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Two kinds of peaked solitary waves of the KdV, BBM and Boussinesq equations

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It is well-known that the celebrated Camassa-Holm equation has the peaked solitary waves, which have been not reported for other mainstream models of shallow water waves. In this letter, the closed-form solutions of peaked solitary waves of the KdV equation, the BBM equation and the Boussinesq equation are given for the first time. All of them have either a peakon or an anti-peakon. Each of them exactly satisfies the corresponding Rankine-Hogoniot jump condition and could be understood as weak solution. Therefore, the peaked solitary waves might be common for most of shallow water wave models, no matter whether or not they are integrable and/or admit breaking-wave solutions.

solitary waves, peaked crest, progressive, discontinuity, weak solution

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1 Introduction

Since the solitary surface wave was discovered by Russell [1] in 1834, many models of solitary waves in shallow water have been developed, such as the Boussinesq equation [2], the Korteweg & de Vries (KdV) equation [3], the Benjamin-Bona-Mahony (BBM) equation [4], the Camassa-Holm (CH) equation [5], and so on. Unlike other models, the celebrated CH equation:

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1}$$

possesses both of the global solutions in time and the wave-breaking solutions, where u denotes wave elevation, x, t are the spatial and temporal variables, the subscript denotes the differentiation, and $0 \leqslant \kappa \leqslant 1/2$ is a constant related to the critical shallow water wave speed, respectively. Especially, when $\kappa = 0$, the CH equation (1) has a peaked solitary wave $u(x,t) = c\ \exp(-|x - c\ t|)$, where c denotes the phase speed. It should be emphasized that, at the wave crest $x = ct$, the above closed-form solution has no continuous derivatives respect to x, thus it does not satisfy eq. (1) (when $\kappa = 0$) at the crest $x = c\ t$. So, it is not a strong solution. However, Constantin and Molinet [6] pointed out that $u(x,t) = c\ \exp(-|x - c\ t|)$ could be understood as a weak solution of the CH equation (1) when $\kappa = 0$.

Note that such kind of discontinuity (or singularity) exists widely in natural phenomena, such as dam breaking [7] in hydrodynamics, shock waves in aerodynamics, black holes described by the general relativity, and so on. In the frame of water waves, Stokes [8] found that the limiting gravity wave has a sharp corner at the crest. In case of the dam breaking, there exist sharp corners of wave elevation at the beginning $t = 0$, and such kind of discontinuity of the derivative of wave elevation does not disappear because of the neglect of the viscosity, as shown by Wu and Cheung [9]. In fact, problems related to such kind of discontinuity belong to the so-called Riemann problem [9–11], which is a classic field of fluid mechanics.

However, such kind of peaked solitary waves have never been found for other mainstream models of shallow water

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waves. Solitary wave solutions in shallow water with a fixed speed c and permanent form were found by Korteweg & de Vries [3] using the KdV equation

$$\zeta_t + \zeta_{xxx} + 6\zeta\, \zeta_x = 0, \tag{2}$$

subject to the boundary conditions

$$\zeta \to 0, \zeta_x \to 0, \zeta_{xx} \to 0, \text{as } |x| \to +\infty, \tag{3}$$

where ζ denotes the wave elevation. The KdV equation (2) admits solitary waves but breaking ones. The traditional solitary wave of KdV equation (2) reads

$$\zeta(x,t) = \frac{c}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(x - c\,t - x_0)\right], \tag{4}$$

where sech denotes a hyperbolic secant function and x_0 may be any a constant. This solitary wave has a smooth crest with always positive elevation $\zeta(x,t) \geqslant 0$, and the phase speed c is dependent upon the wave height, i.e. $c = 2\zeta_{\max}$, so that higher solitary waves propagate faster. To the best of the author's knowledge, no peaked solitary waves of KdV equation (2) have been reported.

The KdV equation (2) is an approximation of the fully nonlinear wave equations that admit the limiting gravity wave with a peaked crest, as pointed out by Stokes [8]. This kind of discontinuity of wave elevation widely exists in hydrodynamic problems such as dam break [7]. In addition, current investigations [13, 14] reveal the close relationships between the CH equation (1) and the KdV equation (2). Therefore, one can assume that the progressive solitary waves of the KdV equation (2) might admit peaked solitary waves as well.

2 Peaked solitary waves of KdV equation

Write $\xi = x - c\,t - x_0$ and $\eta(\xi) = \zeta(x,t)$. The original KdV equation (2) becomes

$$-c\eta' + \eta''' + 6\eta\eta' = 0, \tag{5}$$

subject to the boundary condition

$$\eta \to 0,\ \ \eta' \to 0,\ \ \eta'' \to 0, \text{as } |\xi| \to +\infty, \tag{6}$$

where the prime denotes the differentiation with respect to ξ. Besides, there exists the symmetry condition

$$\eta(-\xi) = \eta(\xi), \qquad \xi \in (-\infty, +\infty), \tag{7}$$

so that we need only consider the solution in the interval $\xi \geqslant 0$. Integrating (5) with (6) gives

$$-c\eta + \eta'' + 3\eta^2 = 0, \qquad \xi \geqslant 0. \tag{8}$$

Multiplying it by $2\eta'$ and then integrating it with eq. (6), we have

$$\eta'^2 = \eta^2(c - 2\eta), \qquad \xi \geqslant 0, \tag{9}$$

which has real solutions only when

$$\eta \leqslant \frac{c}{2}. \tag{10}$$

Under the restriction (10), we have from eq. (9) that

$$\eta' = \pm\eta\ \sqrt{c - 2\eta}, \tag{11}$$

say,

$$\frac{\mathrm{d}\eta}{\eta\ \sqrt{c - 2\eta}} = \pm\mathrm{d}\xi, \quad \xi \geqslant 0. \tag{12}$$

Let us first consider the solitary waves with $\eta \geqslant 0$. Integrating eq. (12) in case of $0 \leqslant \eta \leqslant c/2$, we have

$$\tanh^{-1}\left[\sqrt{1 - \frac{2\eta}{c}}\right] = \pm\frac{\sqrt{c}}{2}\xi + \alpha, \tag{13}$$

where α is a constant. Since $0 \leqslant 2\eta/c \leqslant 1$, the left-hand side of the above equation is non-negative so that it holds for all $\xi \geqslant 0$ if and only if

$$\tanh^{-1}\left[\sqrt{1 - \frac{2\eta}{c}}\right] = \frac{\sqrt{c}}{2}\xi + \alpha, \quad \xi \geqslant 0, \alpha \geqslant 0,$$

which gives

$$\eta(\xi) = \frac{c}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}\xi + \alpha\right], \quad \xi \geqslant 0, \alpha \geqslant 0. \tag{14}$$

Using the symmetry condition (7), it holds

$$\eta(\xi) = \frac{c}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}|\xi| + \alpha\right], \quad -\infty < \xi < +\infty, \alpha \geqslant 0 \tag{15}$$

in the whole interval. Thus, we have the solitary waves of the first kind

$$\zeta(x,t) = \frac{c}{2}\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}|x - c\,t - x_0| + \alpha\right], \tag{16}$$

where $\alpha \geqslant 0$, with the wave height

$$\zeta_{\max} = \left(\frac{c}{2}\right)\mathrm{sech}^2(\alpha), \quad \alpha \geqslant 0. \tag{17}$$

Note that $\alpha \geqslant 0$ is a constant parameter. When $\alpha = 0$, it is exactly the same as the traditional solitary waves of the KdV equation with a smooth crest.

In case of $\eta < 0$, it holds $1 - 2\eta/c > 1$ so that

$$\tanh^{-1}\left[\sqrt{1 - \frac{2\eta}{c}}\right]$$

becomes a complex number which has no physical meanings. This is the reason why the solitary solutions of the KdV equation in case of $\eta < 0$ was traditionally neglected. So, we must be very careful in this case.

In case of $\eta \leqslant 0$, write

$$\sqrt{c - 2\eta} = \sqrt{c}\sqrt{1 - \frac{2\eta}{c}} = \sqrt{c}\ z,$$

where $z \geqslant 1$ and

$$z^2 = 1 - \frac{2\eta}{c} \geqslant 1.$$

Then, we have

$$\eta = \frac{c}{2}(1 - z^2), \quad \mathrm{d}\eta = -cz\mathrm{d}z. \tag{18}$$

Since $z \geqslant 1$, it holds

$$\frac{\mathrm{d}\eta}{\eta\sqrt{c-2\eta}} = \frac{1}{\sqrt{c}}\left(\frac{\mathrm{d}z}{z-1} - \frac{\mathrm{d}z}{z+1}\right) = \mathrm{d}\left[\frac{1}{\sqrt{c}}\ln\left(\frac{z-1}{z+1}\right)\right]. \tag{19}$$

Substituting it into eq. (12) and integrating, we have

$$\ln\left(\frac{z-1}{z+1}\right) = \pm\sqrt{c}\,\xi - 2\beta, \qquad \xi \geqslant 0, \tag{20}$$

where β is a constant. Since $z \geqslant 1$, it holds

$$0 \leqslant \frac{z-1}{z+1} < 1,$$

which gives

$$\ln\left(\frac{z-1}{z+1}\right) < 0.$$

Thus, eq. (20) holds for all $\xi \geqslant 0$ if and only if

$$\ln\left(\frac{z-1}{z+1}\right) = -(\sqrt{c}\,\xi + 2\beta), \quad \xi \geqslant 0, \quad \beta > 0, \tag{21}$$

which gives

$$z = \coth\left[\frac{\sqrt{c}}{2}\,\xi + \beta\right], \quad \xi \geqslant 0, \quad \beta > 0.$$

So, we have

$$\eta(\xi) = \frac{c}{2}(1 - z^2) = -\left(\frac{c}{2}\right)\mathrm{csch}^2\left[\frac{\sqrt{c}}{2}\,\xi + \beta\right], \quad \xi \geqslant 0, \quad \beta > 0,$$

where csch denotes a hyperbolic cosecant function, and $\beta > 0$ is a constant. Using the symmetry condition (7), we have the solitary wave

$$\eta(\xi) = \frac{c}{2}(1 - z^2) = -\left(\frac{c}{2}\right)\mathrm{csch}^2\left[\frac{\sqrt{c}}{2}\,|\xi| + \beta\right],$$
$$-\infty < \xi < +\infty, \quad \beta > 0,$$

which is valid in the whole interval.

Let $d \leqslant \zeta(x,t) \leqslant 0$ denote the restriction of $\zeta(x,t)$. Then, it holds

$$\eta(0) = -\left(\frac{c}{2}\right)\mathrm{csch}^2(\beta) \geqslant d,$$

which gives

$$\beta \geqslant \mathrm{csch}^{-1}\sqrt{-\frac{2d}{c}}.$$

Thus, we have the solitary waves of the second kind:

$$\zeta(x,t) = -\frac{c}{2}\mathrm{csch}^2\left[\frac{\sqrt{c}}{2}|x - c\,t - x_0| + \beta\right], \tag{22}$$

where

$$\beta \geqslant \mathrm{csch}^{-1}\sqrt{-\frac{2d}{c}},$$

with the restriction $d \leqslant \zeta(x,t) \leqslant 0$.

The closed-form solutions (16) and (22) of the solitary waves of the first and second kind exactly satisfy the KdV equation (2) in the whole domain $-\infty < x < +\infty$ and $t \geqslant 0$, but except $x = c\,t + x_0$ (corresponding to the wave crest). This is rather similar to the closed-form solution $u(x,t) = c\exp(-|x - c\,t|)$ of the CH equation (1) when $\kappa = 0$. Thus, the closed-form solutions (16) and (22) should be understood as weak solutions. Moreover, the corresponding Rankine-Hogoniot jump condition [12] must be satisfied, so as to ensure that eqs. (16) and (22) have physical meanings.

To give the corresponding Rankine-Hogoniot jump condition [12] of the KdV equation (2), we rewrite it in the flow

$$\zeta_t + [f(\zeta)]_x = 0, \tag{23}$$

where $f(\zeta) = \zeta_{xx} + 3\zeta^2$. Then, ζ is a weak solution of eq. (23), if

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}[\zeta\varphi_t + f(\zeta)\varphi_x]\,\mathrm{d}x\mathrm{d}t + \int_{-\infty}^{+\infty}\zeta(x,0)\varphi(x,0)\mathrm{d}x = 0 \tag{24}$$

for all smooth functions φ with compact support. Besides, there exists such a theorem that, if ζ is a weak solution of eq. (23) such that ζ is discontinuous across the curve $x = \sigma(t)$ but ζ is smooth on either side of $x = \sigma(t)$, then ζ must satisfy the condition

$$\sigma'(t) = \frac{f(\zeta^-) - f(\zeta^+)}{\zeta^- - \zeta^+}, \tag{25}$$

across the curve of discontinuity, where $\zeta^-(x,t)$ is the limit of ζ approaching (x,t) from the left and $\zeta^+(x,t)$ is the limit of ζ approaching (x,t) from the right.

Then, due to the symmetry of the progressive peaked solitary waves about the crest $x = c\,t + x_0$, we have

$$\begin{aligned} c = \frac{f(\zeta^-) - f(\zeta^+)}{\zeta^- - \zeta^+} &= \frac{(\zeta^-)_{xx} - (\zeta^+)_{xx} + 3(\zeta^-)^2 - 3(\zeta^+)^2}{\zeta^- - \zeta^+} \\ &= \frac{(\zeta^-)_{xx} - (\zeta^+)_{xx}}{\zeta^- - \zeta^+} + 3(\zeta^- + \zeta^+) \\ &= \frac{(\zeta^-)_{xxx} - (\zeta^+)_{xxx}}{(\zeta^-)_x - (\zeta^+)_x} + 3(\zeta^- + \zeta^+), \end{aligned} \tag{26}$$

which provides us the so-called Rankine-Hogoniot jump condition of the KdV equation:

$$c = \frac{(\zeta^-)_{xxx} - (\zeta^+)_{xxx}}{(\zeta^-)_x - (\zeta^+)_x} + 3(\zeta^- + \zeta^+), \qquad \text{as } x \to c\,t + x_0. \tag{27}$$

For the closed-form solution (16) at the crest, we have

$$\zeta^- = \zeta^+ = \frac{c}{2}\mathrm{sech}^2(\alpha), \tag{28}$$

$$(\zeta^-)_x = \frac{c^{3/2}}{2}\mathrm{sech}^2(\alpha)\tanh(\alpha), \quad (\zeta^+)_x = -(\zeta^-)_x, \tag{29}$$

$$(\zeta^-)_{xxx} = \frac{c^{5/2}}{4}\left[\cosh(2\alpha) - 5\right]\mathrm{sech}^4(\alpha)\tanh(\alpha),$$
$$(\zeta^+)_{xxx} = -(\zeta^-)_{xxx}. \tag{30}$$

Substituting them into eq. (27), we have

$$\frac{c}{2}\left[\mathrm{sech}^2(\alpha) + \mathrm{sech}^2(\alpha)\cosh(2\alpha) - 2\right] = 0, \tag{31}$$

which is an identity for not only $\alpha = 0$ (corresponding to the traditional smooth solitary wave) but also *arbitrary* $\alpha > 0$ (to the peaked solitary wave)!

Similarly, for the closed-form solution (22) at the crest, we have

$$\zeta^- = \zeta^+ = -\frac{c}{2}\mathrm{csch}^2(\beta), \tag{32}$$

$$(\zeta^-)_x = -\frac{c^{3/2}}{2}\mathrm{csch}^2(\beta)\coth(\beta), \quad (\zeta^+)_x = -(\zeta^-)_x, \tag{33}$$

$$(\zeta^-)_{xxx} = -\frac{c^{5/2}}{4}\left[\cosh(2\beta) + 5\right]\mathrm{csch}^4(\beta)\coth(\beta),$$
$$(\zeta^+)_{xxx} = -(\zeta^-)_{xxx}. \tag{34}$$

Substituting them into eq. (27), we have

$$\frac{c}{2}\left[\mathrm{csch}^2(\beta)\cosh(2\beta) - \mathrm{csch}^2(\beta) - 2\right] = 0, \tag{35}$$

which is an identity for *arbitrary* constant $\beta > 0$.

Therefore, both of eqs. (16) and (22) satisfy the corresponding Rankine-Hogoniot jump condition and should be understood as weak solutions of the KdV equation (2). It should be emphasized that the traditional solitary wave (4) with a smooth crest is just only a special case of the solitary waves (16) of the first kind when $\alpha = 0$. Note that the solitary waves (16) and (22) of the first and second kind of the KdV equation (2) have a peakon and an anti-peakon, respectively. Since $\alpha \geqslant 0$ and $\beta > 0$ are arbitrary constant, the phase speed of these peaked solitary waves has nothing to do with the wave amplitude, for example as shown in Figures 1 and 2. This is quite different from the traditional smooth solitary wave. To the best of the author's knowledge, such kind of peaked solitary waves with peakon or anti-peakon have never been reported for the KdV equation. All of these reveal the novelty of the peaked solitary waves (16) and (22).

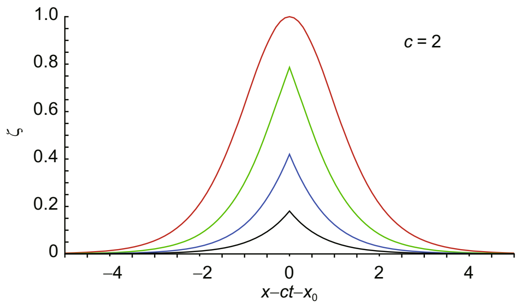

Figure 1 Solitary waves $\zeta(x,t)$ of the first kind of the KdV equation (2) with the same phase speed $c = 2$. Red line: $\alpha = 0$; green line: $\alpha = 1/2$; blue line: $\alpha = 1$; black line: $\alpha = 3/2$.

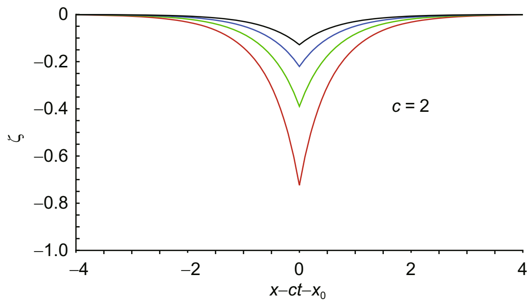

Figure 2 Solitary waves of the second kind of the KdV equation (2) with the same phase speed $c = 2$. Red line: $\beta = 1$; green line: $\beta = 5/4$; blue line: $\beta = 3/2$; black line: $\beta = 7/4$.

3 Peaked solitary waves of modified KdV equation

Similarly, the modified KdV equation [15, 16]

$$\zeta_t + \zeta_{xxx} \pm 6\zeta^2\zeta_x = 0 \tag{36}$$

has the two kinds of solitary waves

$$\zeta(x,t) = \pm\frac{2c}{e^{\sqrt{c}|x-c\,t-x_0|+\alpha} \pm c\, e^{-\sqrt{c}|x-c\,t-x_0|-\alpha}}, \tag{37}$$

where c is the phase speed and $\alpha \geqslant 0$ is a constant.

Note that eq. (25) holds for eq. (23) in general. Now, we have $f(\zeta) = \zeta_{xx} \pm 2\zeta^3$. The corresponding Rankine-Hogoniot jump condition reads

$$\begin{aligned} c &= \frac{f(\zeta^-) - f(\zeta^+)}{\zeta^- - \zeta^+} \\ &= \frac{(\zeta^-)_{xx} - (\zeta^+)_{xx} \pm 2\left[(\zeta^-)^3 - (\zeta^+)^3\right]}{\zeta^- - \zeta^+} \\ &= \frac{(\zeta^-)_{xx} - (\zeta^+)_{xx}}{\zeta^- - \zeta^+} \pm 2\left[(\zeta^-)^2 + \zeta^-\zeta^+ + (\zeta^+)^2\right] \\ &= \frac{(\zeta^-)_{xxx} - (\zeta^+)_{xxx}}{(\zeta^-)_x - (\zeta^+)_x} \pm 2\left[(\zeta^-)^2 + \zeta^-\zeta^+ + (\zeta^+)^2\right], \end{aligned} \tag{38}$$

which provides us the Rankine-Hogoniot jump condition of the modified KdV equation

$$c = \frac{(\zeta^-)_{xxx} - (\zeta^+)_{xxx}}{(\zeta^-)_x - (\zeta^+)_x} \pm 2\left[(\zeta^-)^2 + \zeta^-\zeta^+ + (\zeta^+)^2\right],$$
$$\text{as } x \to c\,t + x_0. \tag{39}$$

Using the closed-form solution (37), it is found that the above condition is satisfied for arbitrary constant $\alpha \geqslant 0$. Therefore, the closed-form solution (37) exactly satisfies the corresponding Rankine-Hogoniot jump condition of the modified KdV equation (36) and could be understood as a weak solution.

4 Peaked solitary waves of the BBM equation

Similarly, the BBM equation [4]

$$\zeta_t + \zeta_x + \zeta\,\zeta_x - \zeta_{xxt} = 0 \tag{40}$$

has the solitary waves of the first kind

$$\zeta(x,t) = 3(c-1)\mathrm{sech}^2\left[\frac{\sqrt{1-c^{-1}}}{2}|\xi| + \alpha\right], \quad \alpha \geqslant 0, \tag{41}$$

and the solitary waves of the second kind

$$\zeta(x,t) = -3(c-1)\mathrm{csch}^2\left[\frac{\sqrt{1-c^{-1}}}{2}|\xi| + \beta\right], \quad \beta > 0, \tag{42}$$

where $\xi = x - c\,t - x_0$, c is the phase speed, and $\alpha \geqslant 0, \beta > 0$ are constants.

Now, we have $f(\zeta) = \zeta + \zeta^2/2 - \zeta_{xt}$. The corresponding Rankine-Hogoniot jump condition reads

$$\begin{aligned} c &= \frac{f(\zeta^-) - f(\zeta^+)}{\zeta^- - \zeta^+} \\ &= \frac{(\zeta^-) - (\zeta^+) + \left[(\zeta^-)^2 - (\zeta^+)^2\right]/2 - [(\zeta^-)_{xt} - (\zeta^+)_{xt}]}{\zeta^- - \zeta^+} \\ &= 1 + \frac{1}{2}(\zeta^- + \zeta^+) - \frac{(\zeta^-)_{xxt} - (\zeta^+)_{xxt}}{(\zeta^-)_x - (\zeta^+)_x}, \end{aligned} \tag{43}$$

which provides us the Rankine-Hogoniot jump condition of the MMB equation

$$c = 1 + \frac{1}{2}(\zeta^- + \zeta^+) - \frac{(\zeta^-)_{xxt} - (\zeta^+)_{xxt}}{(\zeta^-)_x - (\zeta^+)_x}, \quad \text{as } x \to c\,t + x_0. \tag{44}$$

For the solitary waves (41) of the first kind at the crest, we have

$$\zeta^- = 3(c-1)\mathrm{sech}^2(\alpha), \tag{45}$$

$$(\zeta^-)_x = 3c(1-c^{-1})^{3/2}\mathrm{sech}^2(\alpha)\tanh(\alpha), \tag{46}$$

$$\begin{aligned}(\zeta^-)_{xxt} = &-\frac{3}{4}c^2(1-c^{-1})^{5/2}\mathrm{sech}^5(\alpha) \\ &\times[\sinh(3\alpha) - 11\sinh(\alpha)],\end{aligned} \tag{47}$$

and

$$\zeta^+ = \zeta^-, \quad (\zeta^+)_x = -(\zeta^-)_x, \quad (\zeta^+)_{xxt} = -(\zeta^-)_{xxt}.$$

Substituting all of these into eq. (44), it is found that eq. (41) satisfies the Rankine-Hogoniot jump condition (44) for arbitrary constant $\alpha \geqslant 0$. Similarly, eq. (42) satisfies the Rankine-Hogoniot jump condition (44) for arbitrary constant $\beta > 0$. Therefore, both of eqs. (41) and (42) should be understood as weak solutions of the BBM equation (40).

5 Peaked solitary waves of Boussinesq equation

In addition, the Boussinesq equation [2]

$$\frac{\partial^2\zeta}{\partial t^2} - gh\frac{\partial^2\zeta}{\partial x^2} - gh\frac{\partial^2}{\partial x^2}\left(\frac{3\zeta^2}{2h} + \frac{h^2}{3}\frac{\partial^2\zeta}{\partial x^2}\right) = 0 \tag{48}$$

has the solitary waves of the first kind

$$\zeta = h\left(\frac{c^2}{gh} - 1\right)\mathrm{sech}^2\left[\frac{\sqrt{3}}{2h}\sqrt{\frac{c^2}{gh} - 1}\,|\xi| + \alpha\right], \tag{49}$$

and the solitary waves of the second kind

$$\zeta = -h\left(\frac{c^2}{gh} - 1\right)\mathrm{csch}^2\left[\frac{\sqrt{3}}{2h}\sqrt{\frac{c^2}{gh} - 1}\,|\xi| + \beta\right], \tag{50}$$

where $\xi = x - c\,t - x_0$, h denotes the water depth, g the acceleration of gravity, c the phase speed of wave, $\alpha \geqslant 0, \beta > 0$ are constant, respectively.

For progressive solitary wave, we have

$$\frac{\partial}{\partial x} = -\frac{1}{c}\frac{\partial}{\partial t}.$$

Then, eq. (48) can be rewritten in the form

$$\frac{\partial^2\zeta}{\partial t^2} + \frac{gh}{c}\frac{\partial^2\zeta}{\partial x\partial t} + \frac{gh}{c}\frac{\partial^2}{\partial x\partial t}\left(\frac{3\zeta^2}{2h} + \frac{h^2}{3}\frac{\partial^2\zeta}{\partial x^2}\right) = 0. \tag{51}$$

Integrating it with respect to t and using the boundary condition $\zeta(\pm\infty) = 0$, we have

$$\zeta_t + \frac{gh}{c}\left[\zeta + \frac{3\zeta^2}{2h} + \frac{h^2}{3}\zeta_{xx}\right]_x = 0. \tag{52}$$

Thus, we have here

$$f(\zeta) = \frac{gh}{c}\left(\zeta + \frac{3\zeta^2}{2h} + \frac{h^2}{3}\zeta_{xx}\right).$$

According to eq. (25), we have

$$\begin{aligned} c &= \frac{f(\zeta^-) - f(\zeta^+)}{\zeta^- - \zeta^+} \\ &= \left(\frac{gh}{c}\right)\frac{(\zeta^- - \zeta^+) + \frac{3}{2h}\left[(\zeta^-)^2 - (\zeta^+)^2\right] + \frac{h^2}{3}\left[(\zeta^-)_{xx} - (\zeta^+)_{xx}\right]}{\zeta^- - \zeta^+} \\ &= \left(\frac{gh}{c}\right)\left[1 + \frac{3}{2h}(\zeta^- + \zeta^+) + \left(\frac{h^2}{3}\right)\frac{(\zeta^-)_{xx} - (\zeta^+)_{xx}}{\zeta^- - \zeta^+}\right] \\ &= \left(\frac{gh}{c}\right)\left[1 + \frac{3}{2h}(\zeta^- + \zeta^+) + \left(\frac{h^2}{3}\right)\frac{(\zeta^-)_{xxx} - (\zeta^+)_{xxx}}{(\zeta^-)_x - (\zeta^+)_x}\right], \end{aligned} \tag{53}$$

which provides us the Rankine-Hogoniot jump condition of the Boussinesq equation

$$\frac{c^2}{gh} = 1 + \frac{3}{2h}(\zeta^- + \zeta^+) + \left(\frac{h^2}{3}\right)\frac{(\zeta^-)_{xxx} - (\zeta^+)_{xxx}}{(\zeta^-)_x - (\zeta^+)_x}, \quad \text{as } x \to c\,t + x_0. \tag{54}$$

For the peaked solitary wave (49), we have at the crest that

$$\zeta^- = h\left(\frac{c^2}{gh} - 1\right)\operatorname{sech}^2(\alpha), \tag{55}$$

$$(\zeta^-)_x = \sqrt{3}\left(\frac{c^2}{gh} - 1\right)^{3/2} \operatorname{sech}^2(\alpha)\tanh(\alpha), \tag{56}$$

$$(\zeta^-)_{xxx} = \frac{3\sqrt{3}}{4h^2}\left(\frac{c^2}{gh} - 1\right)^{5/2} \operatorname{sech}^2(\alpha) \times [\sinh(3\alpha) - 11\sinh(\alpha)], \tag{57}$$

and

$$(\zeta^+)_x = -(\zeta^+)_x, \quad (\zeta^+)_{xxx} = -(\zeta^+)_{xxx}.$$

Substituting all of these into eq. (54), we have

$$\frac{1}{4}\left(\frac{c^2}{gh} - 1\right)\left[\operatorname{sech}^2(\alpha) + \operatorname{sech}^2(\alpha)\operatorname{csch}(\alpha)\sinh(3\alpha) - 4\right] = 0, \tag{58}$$

which is an identity for arbitrary constant $\alpha \geqslant 0$. Similarly, it is found that eq. (50) satisfies the Rankine-Hogoniot jump condition (54) for arbitrary constant $\beta > 0$. Thus, the peaked solitary waves (49) and (50) should be understood as weak solutions of Boussinesq equation (48).

6 Conclusion and discussion

In this article, we give, for the first time, the closed-form solutions of the peaked solitary waves of the KdV equation [3], the modified KdV equation [15, 16], the BBM equation [4], and Boussinesq equation [2], respectively. All of them exactly satisfy the corresponding Rankine-Hogoniot jump condition for *arbitrary* constant $\alpha \geqslant 0$ or $\beta > 0$. Note that the 1st-derivative ζ_x of the elevation has a jump at the crest. Obviously, for all peaked solitary waves found in this article, $\zeta_x > 0$ on the left of the crest, but $\zeta_x < 0$ on the right of the crest, respectively. In other words, $\zeta_x^- > \zeta_x^+$. So, these peaked solutions also satisfy the so-called entropy condition. Thus, they could be understood as weak solutions.

Note that the traditional smooth solitary waves are just special cases of the peaked solitary waves when $\alpha = 0$. So, these weak solutions are more general. Besides, unlike the smooth solitary waves whose phase speed strongly depends upon wave amplitude, the peaked solitary waves (when $\alpha > 0$ or $\beta > 0$) have nothing to do with the wave amplitude, for example as shown in Figures 1 and 2. This is quite different from the traditional solitary waves. In addition, the solitary waves with negative elevation such as shown in Figure 2 have never been reported for these mainstream models of shallow waves. All of these show the novelty of these peaked solitary waves. If these peaked waves as weak solutions indeed exist and have physical meanings, then nearly all mainstream models for shallow water waves, including the KdV equation [3], the modified KdV equation [15, 16], the BBM equation [4], and Boussinesq equation [2] and the CH equation [5], have the peaked solitary waves, no matter whether or not they are integral and admit breaking-wave solutions. Thus, the peaked solitary waves might be a common property of shallow water wave models. As shown by Liao [17], even the exactly nonlinear water wave equations also admit the peaked solitary waves: this might reveal the origin of the peaked solitary waves in shallow water predicted by these mainstream models.

Certainly, further investigations on these peaked solitary waves are needed, especially the stability of them, the interactions between multiple peaked solitary waves, and so on. Note that these peaked solitary waves have discontinuous 1st derivative at crest so that their higher derivatives at crest tend to infinity and the perturbation theory does not work. Thus, strictly speaking, the validity of the KdV equation [3], the modified KdV equation [15, 16], the BBM equation [4], and Boussinesq equation [2] should be verified carefully in the meaning of the weak solution (24). Note that the Rankine-Hogoniot jump condition and entropy condition are only necessary conditions for a weak solution. So, there are some open questions. For example, how do these peaked solitary waves come into being? How to directly prove that they are indeed weak solutions satisfying (24)?

Note that the peaked solitary waves have never been observed experimentally and in practice. Obviously, it is more difficult to create and remain such kinds of peaked solitary waves than the traditional smooth ones. So, it is an challenging work to observe these peaked solitary waves in experiments.

It should be emphasized that the discontinuity and/or singularity exist widely in natural phenomena, such as dam break in hydrodynamics, shock waves in aerodynamics, black holes described by general relativity and so on. Indeed, the discontinuity and/or singularity are rather difficult to handle by traditional methods. But, the discontinuity and/or singularity can greatly enrich and deepen our understandings about the real world, and therefore should not be evaded easily.

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