0, \tag{45}$$

from which we obtain $\beta\neq 0$ for the first linear operator ($\alpha=1$) and $\beta\neq k\pi$ ($k=0,1,2,3,\ldots$) for the second one ($\alpha=2$). However, they are only necessary conditions for the selection of β.

For the first linear operator ($\alpha=1$) we select $\lambda=0{\cdot}75$, $M=3$ and inspect the histories of errors of the iterative process for various values of β ($\beta=0{\cdot}01$, 1, 3, 5, 10, 25, 50, 100) as shown in Figure 5. It is interesting to note from Figure 5 that for a large range of β ($10^{-2}\leqslant\beta\leqslant 10^2$) the corresponding iterations converge and the proposed BEM is valid, although the iteration for $\beta=100$ converges very slowly. For the second linear operator ($\alpha=2$) the range of β making the corresponding iteration convergent is $0<\beta<\pi$, which is much smaller than that for the first linear operator ($\alpha=1$), as shown in Figure 6. We note from Figure 6 that for all values of $\beta>\pi$ except $\beta\neq k\pi$, such as $\beta=3{\cdot}2$, 4, 25, 50, the corresponding iterations diverge.

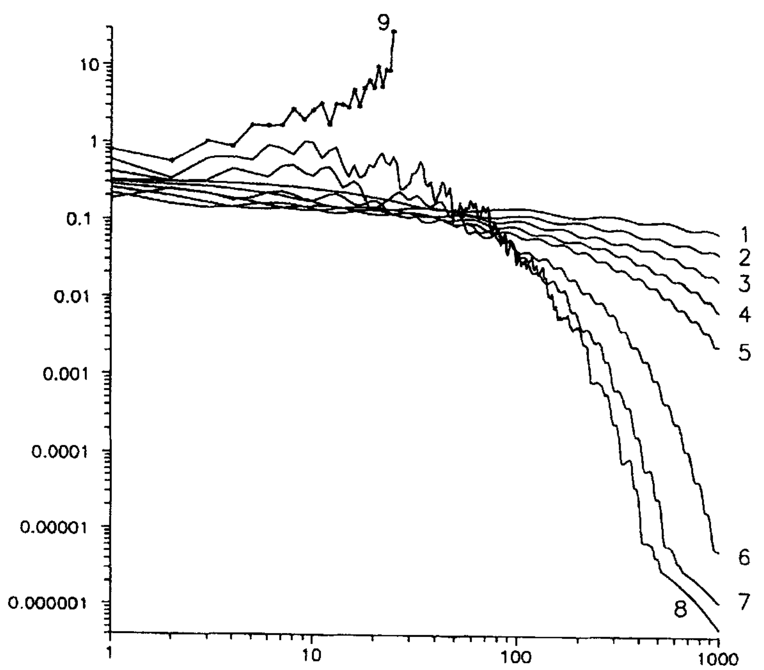

Figure 3. History of errors during iterative process for different values of λ ($\alpha=1$): horizontal axis, iterative times; vertical axis, *RMS*; numerical parameters, $\beta=1$, $M=3$, $N=100$; curve 1, $\lambda=0{\cdot}25$; 2, $\lambda=0{\cdot}5$; 3, $\lambda=0{\cdot}75$; 4, $\lambda=1$; 5, $\lambda=1{\cdot}25$; 6, $\lambda=2$; 7, $\lambda=2{\cdot}5$; 8, $\lambda=2{\cdot}75$; 9, $\lambda=3$

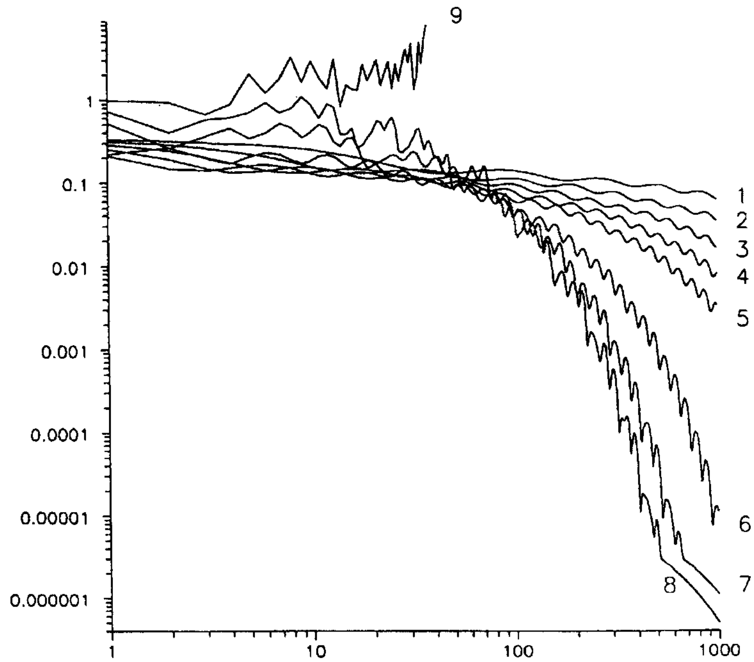

Figure 4. History of errors during iterative process for different values of λ ($\alpha=2$): horizontal axis, iterative times; vertical axis, *RMS*; numerical parameters, $\beta=1$, $M=3$, $N=100$; curve 1, $\lambda=0{\cdot}25$; 2, $\lambda=0{\cdot}5$; 3, $\lambda=0{\cdot}75$; 4, $\lambda=1$; 5, $\lambda=1{\cdot}25$; 6, $\lambda=2$; 7, $\lambda=2{\cdot}5$; 8, $\lambda=2{\cdot}75$; 9, $\lambda=3$

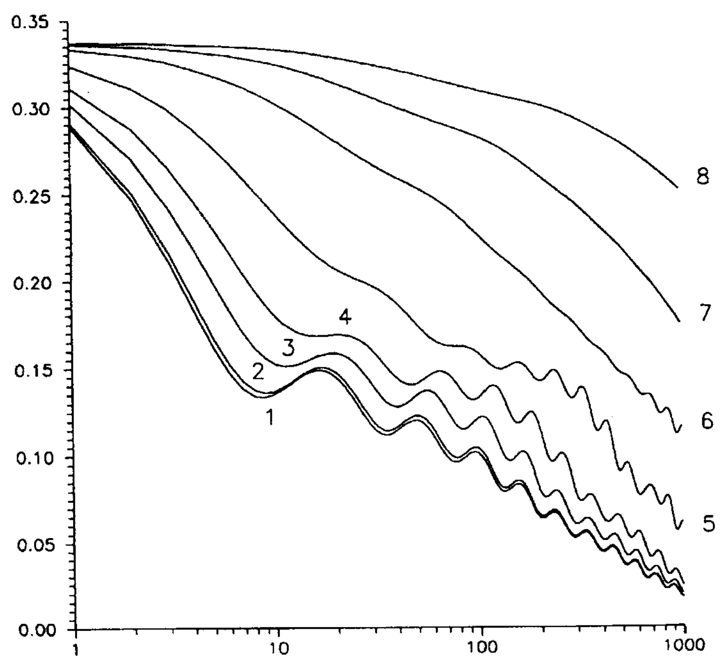

Figure 5. History of errors during iterative process for different linear operators ($\alpha=1$): horizontal axis, iterative times; vertical axis, *RMS*; numerical parameters, $\lambda=0{\cdot}75$, $M=3$, $N=100$; curve 1, $\beta=0{\cdot}1$; 2, $\beta=1$; 3, $\beta=3$; 4, $\beta=5$; 5, $\beta=10$; 6, $\beta=25$; 7, $\beta=50$; 8, $\beta=100$

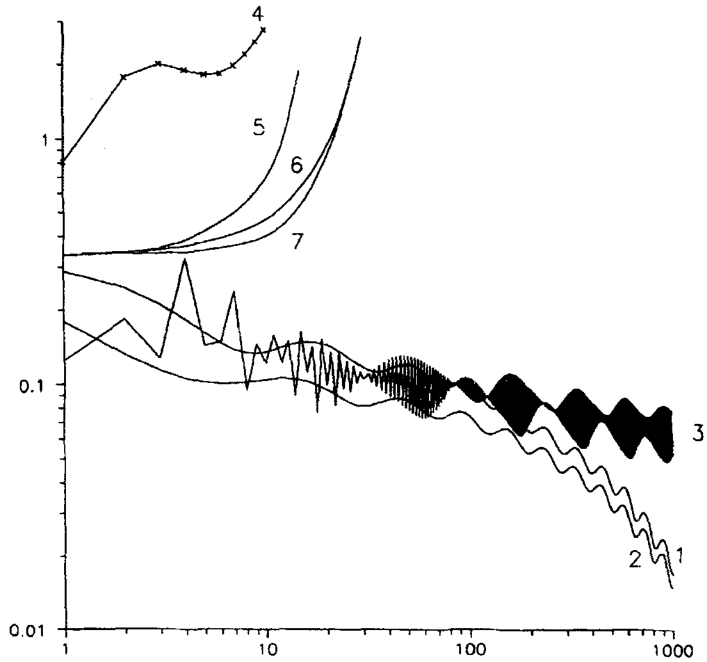

Figure 6. History of errors during iterative process for different linear operators ($\alpha = 2$): horizontal axis, iterative times; vertical axis, *RMS*; numerical parameters, $\lambda = 0{\cdot}75$, $M = 3$, $N = 100$; curve 1, $\beta = 0{\cdot}01$; 2, $\beta = 3$; 3, $\beta = 3{\cdot}1$; 4, $\beta = 3{\cdot}2$; 5, $\beta = 4$; 6, $\beta = 25$; 7, $\beta = 50$

According to the above test computations, many values of β, i.e. $10^{-2} \leqslant \beta \leqslant 10^2$ for the first linear operator L_1 and $0 < \beta < \pi$ for the second linear operator L_2, can make the proposed BEM valid. Note that L_1 and L_2 are the simplest second-order linear differential operators that have nearly no relation with the non-linear equation (10). In Section 1 we pointed out that the linear operator L_0 has special meaning for the traditional BEM—it is very important and absolutely necessary. Moreover, the traditional BEM is invalid for the considered problem, because equation (10) only contains a linear operator $L_0(h) = \tau^4(\partial h/\partial \tau)$ which certainly cannot satisfy two boundary conditions. However, for the proposed new BEM the linear operator is *not* so important because we have much greater freedom to select a familiar linear operator! This kind of freedom is especially important for applying the proposed BEM to solve those non-linear problems whose governing equations and boundary conditions do not even contain any linear terms at all. For the proposed general BEM the linear operator L_0 is not special at all—it is only a common one among many proper linear operators.

3.2. *A new kind of iteration procedure*

We know that for the two linear operators L_1 and L_2, large values of λ in the region $0 < \lambda \leqslant \rho \leqslant 1$ can accelerate the convergence process. However, what are the iteration procedures in the case $\lambda > 1$?

In Figures 3 and 4 we show also the histories of errors during the iterative process for $\lambda = 1{\cdot}25$, 2, 2·5, 2·75, 3 for $\beta = 1$, $M = 3$. It is interesting to note that for both linear operators L_1 and L_2 a large value of λ, especially $\lambda = 2{\cdot}5$ and 2·75, accelerates greatly the convergence process, except for $\lambda = 3$ under which the iteration diverges. The histories of errors during the iterative process at $\lambda = 2{\cdot}5$ ($\beta = 1$) for various orders of formula ($M = 1$, 2, 3) are shown in Figures 7 and 8 for $\alpha = 1$ and 2 respectively. For $\lambda = 2{\cdot}5$ we obtain a similar result to that for $\lambda = 0{\cdot}75$, i.e. a higher-order iterative

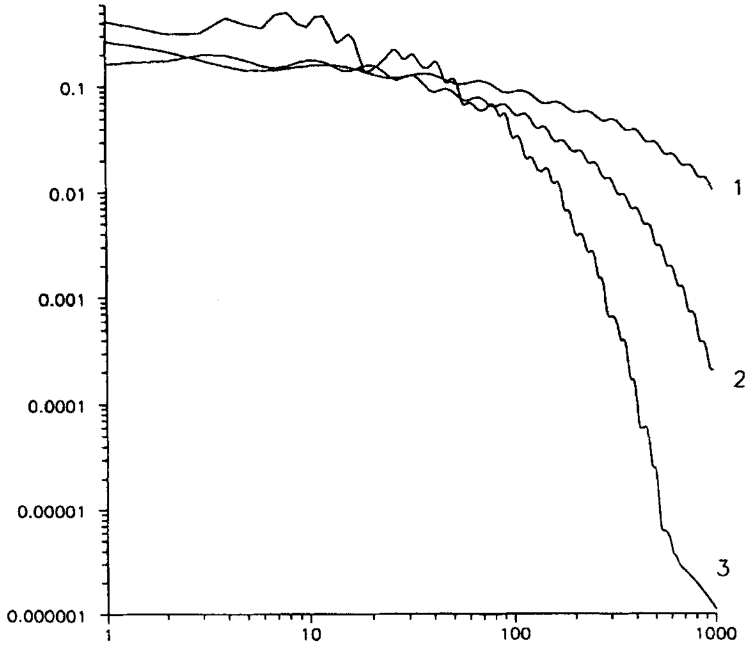

Figure 7. History of errors during iterative process for different order formulae ($\alpha = 1$): horizontal axis, iterative times; vertical axis, *RMS*; numerical parameters, $\beta = 1$, $\lambda = 2{\cdot}5$, $N = 100$; curve 1, first-order formula; 2, second-order formula; 3, third-order formula

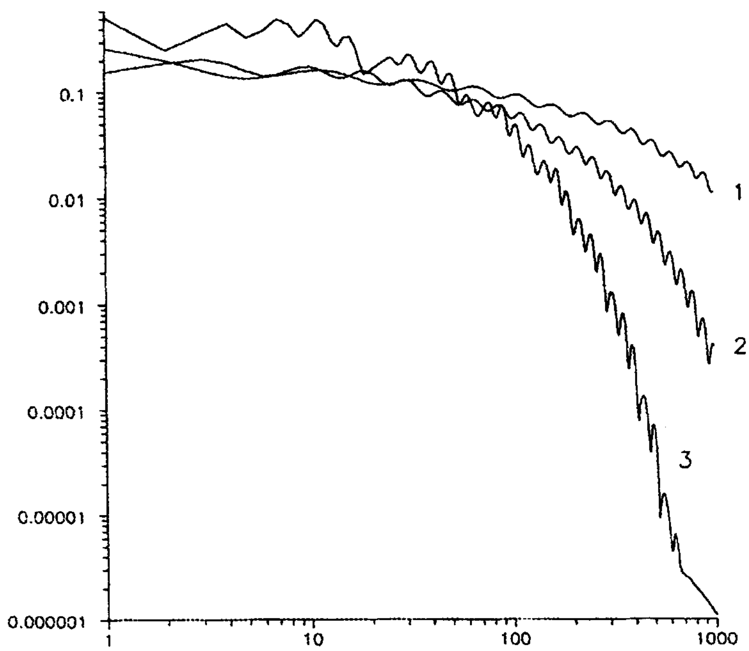

Figure 8. History of errors during iterative process for different order formulae ($\alpha = 2$): horizontal axis, iterative times; vertical axis, *RMS*; numerical parameters, $\beta = 1$, $\lambda = 2{\cdot}5$, $N = 100$; curve 1, first-order formula; 2, second-order formula; 3, third-order formula

formula accelerates convergence. However, this is a little strange. We only know the relation $h(\tau) = \hbar(\tau, 1)$ but do not know anything about $\hbar(\tau, p)$ for $p > 1$ except its continuity. Therefore, in the case $\lambda > 1$, λ has to be considered as a kind of iterative parameter similar to the relaxation factor in the successive overrelaxation (SOR) method. However, λ seems to be different from the classical relaxation factor in the SOR method. The corresponding iteration procedure seems to be different from the traditional iteration procedures such as Seidel iteration, Gauss–Seidel iteration and so on. We call this new kind of iteration the *Taylor homotopy iteration.*

The divergence at $\lambda = 3$, as shown in Figures 3 and 4, means that there should exist a region of λ in which the iteration converges and a best value of λ exists for the fastest convergence. It is interesting to note that this range of λ seems to be the same for the two different linear operators L_1 and L_2 as given in the examples, but we do not know why. Perhaps it is only a coincidence.

3.3. *Numerical results*

In this subsection we use

$$RMS = \sqrt{\left(\frac{\sum_{k=0}^{N} \{\tilde{A}[h(\tau_k)]\}^2}{N+1} \right)} < 10^{-10} \tag{46}$$

as the convergence criterion. For various values of N ($N = 200$, 400, 750, 1000, 1250) we use the third-order formula ($M = 3$, $\lambda = 2{\cdot}5$, $\beta = 1$, $\alpha = 1$) to obtain the convergence results given in Tables I–IV. Obviously, $h(\tau_k)$ tends to a fixed value for any fixed value of τ_k ($k = 0, 1, 2, 3, \ldots, N$) as the value of N increases, as shown in Table I. As a result, $f(\eta)$ and $f'(\eta)$ also tend to their corresponding fixed values as the value of N increases, as shown in Table II and Table III respectively. These results agree very well with those given by Howarth,[5] as shown in Figures 9 and 10 and Tables II and III. The value of $f''(0)$ given by Howarth[5] is 0·33206. We obtain the same value $f''(0) = 0{\cdot}33206$ when large enough N is used, as shown in Table IV. It is interesting to note that the numerical result of equation (3) with boundary conditions $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 0{\cdot}33206$ given by the Runge–Kutta (RK) method ($\Delta\eta = 10^{-4}$) is even closer to our results, as shown in Tables II and III. It seems that the present results are better than those given by Howarth.[5] Note that

$$\lim_{t \to 0} \frac{2t}{(1-t)^3 h(t)} = \frac{2}{h'(0)} = \sqrt{\left(\frac{2}{f''(0)}\right)} \neq 0; \tag{47}$$

Table I. Numerical results of $h(\tau) = f'(\eta)$

τ	$N = 200$	$N = 400$	$N = 750$	$N = 1000$	$N = 1250$
0·0	0·00000	0·00000	0·00000	0·00000	0·00000
0·1	0·09054	0·09054	0·09054	0·09054	0·09054
0·2	0·20347	0·20347	0·20347	0·20347	0·20347
0·3	0·34702	0·34702	0·34702	0·34702	0·34702
0·4	0·53036	0·53036	0·53036	0·53036	0·53036
0·5	0·75233	0·75233	0·75233	0·75233	0·75233
0·6	0·95169	0·95165	0·95164	0·95164	0·95164
0·7	0·99996	0·99995	0·99995	0·99995	0·99995
0·8	1·00000	1·00000	1·00000	1·00000	1·00000
0·9	1·00000	1·00000	1·00000	1·00000	1·00000
1·0	1·00000	1·00000	1·00000	1·00000	1·00000

Table II. Comparisons of $f(\eta)$ with results given by Howarth[5]

η	$f(\eta)$ given by Howarth	$f(\eta)$ $N=200$	$N=400$	$N=750$	$N=1000$	$N=1250$	$f(\eta)$ given by RK method
0·0	0·0	0·00000	0·00000	0·00000	0·00000	0·00000	0·00000
0·4	0·02657	0·02656	0·02656	0·02656	0·02656	0·02656	0·02656
0·8	0·10619	0·10612	0·10611	0·10611	0·10611	0·10611	0·10611
1·2	0·23804	0·23798	0·23796	0·23795	0·23795	0·23795	0·23795
1·6	0·42048	0·42036	0·42033	0·42032	0·42032	0·42032	0·42032
2·0	0·65024	0·65001	0·65004	0·65003	0·65003	0·65003	0·65003
2·4	0·92231	0·92230	0·92229	0·92229	0·92229	0·92229	0·92230
2·8	1·23133	1·23104	1·23100	1·23098	1·23098	1·23098	1·23099
3·2	1·56951	1·56917	1·56912	1·56910	1·56910	1·56910	1·56911
3·6	1·92984	1·92963	1·92956	1·92953	1·92953	1·92953	1·92954
4·0	2·30624	2·30588	2·30577	2·30575	2·30575	2·30575	2·30576
4·4	2·69282	2·69249	2·69240	2·69237	2·69237	2·69237	2·69238
5·0	3·28394	3·28342	3·28331	3·28329	3·28328	3·28328	3·28330
6·0	4·28037	4·27981	4·26967	4·27963	4·27963	4·27963	4·27965
7·0	5·28003	5·27944	5·27929	5·27925	5·27924	5·27924	5·27927
8·0	6·28001	6·27941	6·27926	6·27923	6·27922	6·27922	6·27925

Table III. Comparisons of $f'(\eta)$ with the the results given by Howarth[5]

η	$f'(\eta)$ given by Howarth	$f'(\eta)$ $N=200$	$N=400$	$N=750$	$N=1000$	$N=1250$	$f'(\eta)$ given by RK method
0·0	0·00000	0·00000	0·00000	0·00000	0·00000	0·00000	0·00000
0·4	0·13277	0·13277	0·13276	0·13276	0·13276	0·13276	0·13277
0·8	0·26471	0·26471	0·26471	0·26471	0·26471	0·26471	0·26471
1·2	0·39377	0·39377	0·39378	0·39378	0·39378	0·39378	0·39378
1·6	0·51672	0·51675	0·51676	0·51676	0·51676	0·51676	0·51676
2·0	0·62970	0·62975	0·62976	0·62977	0·62977	0·62977	0·62977
2·4	0·72900	0·72899	0·72898	0·72898	0·72898	0·72898	0·72899
2·8	0·81138	0·81150	0·81150	0·81151	0·81151	0·81151	0·81152
3·2	0·87594	0·87607	0·87607	0·87608	0·87608	0·87608	0·87609
3·6	0·92336	0·92331	0·92332	0·92333	0·92333	0·92333	0·92334
4·0	0·95547	0·95549	0·95552	0·95552	0·95552	0·95552	0·95552
4·4	0·97600	0·97587	0·97587	0·98587	0·97587	0·97587	0·97588
5·0	0·99154	0·99158	0·99155	0·99154	0·99154	0·99154	0·99155
6·0	0·99902	0·99899	0·99898	0·99897	0·99897	0·99897	0·99898
7·0	0·99994	0·99993	0·99992	0·99992	0·99992	0·99992	0·99993
8·0	0·99999	0·99997	0·99997	1·00000	1·00000	1·00000	1·00000

Table IV. Numerical results of $f''(0)$

Order	$N=200$	$N=400$	$N=750$	$N=1000$	$N=1250$
Second	0·332064	0·332061	0·332060	0·332060	0·332060
Third	0·332101	0·332071	0·332064	0·332064	0·332064

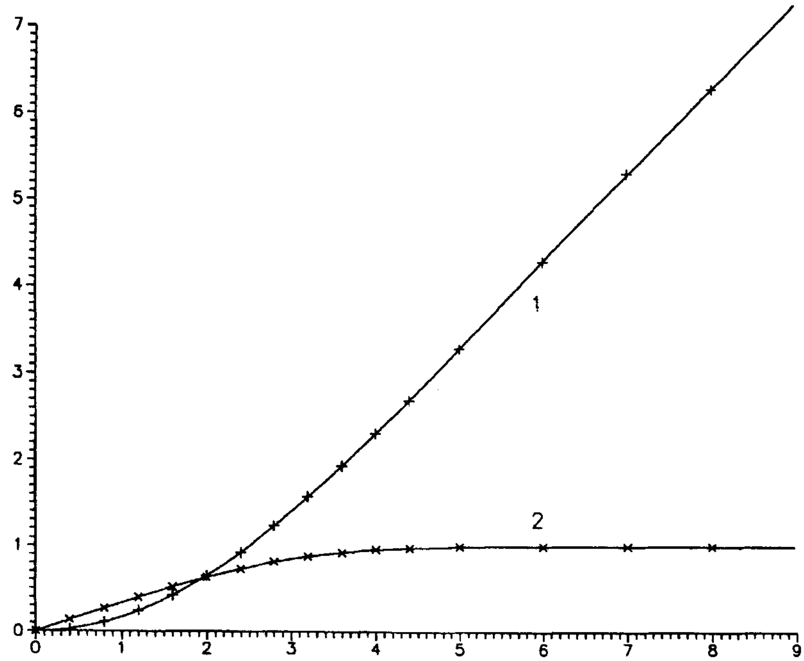

Figure 9. Comparison of numerical results with those given by Howarth:[5] horizontal axis, η; vertical axis, f or f'; curve 1, $f(\eta)$; 2, $f'(\eta)$; $\times$, results given by Howarth; —, present results

therefore the integral (15) is not singular at $t = 0$, so we can simply use Simpson's method for the numerical integral.

Besides the traditional BEM mentioned earlier in this paper, there exist a few other methods, such as the one described in Reference 6, in which a simple non-linear equation $u_{xx} = \alpha u^2$ ($x \in [0, 1]$) with two boundary conditions $u(0) = 1$ and $u(1) = 0{\cdot}25$ is used as an example to show the basic ideas of the method. The domain $x \in [0, 1]$ is first divided into N equal subdomains and then the non-linear equation is linearized in each subdomain. As a result, a set of $2N$ algebraic equations has to be solved,

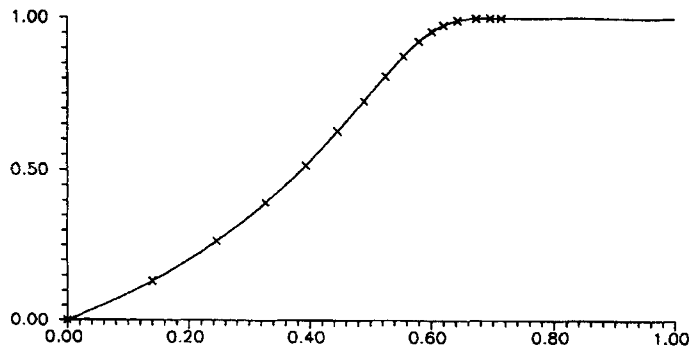

Figure 10. Comparison of numerical results with those given by Howarth:[5] horizontal axis, $\tau = \sqrt{f}/(1 + \sqrt{f})$; vertical axis, $f'(\tau)$; $\times$, results given by Howarth; —, present results

which requires much CPU storage if N is large, say $N = 1000$ as used in this paper. However, the proposed general BEM needs to solve only a set of *two* linear algebraic equations for the same problem. Therefore less CPU storage is needed for each iteration.

The proposed BEM can also be applied to solve more realistic non-linear problems and the higher-order formulae can increase the stability and convergence of the computational scheme. For instance, the basic ideas of the proposed BEM have been successfully used to solve the 2D Navier–Stokes equations and it has been found that the corresponding second-order formulae are still stable even for Reynolds numbers up to 10,000, as described in References 7 and 8. It should be noted that Navier–Stokes equations contain proper linear operators. In this paper we generalize the basic ideas described in References 7 and 8 and consider the cases where no proper linear operators exist.

4. DISCUSSION AND CONCLUSIONS

In this paper a well-known non-linear problem in fluid mechanics is used as an example to illustrate the basic ideas of a kind of general BEM for non-linear differential problems. This new BEM is based on the homotopy technique of topology and therefore has a good mathematical base. The proposed BEM can give us great freedom to select a proper linear operator L, whose fundamental solution is familiar to us, to construct a family of high-order iterative formulae. This kind of freedom is very important, especially for non-linear differential equations which do not contain any linear terms at all, or whose linear operator is so complex that its corresponding fundamental solution is either unknown or difficult to obtain, or whose linear operator is useless because it cannot satisfy all the boundary conditions. It should be emphasized that the non-linear operator $\tilde{A}$ in the zeroth-order deformation equation (20) may be quite general and may even not contain any linear operators. Therefore the proposed BEM can be applied to solve quite general non-linear problems, even including those whose governing equations and boundary conditions do not contain any linear terms at all.

Moreover, the simple example given in the present paper illustrates that a large value of λ ($\lambda < \rho$, the convergence radius) and a high-order iterative formula may accelerate the convergence of iteration. This can be easily understood, because the proposed BEM is based on the Taylor series and the homotopy technique which give the relation $h(\tau) = \hbar(\tau, 1)$, so that a high-order formula with large λ ($\lambda \leqslant \rho \leqslant 1$) gives accurate approximations at every iteration. It should be emphasized that for the given example the iteration can also be accelerated greatly by using large values of λ ($\lambda > 1$), which in fact gives a kind of new iteration procedure that seems to be different from the traditional iteration procedure such as Seidel iteration, Gauss–Seidel iteration, the Gauss–Seidel iteration and so on.

It is true that linear and non-linear differential equations are quite different. However, they can be linked closely by constructing a homotopy similar to equation (20). The homotopy method emphasizes relationships and continuous transformation between quite different things. Note that the mth-order deformation equation (29) with boundary conditions (30) and (31) is linear for any $m \geqslant 1$. *This is a very important property of a homotopy by which we can transfer any non-linear problems into an infinite number of linear problems*, as mentioned in other papers.[7–9] Obviously, linear problems can be solved by the boundary element method. Thus the proposed BEM can be seen as a successful application of a non-linear analysis technique called the homotopy analysis method[7–9] which is proposed to overcome some limitations and restrictions of perturbation techniques.

Similarly to the traditional BEM for non-linear problems, the proposed general BEM also needs the integral over the internal domain Ω. In this paper we discretize the internal domain and use the Gauss numerical integral formula. This decreases greatly the effectiveness of the BEM. However, a boundary element technique called the dual reciprocity boundary element method[10,11] has been developed to overcome the disadvantage of the traditional BEM for non-linear problems by means of transforming the domain integral to the surface. Thus the proposed general BEM might become more effective if it were combined with the dual reciprocity boundary element method.

In this paper we use two different linear operators $L_1(u)=u_{xx}-\beta^2u$ and $L_2(u)=u_{xx}+\beta^2u$ to construct homotopy (20) and obtain successfully in each case the convergent results of the original non-linear equation (3) with boundary conditions (4) and (5). Thus we would like to emphasize that the linear operator L_0 of a non-linear problem, which is very important and absolutely necessary for the traditional BEM, has *no* special meaning at all—it is merely one of the many proper linear operators suited to the proposed BEM, since we now have great freedom to select a familiar linear operator. Certainly, the greater freedom is for the better. This is the reason why the proposed BEM seems to be superior to the traditional one. However, how can we use this kind of freedom? That is to say, how do we know that a selected linear operator is proper and even *better* than another one? Indeed, for any non-linear problem there may exist many proper linear operators suited to the proposed BEM. All these linear operators may construct a mathematical space S in which there exists a best linear operator, but how can we determine the best one? Unfortunately, we know very little about these interesting questions which need further research. We also note that the proposed BEM is based on the assumption that the solution of the so-called zero-order deformation equation exists and is smooth enough for $p\in[0, 1]$. This assumption is also the basis of the continuation method.[12] For the problem under consideration the assumption seems to hold. However, the types of general non-linear problems are so complex that there does not exist a rigorous mathematical proof of the assumption, although the basic ideas of the proposed BEM have been applied successfully to solve some highly non-linear problems such as the 2D Navier–Stokes equations at high Reynolds number ($Re=10{,}000$).[8] Fortunately, it has been proved that the solution of the zero-order deformation equation exists near $p=0$, so the assumption holds as long as the parameter λ is small enough. Thus pure mathematical research in this direction is necessary, since this assumption is crucial to the development of the method. Moreover, although the given example has indeed illustrated the effectiveness of the proposed BEM, it seems to be a little simple. The proposed BEM deserves further research and wide applications to solve more complex 2D and 3D non-linear problems in engineering.

APPENDIX: NOMENCLATURE

A, $\tilde{A}$	non-linear differential operator
B	differential operator on boundary Γ
$c(\vec{r})$	geometric factor
f	real function
$g(\vec{r})$	real function of position vector $\vec{r}$
$h(\tau)$	real function of τ
$h_0^{[m]}(\tau)$	mth-order deformation derivatives at $p=0$
$\hbar(\tau,p)$	homotopy defined by equations (20)–(22)
$\hbar^{[m]}(\tau,p)$	mth-order deformation derivatives
L, L_0, L_1, L_2	linear differential operators
M	order of high-order iterative formula (28)
N	total number of subdomains
p	imbedding parameter
$\vec{r}$	position vector
U	velocity at infinity of 2D laminar viscous flow over flat plate
x, y	Cartesian co-ordinates

Greek letters

α	subscript of linear operators L_1 and L_2
β	parameter of linear operators L_1 and L_2
Γ	boundary of domain Ω
λ	iterative factor
ν	kinematic viscosity coefficient
ρ	radius of convergence
ψ	streamfunction
ω	fundamental solution
Ω	domain

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