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## Homotopy Analysis Method: A New Analytical Technique for Nonlinear Problems<sup>1</sup>

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**Abstract:** In this paper, the basic ideas of a new kind of analytical technique, namely the Homotopy Analysis Method (HAM), are briefly described. Different from perturbation techniques, the HAM does not depend on whether or not there exist small parameters in nonlinear equations under consideration. Therefore, it provides us with a powerful tool to analyse strongly nonlinear problems. A simple but typical example is used to illustrate the validity and the great potential of the HAM. Moreover, a pure mathematical theorem, namely the General Taylor Theorem, is given in appendix, which provides us with some rational knowledge for the validity of this new analytical technique.

**Key Words:** Nonlinear analytical technique, strong nonlinearity, homotopy, topology

### Introduction

Although the rapid development of digital computers makes it easier and easier to numerically solve nonlinear problems, it is still rather difficult to give analytical expressions of them. Most of our nonlinear analytical techniques are unsatisfactory. For instance,

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<sup>1</sup>The paper was received on Apr. 28, 1997

perturbation techniques are widely applied to analyse nonlinear problems in science and engineering. Unfortunately, perturbation techniques are so strongly dependent on small parameters appearing in equations under consideration that they are restricted only to weakly nonlinear problems. For strongly nonlinear problems which don't contain any small parameters, perturbation techniques are completely invalid. So, it seems necessary and worthwhile developing a new kind of analytical technique independent of small parameters.

Liao [1-6] has made some attempts in this direction. Liao [1,2] proposed a new analytical technique in his Ph. D. dissertation, namely the Homotopy Analysis Method (HAM). Based on homotopy of topology, the validity of the HAM is independent of whether or not there exists small parameters in the considered equations. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides us with a possibility to analyse strongly nonlinear problems. Liao [3,4,5,6] used the HAM to successfully solve some nonlinear problems, while he has been making unremitting efforts to improve this method step by step. Moreover, Liao [7,8] and Liao & Chwang [9] applied the basic ideas of the HAM to propose the so-called general boundary element method (GBEM), which is valid even for those nonlinear problems whose governing equations and boundary conditions don't contain any linear terms so that it greatly generalizes the traditional BEM. Furthermore, Liao [10] even applied the HAM to obtain a pure mathematical theorem, namely the General Taylor Theorem, which can be applied to greatly enlarge the convergence radius of a traditional Taylor series. All of these verify the validity and the great potential of the HAM.

In this paper, we use a typical example to briefly introduce the basic ideas of the HAM and to further show its validity and great potential.

## 1. Basic ideas of the Homotopy Analysis Method

Consider the following equation

$$u'(t) + 2tu^2(t) = 0, \quad u(0) = 1, \quad (1)$$

whose exact solution is

$$u(t) = \frac{1}{1+t^2}. \quad (2)$$

Assuming  $t$  is small, we can obtain by perturbation techniques the following power series

$$u(t) = 1 - t^2 + t^4 - t^6 + \dots = \sum_{k=0}^{+\infty} (-1)^k t^{2k}, \quad (3)$$

whose convergence radius is however rather small, say,  $\rho = 1$ , so that the above perturbation approximation is nearly worthless.

Now we apply the HAM to solve it. First, we construct such a continuous mapping  $U(t, p, \hbar) : [0, +\infty) \times [0, 1] \times \mathfrak{R}_0 \rightarrow \mathfrak{R}$ , governed by

$$(1-p) \frac{\partial U(t, p, \hbar)}{\partial t} = \hbar p \left[ \frac{\partial U(t, p, \hbar)}{\partial t} + 2tU^2(t, p, \hbar) \right] \quad p \in [0, 1], \hbar \neq 0 \quad (4)$$

with boundary condition

$$U(0, p, \hbar) = 1 \quad (5)$$

so that

$$U(t, 0, \hbar) = 1, U(t, 1, \hbar) = u(t) = \frac{1}{1+t^2}, \quad (6)$$

where  $\mathfrak{R} = (-\infty, +\infty)$  and  $\mathfrak{R}_0 = (-\infty, 0) \cup (0, +\infty)$ . Note that we here let  $\hbar$  be a nonzero real number. Clearly, as  $p$  increases from 0 to 1,  $U(t, p, \hbar)$  varies continuously from  $U(t, 0, \hbar) = 1$  to the exact solution  $u(t) = (1 + t^2)^{-1}$ . This kind of variation is named *deformation* in topology so that we call Eqs. (4), (5) *the zeroth-order deformation equations*. Second, assuming that  $U(t, p, \hbar)$  is so smooth that

$$u_0^{[k]}(t, \hbar) = \left. \frac{\partial^k U(t, p, \hbar)}{\partial p^k} \right|_{p=0},$$

namely *the  $k$ th-order deformation derivatives* ( $k \geq 1$ ), exists, and moreover, the corresponding Maclaurin series of  $U(t, p, \hbar)$  at  $p = 0$ , say,

$$U(t, 0, \hbar) + \sum_{k=1}^{+\infty} \frac{u_0^{[k]}(t, \hbar)}{k!} p^k, \quad (7)$$

converges at  $p = 1$ , then, we have by (6) that

$$u(t) = 1 + \sum_{k=1}^{+\infty} \frac{u_0^{[k]}(t, \hbar)}{k!}. \quad (8)$$

Here,  $u_0^{[k]}(t, \hbar)$  ( $k \geq 1$ ) is governed by the  $k$ th-order deformation equations

$$\frac{\partial u_0^{[1]}(t, \hbar)}{\partial t} = 2\hbar t \quad (9)$$

$$\frac{\partial u_0^{[k]}(t, \hbar)}{\partial t} = k \left[ (1 + \hbar) \frac{\partial u_0^{[k-1]}(t, \hbar)}{\partial t} + 2t \sum_{m=0}^{k-1} \binom{k-1}{m} u_0^{[m]}(t, \hbar) u_0^{[k-1-m]}(t, \hbar) \right] \quad (k \geq 2) \quad (10)$$

under boundary condition

$$u_0^{[k]}(0, \hbar) = 0 \quad (k \geq 1), \quad (11)$$

which are given by differentiating the zeroth-order deformation equations (4), (5)  $k$  times with respect to  $p$  and then setting  $p = 0$ . The above *linear* first-order differential equation can be rather easily solved. So, substituting these solutions into (8), we obtain

$$u(t) = \lim_{m \rightarrow +\infty} \sum_{k=0}^m [(-1)^k t^{2k}] \Phi_{m,k}(\hbar), \quad (12)$$

where

$$\Phi_{m,k}(\hbar) = \begin{cases} 0, & k > m \\ (-\hbar)^k \sum_{i=0}^{m-k} \binom{m}{m-k-i} \binom{k+i-1}{i} \hbar^i, & 1 \leq k \leq m \\ 1, & k \leq 0 \end{cases} \quad (13)$$

namely *the approaching function*.

The approaching function  $\Phi_{m,k}(\hbar)$  has rather general meanings. Liao [10] rigorously proved that  $\Phi_{m,k}(\hbar)$  ( $1 \leq k \leq m$ ) has the following properties

$$[\Phi_{m,k}(\hbar)]' = (-1)^k k \binom{m}{k} \hbar^{k-1} (1 + \hbar)^{m-k}, \quad (14)$$

$$\Phi_{m+1,k}(\hbar) - \Phi_{m,k}(\hbar) = \binom{m}{k-1} (-\hbar)^k (1+\hbar)^{m-k+1}, \quad (15)$$

$$\Phi_{m,k}(-1) = 1. \quad (16)$$

And moreover, for finite positive integer  $\nu$ , it holds

$$\lim_{m \rightarrow +\infty} \Phi_{m,\nu}(\hbar) = 1, \quad (|1+\hbar| < 1). \quad (17)$$

The above properties hold for complex variable  $\hbar$ . Furthermore, Liao [10] rigorously proved the so-called General Taylor Theorem which is simply recited in Appendix of this paper. Clearly, function  $f(z) = 1/(1+z^2)$  has two singularities  $\xi_1 = i$  and  $\xi_2 = -i$ . So, according to the so-called General Taylor Theorem, expression (12) is valid in the region

$$|t| < \sqrt{\frac{2}{|\hbar|}} - 1, \quad (-2 < \hbar < 0). \quad (18)$$

## 2. Discussions and conclusions

Now, let us compare the perturbation approximation (3) with the solution (12). The former converges only in the region  $|t| < 1$ , but the latter converges in a region which is however a function of  $\hbar$ . First, according to (16) and (18), in case of  $\hbar = -1$ , (12) gives the same result as (3), so that (3) is just a special member of the family of the approximations (12). It means that (12) contains the perturbation approximation (3) in logic. This kind of continuation in logic has proved rather important in both science and mathematics. Second, as  $\hbar$  increases from  $-1$  to  $0$ , the convergence radius of power series (12) becomes larger and larger. In limit  $\hbar \rightarrow 0$ , (12) is valid in the whole real axis! However, as  $\hbar$  decreases from  $-1$  to  $-2$ , the convergence radius becomes smaller and smaller. So, the perturbation approximation (3) is neither the best nor the worst: it's a special but common member among the family of approximations (12). At last, we emphasize that the whole approach mentioned above does not need the small parameter assumption. In other words, the validity of the HAM is independent of whether or not there exist small parameters in the problems considered. In form, the HAM seems able to be applied to any nonlinear problems, although there certainly should exist some restrictions for its applications which we currently don't know clearly.

Different from the perturbation approximation (3), (12) denotes now a *family* of approximations. The convergence radius of (12) is determined by the value of  $\hbar$  which is introduced to construct the so-called zeroth-order deformation equations (4) and (5). Clearly, different values of  $\hbar$  correspond to different continuous mappings, or more precisely, different homotopies. In fact, Eqs. (4) and (5) construct a *family* of homotopies  $U(t, p, \hbar)$  dependent on  $\hbar$ . Certainly, some among them are 'better', some are worse. This can well explain why the convergence radius of (12) is a function of  $\hbar$ . The introducing of parameter  $\hbar$  provides us with larger freedom to get more and even better approximations.

The foregoing example verifies the validity and great potential of the Homotopy Analysis Method, although it certainly needs more applications and further improvements.

## Appendix

**General Taylor Theorem** Let  $\nu$  be finite positive integer and  $\hbar$  be complex number.

If complex function  $f(z)$  is analytical at  $z = z_0$ , then

$$\begin{aligned} f(z) &= \lim_{m \rightarrow +\infty} \sum_{k=0}^m \left[ \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] \Phi_{m,k}(\hbar) \\ &= \lim_{m \rightarrow +\infty} \sum_{k=0}^{m+\nu} \left[ \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] \Phi_{m,k-\nu}(\hbar) \\ &= \lim_{m \rightarrow +\infty} \sum_{k=0}^m \left[ \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] \Phi_{m+\nu,k+\nu}(\hbar) \end{aligned}$$

holds in the region

$$\bigcap_{k \in I} \left| 1 + \hbar - \hbar \left( \frac{z - z_0}{\xi_k - z_0} \right) \right| < 1, \quad (|1 + \hbar| < 1),$$

where  $\Phi_{m,k}(\hbar)$  is defined by (13) and  $\xi_k$  ( $k \in I$ ) denotes the set of all singular points of  $f(z)$ . Moreover, it becomes just the traditional Taylor theorem in case  $\hbar = -1$ .

**General Newtonian Binomial Theorem** Let  $\alpha$ ,  $t$  and  $\hbar$  be real numbers. The equality

$$(1+t)^\alpha = 1 + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \left[ \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} t^k \right] \Phi_{m,k}(\hbar) \quad (\alpha \neq 0, 1, 2, 3, \dots)$$

holds in the region

$$-1 < t < \frac{2}{|\hbar|} - 1, \quad (-2 < \hbar < 0),$$

where  $\Phi_{m,k}(\hbar)$  is defined by (13).

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## An Approach in Modeling Cavitating Flows with Gravity Effect<sup>1,2</sup>

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**Abstract:** Numerical investigations on submerged cavitating slender axisymmetric bodies with gravity effect are conducted in this paper. The developed cavity shape and the hydrodynamic characteristics on cavitating axisymmetric bodies are calculated. While these cavities originate from both the shoulder and the tail of bodies, we focus our main attention on the cavities originated from the shoulder. The boundary element method is used to solve the potential flow problem, and a new iteration method named Adaptive Modified Newton Iteration is developed to determine the cavity shape.

**Key words:** partial cavitation, axisymmetric bodies, iteration method, gravity effect

### 1. Introduction

The hydrodynamic characteristics on a submerged, fully cavitating bodies pose several difficult problems. Many factors work together to make this problem more complex. Lemonnier, H. & Rowe, A.<sup>[1]</sup> have studied cavitating flows in which allowance was made for the presence of an axial gravity field, but no detailed exposition was provided.

A cavitating-flow calculation method is presented, which is an attempt at considering the gravity effect. The cavitating flow around an axisymmetrical body is in general a three-dimensional problem. The study is limited to the case of zero angle of attack and assumes that the symmetry axis of the body is identical to the direction of the gravity.

### 2. Problem Formulation

Under the assumption of incompressible, irrotational and inviscid flow, the governing equation for the potential function  $\Phi_{tot}$  is given by Laplace's equation. The potential function can be subdivided into two parts, one part  $\Phi_s$  due to the free stream, and the other part  $\Phi$  due to the perturbation caused by the body and the cavity. The governing equation for the perturbation part can be written as follows:

$$\nabla^2 \Phi = 0 \quad (1)$$

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<sup>1</sup>The paper was received on Feb. 26, 1997

<sup>2</sup>Sponsored by the National Natural Science Foundation of China, the Scientific Research Foundation of Doctoral Discipline of National Education Committee and the National Defence Foundation of Science and Technology Pre-research.