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On cusped solitary waves in finite water depth

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ABSTRACT

It is well-known that the Camassa–Holm (CH) equation admits both of the peaked and cusped solitary waves in shallow water. However, it was an open question whether or not the exact wave equations can admit them in finite water depth. Besides, it was traditionally believed that cusped solitary waves, whose 1st-derivative tends to infinity at crest, are essentially different from peaked solitary ones with finite 1st-derivative. Currently, based on the symmetry and the exact water wave equations, Liao [1] proposed a unified wave model (UWM) for progressive gravity waves in finite water depth. The UWM admits not only all traditional smooth progressive waves but also the peaked solitary waves in finite water depth: in other words, the peaked solitary progressive waves are consistent with the traditional smooth ones. In this paper, in the frame of the linearized UWM, we give, for the first time, some explicit expressions of cusped solitary waves in finite water depth, and besides reveal a close relationship between the cusped and peaked solitary waves: a cusped solitary wave is consist of an infinite number of peaked solitary ones with the same phase speed, so that it can be regarded as a special peaked solitary wave. This also well explains why and how a cuspon has an infinite 1st-derivative at crest. Besides, it is found that, when wave height is small enough, the effect of nonlinearity is negligible for the interaction of peaked waves so that these explicit expressions are good enough approximations of peaked/cusped solitary waves in finite water depth. In addition, like peaked solitary waves, the vertical velocity of a cusped solitary wave in finite water depth is also discontinuous at crest ($x = 0$), and especially its phase speed has nothing to do with wave height, too. All of these would deepen and enrich our understandings about the cusped solitary waves.

1. Introduction

The smooth solitary surface wave was first reported by John Scott Russell [2] in 1844. Since then, various types of solitary waves have been found. The mainstream models of shallow water waves, such as the Boussinesq equation [3], the KdV equation [4], the BBM equation [5] and so on [6,7], admit dispersive *smooth* periodic/solitary progressive waves with permanent form: the wave elevation is *infinitely* differentiable *everywhere*. Especially, the phase speed of the smooth waves is highly dependent upon wave height: the larger the wave height of a smooth progressive wave, the faster it propagates. The only

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exception is the limiting wave with the highest amplitude, which has a sharp crest, as pointed out by Stokes. However, Stokes limiting wave has never been observed in practice. Nowadays, the smooth amplitude-dispersive periodic/solitary waves are the mainstream of researches in water waves.

In 1993, Camassa and Holm [8] proposed the celebrated Camassa–Holm (CH) equation for shallow water waves, and first reported the so-called peaked solitary wave, called peakon, which has a peaked crest with a discontinuous (but finite) 1st-order derivative at crest. This is a breakthrough in water wave theories, since it opens a new field of research in the past 20 years. Physically, different from the KdV equation and Boussinesq equation, the CH equation can model phenomena of not only soliton interaction but also wave breaking [9]. Mathematically, the CH equation is integrable and bi-Hamiltonian, therefore possesses an infinite number of conservation laws in involution [8]. Besides, it is associated with the geodesic flow on the infinite dimensional Hilbert manifold of diffeomorphisms of line [9]. Thus, the CH equation has lots of intriguing physical and mathematical properties. It is even believed that the CH equation "has the potential to become the new master equation for shallow water wave theory" [10]. In addition, Kraenkel and Zenchuk [11] reported the cusped solitary waves of the CH equation, called cuspon. The so-called cuspon is a kind of solitary wave with the 1st derivative going to *infinity* at crest. Note that, unlike a peakon that has a *finite* 1st derivative, a cuspon has an *infinite* 1st derivative at crest. Thus, it was traditionally believed that peakons and cuspons are completely *different* two kinds of solitary waves.

However, the CH equation is a simplified model of water waves in *shallow* water. It was an open question whether or not the exact wave equations admit the peaked and cusped solitary waves in *finite* water depth. For example, the velocity distribution of peaked/cusped solitary waves in the vertical direction was unknown, since it can not be determined by a wave model in shallow water (such as the CH equation). Currently, based on the symmetry and the exact wave equations, Liao proposed a unified wave model (UWM) for progressive gravity waves in finite water depth with permanent form [1]. It was found that the UWM admits not only all traditional smooth periodic/solitary waves but also the peaked solitary waves in finite water depth, even including the famous peaked solitary waves of the CH equation as its special case. Therefore, the UWM unifies both of the smooth and peaked solitary waves in finite water depth, for the first time. In other words, the progressive peaked solitary waves in finite water depth are consistent with the traditional smooth waves, and thus are as acceptable and reasonable as the smooth ones.

In this article, we first give an closed-form expression of cusped solitary waves in finite water depth by means of the linearized UWM. Then, we show that, when wave height is small, the effect of nonlinearity is small and only near the crest so that it is negligible. Thus, a cusped solitary wave might be regarded as an infinite number of peaked solitary ones. This reveals, for the first time to the best of my knowledge, a simple but elegant relationship between the peaked and cusped solitary waves in finite water depth.

2. Cusped solitary waves in finite water depth

Let us first describe "the unified wave model" (UWM) [1] briefly. Consider a progressive gravity wave propagating on a horizontal bottom in a *finite* water depth D, with a constant phase speed c and a permanent form. For simplicity, the problem is solved in the frame moving with the phase speed c. Let x, z denote the horizontal and vertical dimensionless co-ordinates (using the water depth D as the characteristic length), with $x = 0$ corresponding to the wave crest, $z = -1$ to the bottom, and the z axis upward, respectively. Assume that the wave elevation $\eta(x)$ has a symmetry about the crest, the fluid in the interval $x > 0$ (and $x < 0$) is inviscid and incompressible, the flow in $x > 0$ (and $x < 0$) is irrotational, and surface tension is neglected. Here, it should be emphasized that, different from all traditional wave models, the flow at $x = 0$ is *not* absolutely necessary to be irrotational. Let $\phi(x,z)$ denote the velocity potential. All of them are dimensionless using D and $\sqrt{gD}$ as the characteristic scales of length and velocity, where g is the acceleration due to gravity. In the frame of the UWM, the velocity potential $\phi(x,z)$ and the wave elevation $\eta(x)$ are first determined by the exact wave equations (i.e. the Laplace equation $\nabla^2\phi = 0$, the two nonlinear boundary conditions on the unknown free surface η, the bed condition and so on) only in the interval $x \in (0, +\infty)$, and then extended to the whole interval $(-\infty, +\infty)$ by means of the symmetry

$$\eta(-x) = \eta(x), \quad u(-x,z) = u(x,z), \quad v(-x,z) = -v(x,z),$$

which enforces the additional restriction condition $v(0,z) = 0$. It should be emphasized that, in the frame of the UWM, the flow at $x = 0$ is *not* necessarily irrotational, so that the UWM is more general: this is the reason why the UWM can admit both of the smooth and peaked solitary waves. For details, please refer to Liao [1].

When wave height is small enough, the effect of nonlinearity is small and besides only near the crest, so that the nonlinearity is negligible, as shown in Section 3. Therefore, we first consider the linearized UWM in this section.

In the interval $(0, +\infty)$, the governing equation $\nabla^2\phi(x,z) = 0$ with the bed condition $\phi_z(x,-1) = 0$ has two kinds of general solutions [12], where the subscript denotes the differentiation with respect to z. One is

$$\cosh[nk(1+z)]\sin(nkx),$$

corresponding to the smooth periodic waves with the dispersive relation

$$\alpha^2 = \frac{\tanh(k)}{k} \leqslant 1, \tag{1}$$

where $\alpha = c/\sqrt{gD}$ is the dimensionless phase speed, k is wave number and n is an integer, respectively. The other is

$\cos[nk(z+1)]\exp(-nkx)$,

corresponding to the peaked solitary waves in finite water depth [1], with the relation

$$\alpha^2 = \frac{\tan(k)}{k} \geqslant 1, \tag{2}$$

where k has nothing to do with wave number. Here, the linear boundary conditions on free surface are used. Given $\alpha \leqslant 1$ for the smooth periodic waves, the transcendental Eq. (1) has a *unique* solution, as mentioned in the textbook [12]. However, given $\alpha \geqslant 1$ for the peaked solitary waves, the transcendental Eq. (2) has an *infinite* number of solutions:

$$\alpha^2 = \frac{\tan k_n}{k_n}, \quad n\pi \leqslant k_n \leqslant n\pi + \frac{\pi}{2}, \quad n \geqslant 0, \tag{3}$$

corresponding to an infinite number of peaked solitary waves [1]

$$\eta_n(x) = A_n \exp(-k_n|x|) \tag{4}$$

in the frame of the linear UWM, where A_n denotes its wave height. For example, when $\alpha^2 = 2\sqrt{3}/\pi$, the transcendental Eq. (2) has an infinite number of solutions $k_0 = \pi/6, k_1 = 4.51413, k_2 = 7.73730, k_3 = 10.91266, k_4 = 14.07281, k_5 = 17.22616, k_6 = 20.37587, k_7 = 23.52341, k_8 = 26.66955, k_9 = 29.81472, k_{10} = 32.95921$, and the asymptotic expression

$$k_n \approx (n+0.5)\pi, \quad n > 10, \tag{5}$$

with less than 0.08% error. In general, $k_n \approx (n+0.5)\pi$ is a rather accurate approximation of k_n for large enough integer n.

Obviously, the peaked solitary wave (4) is not smooth at crest, i.e. its first derivative is discontinuous. Note that the well-known peaked solitary wave $\eta = c\exp(-|x|)$ of the CH equation is only a special case of (4) when $A_n = c$ and $k_n = 1$. However, unlike $\eta = c\exp(-|x|)$ that is a weak solution of the CH equation, it is *unnecessary* to consider whether or not the peaked solitary wave (4) is a kind of weak solution, because, unlike the CH equation that is defined in the *whole* domain $-\infty < x < +\infty$, the governing equation of the UWM is defined only in $0 < x < +\infty$. Physically, unlike the CH equation and other traditional wave models, waves in the frame of the UWM are *not* necessary to be irrotational at $x = 0$, therefore the governing equation holds only in the domain $0 < x < +\infty$, since the solution in the interval $-\infty < x < 0$ is gained by means of the symmetry. Mathematically, $x = 0$ is a boundary of the governing equation, and it is well-known that solutions of differential equations can be non-smooth at boundary, like a beam with discontinuous cross sections acted by a constant bending moment. Therefore, in the frame of the UWM, it is *unnecessary* to consider whether the peaked solitary waves (4) are weak solutions or not. This is the reason why, unlike the well-known peaked solitary wave $\eta = c\exp(-|x|)$ of the CH equation whose phase speed is *always* equal to its wave height, the phase speed of the peaked solitary waves (4) given by the UWM has *nothing* to do with wave height [1]. This is the most attractive novelty of the UWM, which might provide us a simple, elegant relationship between peaked and cusped solitary waves, as described below.

The above peaked solitary waves in finite water depth have some unusual characteristics, as revealed by Liao [1]. First, it has a peaked crest with a discontinuous vertical velocity v at crest. Besides, unlike the smooth waves whose horizontal velocity u decays exponentially from free surface to bottom, the horizontal velocity u of the peaked solitary waves at bottom is always larger than that on free surface. Especially, different from the smooth waves whose phase speed depends upon wave height, the phase speed of the peaked solitary waves in finite water depth has nothing to do with wave height, i.e. it is *non-dispersive*.

Thus, in the frame of the linear UWM [1], given a dimensionless phase speed $\alpha \geqslant 1$, there exist an *infinite* number of peaked solitary waves $A_n\exp(-k_n|x|)$ with the *same* phase speed α but different wave amplitudes A_n. Thus, we may have such peaked solitary waves

$$\eta(x) = \sum_{n=0}^{\infty} A_n \exp(-k_n|x|),$$

where A_n is a constant, which can be chosen with great freedom, as long as the above infinite series is convergent in the whole interval $(-\infty, +\infty)$. As a special case of it, let us consider such a one-parameter family of wave elevations

$$\eta(x) = \frac{H_w}{\zeta(\beta)} \sum_{n=1}^{+\infty} \frac{1}{n^\beta} \exp(-k_{n-1}|x|), \quad \beta > 1, \tag{6}$$

where H_w denotes wave height, $\beta > 1$ is a constant, $\zeta(\beta)$ is the Riemann zeta function, and k_n is determined by (2) for the given $\alpha \geqslant 1$, respectively. Since $\beta > 1$, we have $\sum_{n=1}^{+\infty} n^{-\beta} = \zeta(\beta)$ so that the above infinite series converges to the wave height H_w at $x = 0$, and besides it is convergent in the whole interval $(-\infty, +\infty)$. However, its 1st derivative at $x = 0$, i.e.

$$\eta'(0) \pm \frac{H_w}{\zeta(\beta)} \sum_{n=1}^{+\infty} \frac{k_{n-1}}{n^\beta}, \tag{7}$$

is convergent to a *finite* value when $\beta > 2$, but tends to *infinity* when $1 < \beta \leqslant 2$, because $k_{n-1} \approx (n-0.5)\pi$ for large enough integer n and the series $\sum 1/n^{\beta-1}$ is convergent when $\beta > 2$ but tends to infinity when $0 < \beta \leqslant 2$. So, the infinite series (6) defines a *cusped* solitary wave in finite water depth when $1 < \beta \leqslant 2$ and a *peaked* solitary wave when $\beta > 2$. Therefore, in essence, a cusped solitary wave in *finite* water depth might be consist of an *infinite* number of peaked solitary waves (when $1 < \beta \leqslant 2$) with the same phase speed! To the best of the author's knowledge, this reveals a simple but elegant relationship between the peaked and cusped solitary waves in finite water depth! In addition, the infinite series (6) illustrates the consistency of the peaked and cusped solitary waves, and besides explains *why* and *how* a cuspon has an infinite 1st-derivative at crest. Since the phase speed of peaked solitary waves (4) in finite water depth has nothing to do with the wave height, it is straight forward that the phase speed of a cusped solitary wave in *finite* water depth also has *nothing* to do with the wave height, too.

Note that, according to the definition of the wave elevation (6), given a dimensionless phase velocity $\alpha \geqslant 1$ and an arbitrary (but small enough) wave height H_w, there exist an *infinite* number of cusped solitary waves, dependent upon $\beta \in (1, 2]$. For example, the two cusped solitary waves in finite water depth defined by the infinite series (6) in the case of $\alpha^2 = 2\sqrt{3}/\pi, H_w = 1/10$ when $\beta = 1.5$ and $\beta = 1.9$ are as shown in Fig. 1. It should be emphasized that the *same* expression (6) can define an *infinite* number of peaked solitary waves in *finite* water depth, too, depending on $\beta \in (2, +\infty)$. For example, the two peaked solitary waves in finite water depth in the case of $\alpha^2 = 2\sqrt{3}/\pi, H_w = 1/10$ when $\beta = 5/2$ and $\beta = 10$ are as shown in Fig. 2. This well illustrates the consistency of the peaked and cusped solitary waves in *finite* water depth.

Theoretically speaking, given an arbitrary wave height H_w and a dimensionless phase speed $\alpha \geqslant 1$, there are many different types of peaked/cusped solitary waves in finite water depth. For example, a more generalized, two-parameter family of peaked/cusped solitary waves in finite water depth reads

$$\eta(x) = \frac{H_w}{\zeta(\beta, \gamma)} \sum_{n=0, n\neq -\gamma}^{+\infty} \frac{1}{(n+\gamma)^\beta} \exp(-k_n|x|), \tag{8}$$

where $\beta > 1$ and $\gamma \neq 0$ are constants to be chosen with great freedom, $\zeta(\beta, \gamma)$ is a generalized Riemann zeta function, and k_n is determined by (2) for the given $\alpha \geqslant 1$, respectively. Since $k_n \approx (n + 0.5)\pi$ for large enough integer n, the above infinite series defines a *cusped* solitary wave when $1 < \beta \leqslant 2$ and a *peaked* ones when $\beta > 2$, respectively. This illustrates once again the consistency of the peaked and cusped solitary waves in finite water depth.

According to the linearized UWM [1], the velocity potential ϕ^+ (defined only in the interval $x > 0$) corresponding to the peaked/cusped solitary wave elevation (6) reads

$$\phi^+ = -\frac{\alpha H_w}{\zeta(\beta)} \sum_{n=1}^{+\infty} \frac{\cos[k_{n-1}(z+1)] \exp(-k_{n-1}x)}{n^\beta \sin(k_{n-1})}, \tag{9}$$

which gives, using the symmetry, the corresponding horizontal velocity

$$u = \frac{\alpha H_w}{\zeta(\beta)} \sum_{n=0}^{+\infty} \frac{k_n \cos[k_n(z+1)] \exp(-k_n|x|)}{(n+1)^\beta \sin(k_n)} \tag{10}$$

in the whole interval $x \in (-\infty, +\infty)$, the vertical velocity

$$v^+ = \frac{\alpha H_w}{\zeta(\beta)} \sum_{n=0}^{+\infty} \frac{k_n \sin[k_n(z+1)] \exp(-k_n x)}{(n+1)^\beta \sin(k_n)} \tag{11}$$

in the interval $x \in (0, +\infty)$, and the vertical velocity

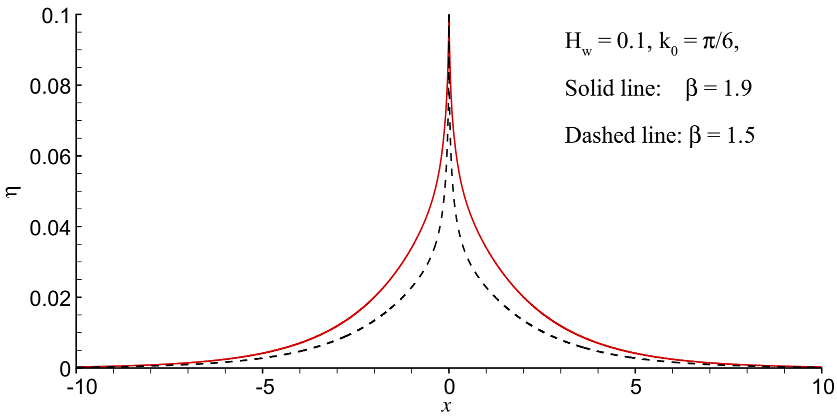

Fig. 1. Cusped solitary waves in finite water depth defined by (6) when $H_w = 0.1$ and $k_0 = \pi/6$ (corresponding to $\alpha = 12^{1/4}/\sqrt{\pi}$). Solid line: $\beta = 1.9$; dashed line: $\beta = 1.5$.

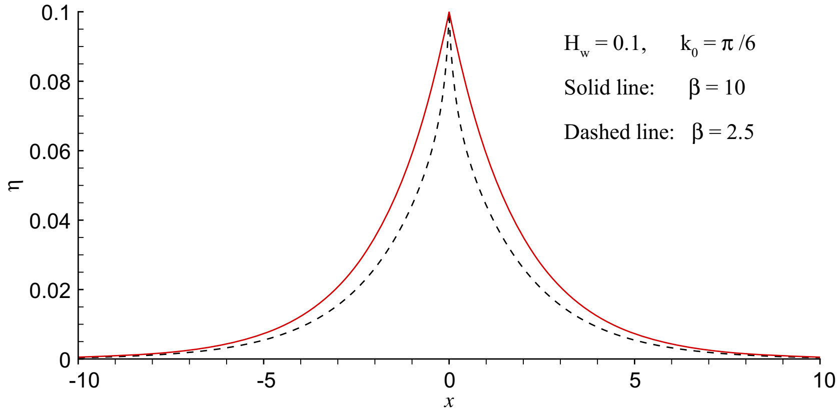

Fig. 2. Peaked solitary waves in finite water depth defined by (6) when $H_w = 0.1$ and $k_0 = \pi/6$ (corresponding to $\alpha = 12^{1/4}/\sqrt{\pi}$). Solid line: $\beta = 10$; dashed line: $\beta = 2.5$.

$$v^- = -\frac{\alpha H_w}{\zeta(\beta)} \sum_{n=0}^{+\infty} \frac{k_n \sin[k_n(z+1)] \exp(k_n x)}{(n+1)^\beta \sin(k_n)} \tag{12}$$

in the interval $(-\infty, 0)$, respectively. Obviously, it holds

$$\lim_{x\to 0} v^+ = -\lim_{x\to 0} v^- \neq 0$$

for the cusped ($1 < \beta \leqslant 2$) and peaked ($\beta > 2$) solitary waves, although we always have $v = 0$ at $x = 0$. Thus, like a peakon in finite water depth, a cuspon in finite water depth has the velocity discontinuity at $x = 0$, too.

Especially, at $z = 0$ and as $x \to 0$, the corresponding vertical velocity reads

$$\lim_{x\to 0} v^+(x, 0) = \frac{\alpha H_w}{\zeta(\beta)} \sum_{n=0}^{+\infty} \frac{k_n}{(n+1)^\beta}$$

which is finite when $\beta > 2$ but tends to infinity when $1 < \beta \leqslant 2$, since $k_n \approx (n + 0.5)\pi$ for large enough integer n. Thus, unlike a peaked solitary wave in finite water depth whose v is always finite, the vertical velocity of the cusped solitary waves in finite water depth tends to infinity at $z = 0$ as $x \to 0$. Mathematically, this is acceptable, since it is traditionally believed that a cuspon has a higher singularity than a peakon. Such kind of singularity leads to a more strong vortex sheet at $x = 0$ near $z = 0$. Physically, in reality such kind of singularity and discontinuity, "if it could ever be originated, would be immediately abolished by viscosity", as mentioned by Lamb [13].

3. Effect of nonlinearity

To consider the effect of the nonlinearity, the fully nonlinear boundary conditions on free surface is used. In the frame of the UWM, we solve the exact wave equation in the $x > 0$ and then use the symmetry to extend the solution to the whole domain. For detailed mathematical formulations, please refer to Liao [1].

Without loss of generality, let us consider here a wave system consist of M progressive peaked ones with the *same* phase speed α, whose surface elevation given by the linear UWM reads

$$\eta(x) = \frac{H}{\zeta(\beta)} \sum_{n=1}^{M} \frac{1}{n^\beta} \exp(-k_{n-1}|x|), \quad \beta > 1 \tag{13}$$

where $\beta > 1$ and $H > 0$ are constants, and

$$\alpha^2 = \frac{\tan(k_{n-1})}{k_{n-1}}, \quad n = 1, 2, \ldots, M$$

Since each wave component has the same phase speed, they as a whole construct one progressive peaked wave with a single wave crest. Obviously, there exist nonlinear interactions between wave components.

According to the linear UWM [1], the velocity potential corresponding to the wave elevation (13) reads

$$\phi^+ = -\frac{\alpha H}{\zeta(\beta)} \sum_{n=1}^{M} \frac{\cos[k_{n-1}(z+1)] \exp(-k_{n-1}x)}{n^\beta \sin(k_{n-1})}, \quad x > 0 \tag{14}$$

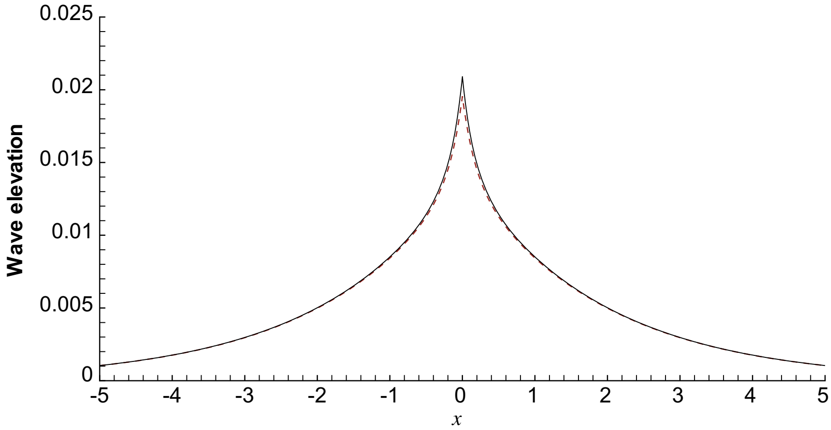

Fig. 3. Comparison of the linear wave (13) with the corresponding exact ones satisfying the fully nonlinear free surface boundary conditions when $H = 1/40, \beta = 1.9, M = 5$ and $k_0 = \pi/6$. Solid line: the linear wave (13); dashed line: the exact wave given by the fully nonlinear free surface conditions.

In the frame of the HAM, the above expression is now used as the initial guess of the velocity potential. Almost all are the same as those in [1], except that the velocity potential always contains the above-mentioned part with the same coefficients and that the wave height are not given. In addition, we choose the same auxiliary linear operator as that in [1], and proper values of the so-called convergence-control parameters to guarantee the convergence of series solution. For details, please refer to Liao [1].

Without loss of generality, let us consider the case $k_0 = \pi/6, k_1 = 4.51413, k_2 = 7.73730$ and so on. It is found that, when $H = 1/40, \beta = 1.9$ and $M = 2, 3, 4$ and so on, respectively, the corresponding wave elevations given by the fully nonlinear surface conditions are very close to the above linear results, as shown in Fig. 3 (when $M = 5$). It is found that the two wave elevations are almost the same, except near the crest. This is because the peaked wave components with large k_n decay very quickly, since each decays exponentially. Besides, the wave component with larger k_n has much smaller wave amplitude: for example, when $M = 10$ and $\beta = 1.9$, the wave amplitude of the last component (with $k_9 > 8\pi$) is only about 1.25% of the first. This example suggests that, when the wave height is small enough, the effect of the nonlinear interaction should be negligible.

Thus, the explicit expressions (6) and (8) should be good enough approximations of peaked/cusped solitary waves in finite water depth, when wave height is small enough.

4. Concluding remarks

In summary, in the frame of the linearized UWM [1], we give, for the first time, some explicit expressions of cusped solitary waves in *finite* water depth by means of the linearized UWM, and reveal that a cuspon might be regarded as an infinite number of peaked solitary waves with the same phase speed. This kind of consistency also well explains why and how the 1st-derivative of a cusped solitary wave tends to infinity at crest. It is found that, like a peakon in finite water depth, the vertical velocity of a cuspon is also discontinuous at $x = 0$, and besides, its phase speed also has nothing to do with wave height, too. All of these would deepen and enrich our understandings about the peaked and cusped solitary waves.

It is found that, when wave height is small enough, the effect of nonlinearity is negligible for the interaction of peaked waves, as shown in Section 3. Therefore, the explicit expressions (6) and (8) should be good enough approximations of peaked/cusped solitary waves in finite water depth, when wave height is small enough.

It should be emphasized that, in the frame of the UWM, the governing equation is defined only in the domain $0 < x < +\infty$, since the solution at $-\infty < x < 0$ is given by means of the symmetry. This is quite *different* from other wave models such as the CH equation, which are defined in the *whole* domain $-\infty < x < +\infty$. Physically, it means that the flow at crest is *not* absolutely necessary to be irrotational. Thus, mathematically, we need not consider whether the peaked/cusped solitary waves are weak solutions or not. This is the reason why, unlike the well-known peaked solitary wave $\eta = c\,\exp(-|x|)$ of the CH equation, whose phase speed is *always* equal to wave height, the phase speed of the peaked/cusped solitary waves (4) given by the UWM has *nothing* to do with wave height! This is the most attractive novelty of the UWM, which provides us a simple but elegant relationship between peaked and cusped solitary waves in finite water depth. Vice versa, such kind of simple and elegant relationship further reveals the reasonableness of the UWM.

Finally, we emphasize that the HAM is successfully applied as an easy-to-use, powerful analytic tool to find out the peaked and cusped solitary waves in finite water depth governed by the UWM. This is a good example to illustrate that a truly new method can indeed bring us somethings novel and/or different.

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