Series solution of nonlinear eigenvalue problems by means of the homotopy analysis method

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ABSTRACT

A general analytic approach for nonlinear eigenvalue problems is described. Two physical problems are used as examples to show the validity of this approach for eigenvalue problems with either periodic or non-periodic eigenfunctions. Unlike perturbation techniques, this approach is independent of any small physical parameters. Besides, different from all other analytic techniques, it provides a simple way to ensure the convergence of series of eigenvalues and eigenfunctions so that one can always get accurate enough approximations. Finally, unlike all other analytic techniques, this approach provides great freedom to choose an auxiliary linear operator so as to approximate the eigenfunction more effectively by means of better base functions. This approach provides us a new way to investigate eigenvalue problems with strong nonlinearity.

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1. Introduction

There exist many nonlinear eigenvalue problems in science and engineering. Generally speaking, nonlinear eigenvalue problems are much more difficult to solve than linear ones. Many nonlinear eigenvalue problems have multiple eigenvalues and eigenfunctions. However, even by means of numerical techniques, it is difficult to find all multiple solutions of a nonlinear differential equation.

There are some analytic techniques for nonlinear eigenvalue problems, which are based on either perturbation techniques [1–6], or traditional non-perturbation methods such as the Adomian decomposition method [7–10], Lyapunov artificial small parameter method [11], and so on. It is well known that perturbation techniques are too strongly dependent upon small physical parameters. Besides, convergence radius of perturbation series is often small, so that perturbation approximations are valid in general only for problems with weak nonlinearity. In form, the traditional non-perturbation techniques, such as the Adomian decomposition method and the Lyapunov artificial small parameter method, seem independent of small physical parameters. However, like perturbation techniques, they are in fact valid only for weakly nonlinear problems. This is mainly because both the perturbation techniques and the traditional non-perturbation methods do not provide a way to ensure the convergence of solution series. Besides, by means of perturbation and non-perturbation techniques, one has no freedom to choose better base functions so as to express the eigenfunction more effectively.

Homotopy is a fundamental concept in topology and differential geometry, which can be traced back to Poincaré [12]. Based on homotopy, an analytic technique, namely the homotopy analysis method (HAM) [13–17], is proposed and widely applied to solve strongly nonlinear problems in science, engineering and finance [18–27]. Unlike perturbation techniques, the homotopy analysis method is independent of any small physical parameters. Besides being different from perturbation and traditional non-perturbation methods, it provides a simple way to ensure the convergence of solution series so that one can always get accurate enough approximations even for strongly nonlinear problems. Furthermore, unlike all other analytic techniques, the homotopy analysis method provides great freedom to choose the so-called auxiliary linear operator

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so that one can approximate a nonlinear problem more effectively by means of better base functions [16]. By means of the homotopy analysis method, a few new solutions of some nonlinear problems [28,29] are found, which are neglected by all other analytic methods and even by numerical techniques. In sum, the homotopy analysis method is independent of small physical parameters, and is valid for strongly nonlinear problems.

In this paper, a general analytic approach for nonlinear eigenvalue problems is described by means of two examples. In Section 2, the famous Euler problem of a beam is solved, whose eigenfunction is periodic. The closed-form exact solution [30] of the eigenvalue and eigenfunction of this problem can be expressed by elliptic integrals and Jacobi elliptic functions, so that our series solutions can be compared with the exact solutions. In Section 3, a nonlinear eigenvalue problem related to the collapse of optical pulse is investigated, whose eigenfunction is non-periodic. To show the validity of our approach, our series solutions are compared with exact or numerical ones in Sections 2 and 3. In Section 4, discussions and conclusions are given.

2. Problems with periodic eigenfunction

For an example of periodic eigenvalues, let us consider a beam with uniform cross-section acted by an axial load P, governed by

$$\frac{d\theta}{d\xi} = -\frac{Pu}{EI}, \qquad u(0) = u(l) = 0,$$

where θ denotes the rotation of cross-section and u its deflection, ξ is the arc-coordinate of the natural axis, l the length of the beam, I the moment of inertia, E the Young's Modulus of the beam material, respectively. Differentiating the above equation with ξ and using the relationship $\sin\theta = du/d\xi$, we have

$$\theta''(\xi) + \frac{P}{EI}\sin[\theta(\xi)] = 0, \qquad \theta'(0) = \theta'(l) = 0.$$

Defining the dimensionless arc-coordinate of the natural axis

$$s = \left(\frac{\xi}{l}\right)\pi,$$

we have

$$\theta''(s) + \lambda\sin[\theta(s)] = 0, \qquad \theta'(0) = \theta'(\pi) = 0, \tag{1}$$

where the eigenvalue λ is defined by

$$\lambda = \frac{P}{EI}\left(\frac{l}{\pi}\right)^2. \tag{2}$$

So, as long as the eigenvalue λ is given, one has the axial load

$$P = \lambda(EI)\left(\frac{\pi}{l}\right)^2.$$

2.1. Approach based on the HAM

Let us consider the non-zero solution $\theta(s)$ with $\theta(0) = \alpha$. For a given non-zero value of α, we should find the corresponding eigenfunction $\theta(s)$ and the eigenvalue λ that is related to the axial load P through (2). Without loss of any generality, we rewrite the equation as

$$\theta''(s) + \lambda\sin[\theta(s)] = 0, \quad 0 < s < \pi, \tag{3}$$

subject to the boundary conditions

$$\theta'(0) = \theta'(\pi) = 0, \quad \theta(0) = \alpha. \tag{4}$$

The above nonlinear differential equation has a closed-form exact solution[1]:

$$\lambda = \frac{4\kappa^2 K^2(m^*)}{\pi^2}, \quad \kappa = 1, 2, 3, \ldots, \tag{5}$$

$$\theta(s) = -2\sin^{-1}\left[m^*\operatorname{sn}\left(\sqrt{\lambda}\,s - K(m^*), m^*\right)\right], \tag{6}$$

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[1] This closed-form solution was given by the anonymous reviewer. I would like to express my sincere thanks to the reviewer for this.

where $m^* = \sin(\alpha/2)$, $K(m^*)$ is the complete elliptic integral of the first kind, and sn is a kind of Jacobi elliptic function. For details, please refer to Timoshenko and Gere [30].

According to Eqs. (3) and (4), it is obvious that the eigenfunction $\theta(s)$ can be expressed by the periodic base functions

$$\{\cos(n \kappa s) \mid n \geq 1, \kappa \geq 1, n \in \mathbf{N}, \kappa \in \mathbf{N}\} \tag{7}$$

as

$$\theta(s) = \sum_{n=1}^{+\infty} b_n \cos(n\kappa s), \tag{8}$$

where b_n is a coefficient, and $\kappa \geq 1$ is an integer. Note that the above expression automatically satisfies the boundary condition $\theta'(0) = \theta'(\pi) = 0$. It provides us the so-called *Solution Expression* of $\theta(s)$.

According to Eq. (3), we define a nonlinear operator

$$\mathcal{N}[\phi(s; q), \Lambda(q)] = \frac{\partial^2 \phi(s; q)}{\partial s^2} + \Lambda(q) \left(\frac{\sin[q\phi(s; q)]}{q} \right), \tag{9}$$

where $q \in [0, 1]$ denotes the embedding parameter. Let $\theta_0(s)$ denote a initial guess of the eigenfunction, which satisfies the boundary conditions (4). Besides, let $\mathcal{L}$ denote an auxiliary linear operator, and $\hbar$ a non-zero auxiliary parameter, respectively. All of $\theta_0(s)$, $\mathcal{L}$ and $\hbar$ will be chosen later with great freedom. Then, we construct a two-parameter family of differential equations

$$(1 - q)\mathcal{L}[\phi(s; q) - \theta_0(s)] = q \hbar \mathcal{N}[\phi(s; q), \Gamma(q)], \quad 0 < s < \pi, \tag{10}$$

subject to the boundary conditions

$$\frac{\partial \phi(s; q)}{\partial s}\bigg|_{s=0} = \frac{\partial \phi(s; q)}{\partial s}\bigg|_{s=\pi} = 0, \qquad \phi(0; q) = \alpha. \tag{11}$$

Obviously, when $q = 0$, because of the property $\mathcal{L}(0) = 0$ of any linear operator $\mathcal{L}$, Eqs. (10) and (11) have the solution

$$\phi(s; 0) = \theta_0(s). \tag{12}$$

When $q = 1$, since $\hbar \neq 0$, Eqs. (10) and (11) are equivalent to the original ones, (3) and (4), provided

$$\phi(s; 1) = \theta(s), \qquad \Lambda(1) = \lambda. \tag{13}$$

Write

$$\Lambda(0) = \lambda_0, \tag{14}$$

where λ_0 denotes the initial guess of the eigenvalue. Then, as q increases from 0 to 1, $\phi(s; q)$ varies (or deforms) from the initial guess $\theta_0(s)$ to the eigenfunction $\theta(s)$, so does $\Lambda(q)$ from the initial guess λ_0 to the eigenvalue λ. These kinds of deformations $\phi(s; q)$ and $\Lambda(q)$ are totally determined by the so-called *zeroth-order deformation equations* (10) and (11).

Then, expanding $\phi(s; q)$ and $\Lambda(q)$ into Taylor series with respect to the embedding parameter q, we have, using (12) and (14), the so-called *homotopy-series*

$$\phi(s; q) = \theta_0(s) + \sum_{m=1}^{+\infty} \theta_m q^m, \tag{15}$$

$$\Lambda(q) = \lambda_0 + \sum_{m=1}^{+\infty} \lambda_m q^m, \tag{16}$$

where

$$\theta_m(s) = D_m[\phi(s; q)], \qquad \lambda_m = D_m[\Lambda(q)], \tag{17}$$

in which

$$D_m f = \frac{1}{m!} \frac{\partial^m f}{\partial q^m}\bigg|_{q=0} \tag{18}$$

is called *the mth-order homotopy-derivative* of f.

Note that, in general, the radius of convergence of a power series is finite. Fortunately, the homotopy-series (15) and (16) contain an auxiliary parameter $\hbar$, and besides we have great freedom to choose the auxiliary linear operator $\mathcal{L}$, as illustrated by Liao [16]. Assume that the auxiliary linear operator $\mathcal{L}$ and the auxiliary parameter $\hbar$ are properly chosen so

that the homotopy-series (15) and (16) are convergent at $q = 1$. Then, using (13), we have the so-called *homotopy-series solution*

$$\theta(s) = \theta_0(s) + \sum_{m=1}^{+\infty} \theta_m,\tag{19}$$

$$\lambda = \lambda_0 + \sum_{m=1}^{+\infty} \lambda_m.\tag{20}$$

According to the fundamental theorems in calculus, each coefficient of Taylor series of a function is unique. Thus, $\theta_m(s)$ and λ_m are unique, and are determined by $\phi(s; q)$ and $\Lambda(q)$, respectively. Therefore, the governing equations and boundary conditions of $\theta_m(s)$ and λ_m can be deduced from the zeroth-order deformation equations (10) and (11). For simplicity, write

$$\vec{\theta}_n = \{\theta_0(s), \theta_1(s), \theta_2(s), \ldots, \theta_n(s)\},$$
$$\vec{\lambda}_n = \{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}.$$

Differentiating the zeroth-order deformation equations (10) and (11) m times with respect to q, then setting $q = 0$, and finally dividing by $m!$, we have the so-called *mth-order deformation equation*:

$$\mathcal{L}[\theta_m(s) - \chi_m \theta_{m-1}(s)] = \hbar R_m(\vec{\theta}_{m-1}, \vec{\lambda}_{m-1}),\tag{21}$$

subject to the boundary conditions

$$\theta'_m(0) = \theta'_m(\pi) = 0, \qquad \theta_m(0) = 0, \quad m \geq 1\tag{22}$$

where

$$R_m(\vec{\theta}_{m-1}, \vec{\lambda}_{m-1}) = \frac{1}{(m-1)!} \frac{d^{m-1}\mathcal{N}[\phi(s; q), \Gamma(q)]}{dq^{m-1}}\bigg|_{q=0}\tag{23}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}\tag{24}$$

Using the definitions (9) and (23), and by means of Leibnitz's rule for derivatives of product, we have the explicit expression

$$R_m(\vec{\theta}_{m-1}, \vec{\lambda}_{m-1}) = \theta''_{m-1}(s) + \sum_{n=0}^{m-1} \lambda_n S_{m-n}(s),\tag{25}$$

with the following recurrence formulas

$$S_n = \frac{(A_n - B_n)}{2i}, \quad n \geq 0,$$
$$A_0 = 1, \quad B_0 = 1,$$
$$A_n = i \sum_{m=0}^{n-1} \left(1 - \frac{m}{n}\right) A_m \theta_{n-m-1}, \quad n \geq 1,$$
$$B_n = -i \sum_{m=0}^{n-1} \left(1 - \frac{m}{n}\right) B_m \theta_{n-m-1}, \quad n \geq 1,$$

in which $i = \sqrt{-1}$. For details, please refer to Liao [17]. According to the above recurrence formula, we have

$$R_1(\vec{\theta}_0, \vec{\lambda}_0) = \theta''_0(s) + \lambda_0 \theta_0(s),\tag{26}$$

$$R_2(\vec{\theta}_1, \vec{\lambda}_1) = \theta''_1(s) + \lambda_0 \theta_1(s) + \lambda_1 \theta_0(s),\tag{27}$$

$$R_3(\vec{\theta}_2, \vec{\lambda}_2) = \theta''_2(s) + \lambda_0 \theta_2(s) + \lambda_1 \theta_1(s) + \lambda_2 \theta_0(s) - \frac{\lambda_0}{6} \theta_0^3(s),\tag{28}$$

$$\vdots$$

and so on. So, by means of symbolic computation software such as Mathematica, Maple, MathLab and so on, it is not difficult to get $R_m(\vec{\theta}_{m-1}, \vec{\lambda}_{m-1})$ for large value of m.

Note that the mth-order deformation equations (21) and (22) are linear ODEs. So, according to (19) and (20), the original nonlinear eigenvalue problem is transferred into an infinite number of linear ODEs (21) and (22). However, unlike

perturbation techniques, we do *not* need any small physical parameters to do such a kind of transformation. Besides, unlike the traditional "non-perturbation techniques", we have great freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $\theta_0(s)$.

Both the auxiliary linear operator $\mathcal{L}$ and the initial guess $\theta_0(s)$ are chosen under the so-called *Rule of Solution Expression*: the auxiliary linear operator $\mathcal{L}$ and the initial guess $\theta_0(s)$ must be chosen so that the solutions of the high-order deformation equations (21) and (22) exist and besides they obey the Solution Expression (8). So, for the solutions to obey the Solution Expression (8) and the boundary condition (4), we choose the initial guess of the eigenfunction:

$$\theta_0(s) = \alpha \cos(\kappa s). \tag{29}$$

Because the original equation (3) is of 2nd order, it is natural for us to choose such a 2nd-order auxiliary linear operator

$$\mathcal{L}u = u'' + A_1(s)u' + A_0(s)u, \tag{30}$$

where $A_0(s)$ and $A_1(s)$ are unknown real functions to be determined. Let $f_1(s), f_2(s)$ be the solutions of $\mathcal{L}u = 0$, i.e.

$$\mathcal{L}[f_1(s)] = 0, \qquad \mathcal{L}[f_2(s)] = 0. \tag{31}$$

Let $\hat{\theta}_m(s)$ denote a special solution of Eq. (21), then its general solution is expressed by

$$\theta_m(s) = \hat{\theta}_m(s) + \chi_m \theta_{m-1}(s) + C_1 f_1(s) + C_2 f_2(s).$$

For the solution to obey the Solution Expression (8), we must choose $f_1(s) = \cos(\kappa s)$. Because the coefficients $A_0(s)$ and $A_1(s)$ are real functions, we must choose $f_2(s) = \sin(\kappa s)$, according to the fundamental theorems of linear differential equations. Substituting $f_1(s) = \cos(\kappa s)$ and $f_2(s) = \sin(\kappa s)$ into (31) with the definition (30), we have

$$[A_0(s) - \kappa^2] \cos(\kappa s) - \kappa A_1(s) \sin(\kappa s) = 0,$$
$$[A_0(s) - \kappa^2] \sin(\kappa s) + \kappa A_1(s) \cos(\kappa s) = 0,$$

which give

$$A_0(s) = \kappa^2, \qquad A_1(s) = 0.$$

Substituting them into (30), we have the auxiliary linear operator

$$\mathcal{L}(u) = u'' + \kappa^2 u, \tag{32}$$

with the property

$$\mathcal{L}[C_1 \cos(\kappa s) + C_2 \sin(\kappa s)] = 0. \tag{33}$$

Then, the general solution of Eq. (21) is given by

$$\theta_m(s) = \hat{\theta}_m(s) + \chi_m \theta_{m-1}(s) + C_1 \cos(\kappa s) + C_2 \sin(\kappa s).$$

Using the boundary condition $\theta'_m(0) = \theta'_m(\pi) = 0$, we have $C_2 = 0$. To satisfy the boundary condition $\theta_m(0) = 0$, we have

$$C_1 = -\hat{\theta}_m(0) - \chi_m \theta_{m-1}(0).$$

Using the definition (29) of $\theta_0(s)$, one has

$$R_1(\vec{\theta}_0, \vec{\lambda}_0) = \alpha(\lambda_0 - \kappa^2) \cos(\kappa s).$$

Due to the property (33) of the auxiliary operator $\mathcal{L}$, if $\lambda_0 \neq \kappa^2$, the 1st-order deformation equation contains the so-called secular term $s \cos(\kappa s)$, which is non-periodic and thus disobeys the periodic Solution Expression (8). Thus, for the solution to obey the Solution Expression (8), it must hold

$$\lambda_0 = \kappa^2. \tag{34}$$

Then, it is easy to get the solution $\theta_1(s) = 0$ of the 1st-order deformation equations (21) and (22). Similarly, one can get λ_1 and $\theta_2(s)$, and so on. In general, it holds

$$R_m(\vec{\theta}_{m-1}, \vec{\lambda}_{m-1}) = A_{m,1}(\vec{\lambda}_{m-1}) \cos(\kappa s) + A_{m,2}(\vec{\lambda}_{m-1}) \cos(2\kappa s) + \cdots,$$

where $A_{m,n}(\vec{\lambda}_{m-1})$ is a coefficient dependent upon $\vec{\lambda}_{m-1}$. In general, for the solution to obey the Solution Expression (8), we had to enforce $A_{m,1}(\vec{\lambda}_{m-1}) = 0$, which gives an additional algebraic equation to determine λ_{m-1}. Thereafter, one can get $\theta_m(s)$ by solving the mth-order deformation equations (21) and (22), as mentioned above. In this way, we obtain $\lambda_0, \theta_1, \lambda_1, \theta_2, \lambda_3, \theta_3(s), \ldots$, one after the other. This is easy to do by means of the symbolic computation software.

Note that the linear term of the governing equation (3) is θ'', which corresponds to the linear operator $\mathcal{L}u = u''$. However, we choose the auxiliary linear operator $\mathcal{L}u = u'' + \kappa^2 u$, which has no close relationship with the original nonlinear equation (3). This is mainly because, different from all other analytic techniques, our approach provides us with great freedom to choose the auxiliary linear operator $\mathcal{L}$ so as to approximate the eigenfunction by the chosen base functions.
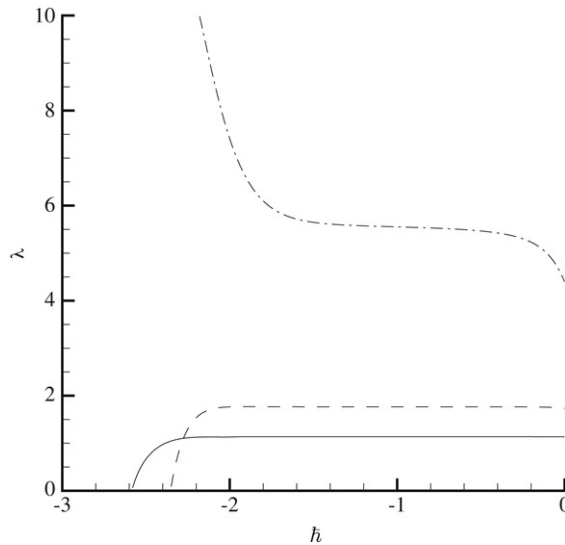
Fig. 1. Curves of $\lambda \sim \hbar$ at the 20th order of approximation. Solid line: $\kappa = 1, \alpha = 1$; Dashed line: $\kappa = 1, \alpha = 2$; Dash-dotted line: $\kappa = 1, \alpha = 3$.

2.2. Solution series

Let (x, u) denote the coordinate of a point of the beam, where $u(x)$ is the displacement of the beam. Then, it holds

$$x(s) = \int_0^s \cos[\theta(s)] \, ds, \qquad u(s) = \int_0^s \sin[\theta(s)] \, ds,$$

where $0 \le s \le \pi$ is the arc length of the beam. Obviously, if $\theta(s)$ is a solution for given $\alpha = \beta$, then $-\theta(s)$ is also a solution for $\alpha = -\beta$, with the same eigenvalue λ for both $\alpha = \pm\beta$. Thus, for the solution $\theta(s)$ of any value of $\alpha > 0$, there always exists another solution $\theta^*(s) = -\theta(s)$. So, we discuss here only the case of $\alpha > 0$.

Note that we have great freedom to choose the value of the auxiliary parameter $\hbar$. Mathematically, for given α and κ, the value of λ at any *finite* order of approximation is dependent upon the auxiliary parameter $\hbar$, because the zeroth and high-order deformation equations contain $\hbar$. Let $\mathbf{R}_\hbar^\lambda$ denote the set of all values of $\hbar$ which ensure the convergence of the series (20) of the eigenvalue λ. According to Liao's proof [13], all of these series solutions must converge to the physical eigenvalue of the original equations (3) and (4). On the other hand, physically, for given α and κ, λ has only one value, because the corresponding axial load P is unique. Thus, the *limit* of all series solutions of λ is the *same* for given α and κ, as long as they are convergent. Let $\hbar$ be the variable of the horizontal axis and the *limit* of the series solution (20) of λ be the variable of vertical axis. Plot the curve $\lambda \sim \hbar$, where λ denotes the limit of the series (20) of the eigenvalue. Because the *limit* of all convergent series solutions (20) of the eigenvalue is the *same* for given α and λ, there exists a horizontal line segment above the region $\hbar \in \mathbf{R}_\hbar^\lambda$. In other words, all values of $\hbar \in \mathbf{R}_\hbar^\lambda$ give the same, convergent eigenvalue for the given α and κ. So, by plotting the curve $\lambda \sim \hbar$ at a high enough order approximation, one can find an approximation of the set $\mathbf{R}_\hbar^\lambda$, as shown in Fig. 1 in the case of $\alpha = 1$. Then, one can choose a proper value of $\hbar \in \mathbf{R}_\hbar^\lambda$ to get a convergent series of λ. For example, from Fig. 1, it is clear that, when $\alpha = 1$, the series of λ is convergent by means of $\hbar = -1.5$, as shown in Table 1. Similarly, there exists a region $\hbar \in \mathbf{R}_\hbar^\theta$, and each value of $\hbar$ in this region ensures the convergence of the series of the eigenfunction $\theta(s)$. It is found that $\mathbf{R}_\hbar^\lambda = \mathbf{R}_\hbar^\theta$. Thus, as long as the series solution of λ is convergent, the corresponding series solution of the eigenfunction $\theta(s)$ is also convergent to the exact solution, as shown in Figs. 2 and 3. In a similar way, for any given values of α and κ, we can always find a proper value of $\hbar$ to ensure the convergence of the series of both the eigenvalue λ and the eigenfunction $\theta(s)$.

According to our series solutions, when $\alpha = 1$, it holds $\lambda/\kappa^2 = 1.137069$, $x(\pi) = 2.395950$ and $\kappa \, y\left(\frac{\pi}{2\kappa}\right) = 0.899203$ for *all* integer $\kappa \ge 1$, as shown in Table 1. Besides, for any a given value of α, all of λ/κ^2, $x(\pi)$ and $\kappa \, y\left(\frac{\pi}{2\kappa}\right)$ are the same for every integer $\kappa \ge 1$, i.e. they are only dependent upon the value of α, as shown in Table 2. All of these series results completely agree with the exact solution (5) and (6), as shown in Figs. 2 and 3. This indicates the validity of the analytic approach for nonlinear eigenvalue problems with periodic eigenfunction.

Let $P_\kappa(\alpha)$ and $E_\kappa(\alpha)$ denote the axial load and the inner energy of the beam for given α and κ. According to our series solutions, it holds

$$P_\kappa(\alpha) = \kappa^2 \, P_1(\alpha), \qquad E_\kappa(\alpha) = \kappa^2 \, E_1(\alpha). \tag{35}$$

So, for given α, there exist an infinite number of eigenfunctions and eigenvalues with different κ, but λ/κ^2, P_κ/κ^2 and E_κ/κ^2 are *independent* of κ. Thus, for a large value of $\kappa \ge 2$, one needs much larger axial load P and much more energy. That is the reason why the deformations of a beam in the case of $\kappa \ge 2$ are hardly observed in practice.

Table 1
Convergent series solutions of Example 1 when $\alpha = 1$ by means of $\hbar = -3/2$

κ	$x(\pi)$	$\kappa \, y\left(\frac{\pi}{2\kappa}\right)$	λ/κ^2
1	2.39590	0.899203	1.137069
2	2.39590	0.899203	1.137069
3	2.39590	0.899203	1.137069
4	2.39590	0.899203	1.137069
8	2.39590	0.899203	1.137069
12	2.39590	0.899203	1.137069
16	2.39590	0.899203	1.137069
20	2.39590	0.899203	1.137069
50	2.39590	0.899203	1.137069
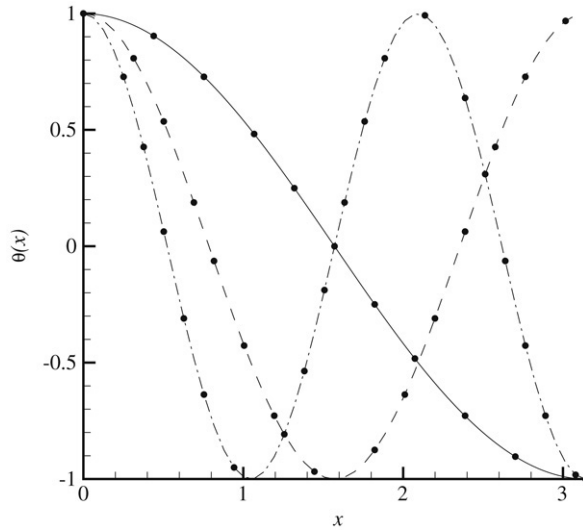
Fig. 2. The eigenfunctions $\theta(x)$ when $\alpha = 1$ by means of $\hbar = -2$. Solid line: 10th-order approximation when $\kappa = 1$; Dashed line: 10th-order approximation when $\kappa = 2$; Dash-dotted line: 10th-order approximation when $\kappa = 3$; Symbols: the corresponding exact solution given by (5) and (6).
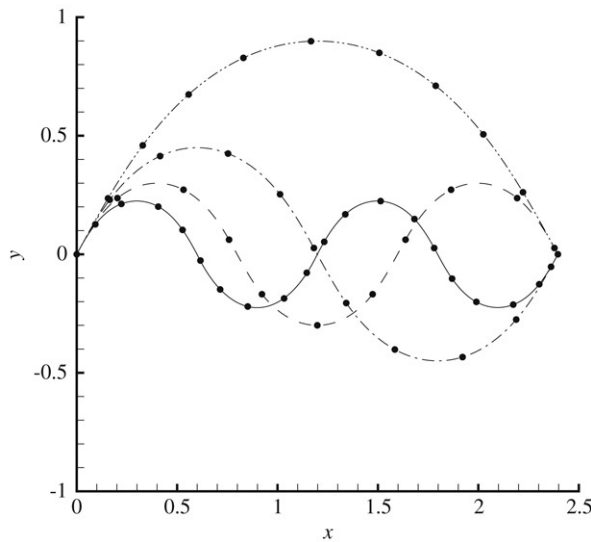

Fig. 3. The deformation of the beam when $\alpha = 1$ ($\lambda/\kappa^2 = 0.899203$) by means of $\hbar = -3/2$. Solid line: 10th-order approximation when $\kappa = 4$; Dashed line: 10th-order approximation when $\kappa = 3$; Dash-dotted line: 10th-order approximation when $\kappa = 2$; Dash-dot-dotted line: 10th-order approximation when $\kappa = 1$; Symbols: the corresponding exact solution given by (5) and (6).

Table 2
Convergent series solution of Example 1 for different values of α

α	$x(\pi)$	$\kappa\, y\left(\frac{\pi}{2\kappa}\right)$	λ/κ^2
0.25	3.09266	0.24838	1.007856
0.50	2.94778	0.48709	1.031956
0.75	2.71254	0.70686	1.073986
1	2.39590	0.89920	1.137069
1.25	2.00989	1.05669	1.226361
1.50	1.56848	1.17325	1.350171
1.75	1.08712	1.24423	1.522165
2	0.58130	1.26641	1.765987
2.05	0.47848	1.26479	1.826701
2.10	0.37535	1.26110	1.892437
2.20	0.16848	1.24750	2.041440
2.25	0.06490	1.23755	2.126220
2.28	0.00273	1.23055	2.180905
2.281	0.00066	1.23031	2.18278
2.2812	0.00025	1.23026	2.18315
2.28132	0.00000	1.23023	2.18338
2.282	−0.00141	1.23006	2.18466
2.30	−0.03871	1.22546	2.21910
2.50	−0.45305	1.15520	2.69940

Let

$$\theta(s) = \sum_{m=1}^{+\infty} B_m \, \cos(m\kappa s)$$

denote the solution of Eqs. (3) and (4), where B_m is a coefficient and $\kappa \geq 1$ is an integer. According to our series solution, the coefficient B_m is only dependent upon the value of α, i.e. it is *independent* of κ. So, as long as one gets the series solution $\theta(s)$ in the case of $\kappa = 1$, one has the series solutions for *all* integers $\kappa \geq 1$. For example, when $\kappa = 2$, the deformation of the beam has two similar parts: one is in the range of $0 \leq s \leq \pi/2$, the other is in $\pi/2 \leq s \leq \pi$, where s is the arc length of the beam, and the 2nd is the inverse of the 1st. When $\kappa = 3$, the deformation of the beam has three similar parts in the range of $0 \leq s \leq \pi/3$, $\pi/3 \leq s \leq 2\pi/3$, $2\pi/3 \leq s \leq \pi$, respectively, as shown in Fig. 3. So, there exists a kind of similarity of eigenfunctions: for a given value of α, the deformation of the beam has κ similar parts for the κth eigenfunction. In general, it holds

$$y(s) = \kappa\, y\left(\frac{s}{\kappa}\right), \quad s \in [0, \pi]. \tag{36}$$

Note that the above relation is not obvious in the exact solution (5) and (6). In some cases, the eigenfunctions look elegant and beautiful, as shown in Fig. 4 for $\alpha = 2$, Fig. 5 for $\alpha = 2.05$, Fig. 6 for $\alpha = 2.2$, Fig. 7 for $\alpha = 2.5$, and Fig. 8 for $\alpha = 2.25$, respectively. All of these curves are obtained by our analytic approach, and agree well with the exact solution.

In this section, we use a beam with uniform cross-section acted by an axial load as an example to show the validity of our analytic approach for nonlinear eigenvalue problems with periodic eigenfunctions.

3. Problems with non-periodic eigenfunction

As an example of eigenvalue problems with non-periodic eigenfunctions, let us consider here the eigenvalue problem governed by

$$u_{rr} + \left(\frac{d-1}{r}\right) u_r - 2\lambda u + 2u^3 = 0, \qquad u(0) = 1, u(+\infty) = 0,$$

where λ is an unknown eigenvalue, $u(r)$ is the eigenfunction, $d = 1, 2, 3$ is the number of dimensions, and the subscript denotes the derivatives with respect to the independent variable r, respectively. This nonlinear eigenvalue problem was solved numerically by Silberberg [31], who investigated an optical pulse that collapses simultaneously and symmetrically in time and space under the combined effect of diffraction, anomalous dispersion, and nonlinear refraction.

According to the boundary conditions, the eigenfunction $u(r)$ is obviously non-periodic. Besides, it is easy to find that $u(r)$ is an even function, i.e. $u(-r) = u(r)$. So, we introduce the transformation

$$u(r) = w(\xi), \qquad \xi = \frac{1}{1 + \gamma\, r^2},$$

and the original equation becomes

$$2\gamma(1-\xi)\xi^3 w'' + \gamma\xi^2[4(1-\xi) - d]\, w' - \lambda\, w + w^3 = 0, \qquad w(0) = 0, \qquad w(1) = 1, \tag{37}$$
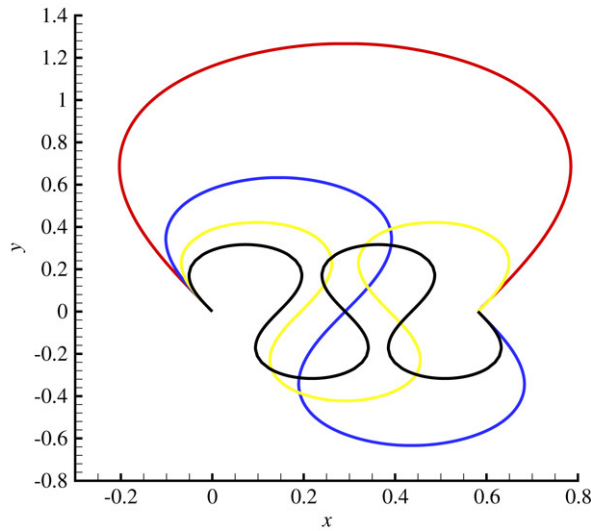
Fig. 4. The deformation of the beam when $\alpha = 2$ ($\lambda/\kappa^2 = 1.765987$) by means of $\hbar = -1$. Solid line in red: $\kappa = 1$; Solid line in blue: $\kappa = 2$; Solid line in yellow: $\kappa = 3$; Solid line in black: $\kappa = 4$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
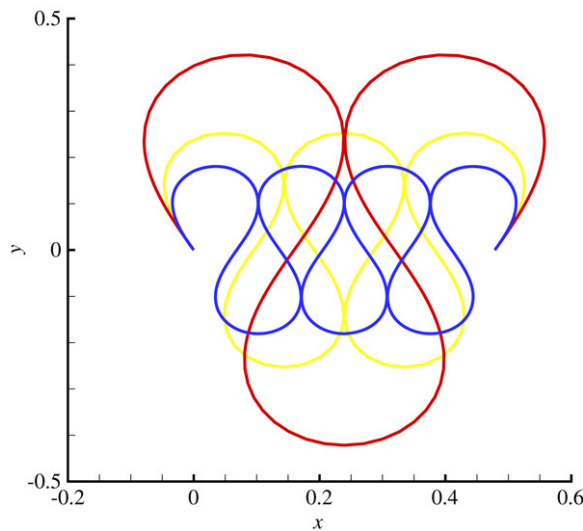

Fig. 5. The deformation of the beam when $\alpha = 2.05$ ($\lambda/\kappa^2 = 1.826701$) by means of $\hbar = -1$. Solid line in red: $\kappa = 3$; Solid line in yellow: $\kappa = 5$; Solid line in blue: $\kappa = 7$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

where the prime denotes the differentiation with respect to ξ. Note that both the eigenfunction $w(\xi)$ and the eigenvalue λ are unknown, but we have only one differential equation for $w(\xi)$. Thus, an additional algebraic equation is necessary to determine the eigenvalue λ. To do so, we substitute the boundary condition $w(1) = 1$ into (37) and obtain the following algebraic equation

$$\lambda = 1 - \gamma \, d \, w'(1), \tag{38}$$

which provides us a relation between the eigenvalue λ and the eigenfunction $w(\xi)$.

3.1. Approach based on the HAM

It is reasonable to assume that $w(\xi)$ can be expressed by the set of non-periodic base functions

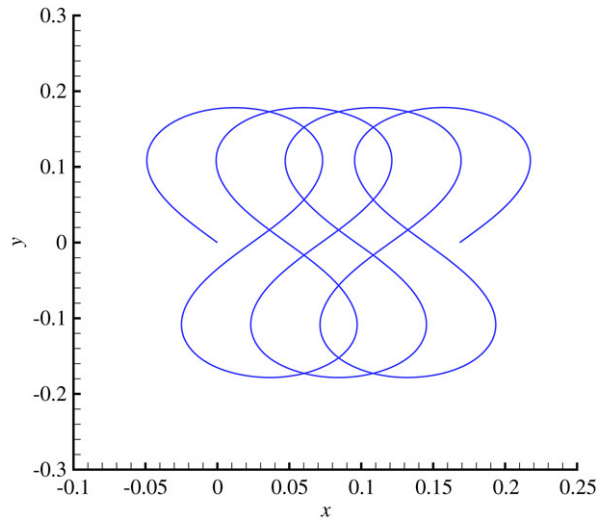$$\left\{ \xi^m \mid m \geq 1, m \in \mathbf{N} \right\}$$

Fig. 6. The deformation of the beam when $\alpha = 2.2$ and $\kappa = 7$ with $\lambda/\kappa^2 = 2.041440$.
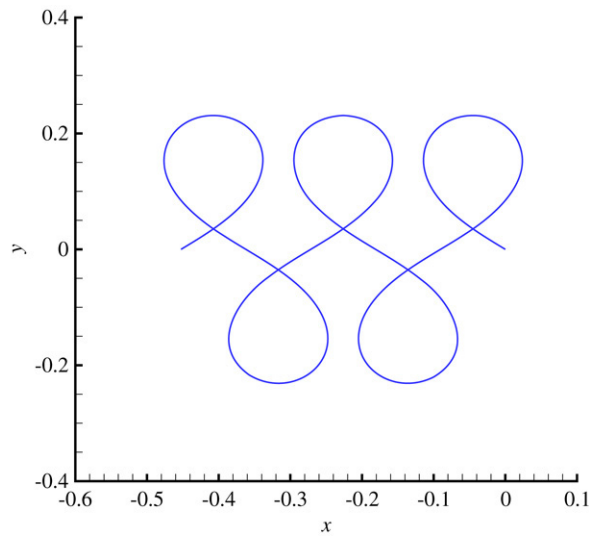

Fig. 7. The deformation of the beam when $\alpha = 2.5$ and $\kappa = 5$ with $\lambda/\kappa^2 = 2.6994$.

as follows:

$$u(\xi) = \sum_{m=1}^{+\infty} a_m \, \xi^m, \tag{39}$$

where a_m is a coefficient.

Let $w_0(\xi)$ denote an initial guess of $w(\xi)$ that satisfies the boundary conditions, $\mathcal{L}$ an auxiliary linear operator, $\hbar$ a non-zero auxiliary parameter, and $q \in [0, 1]$ an embedding parameter, respectively. For simplicity, define the nonlinear operator

$$\mathcal{N}[W(\xi; q), \Lambda(q)] = 2\gamma \, \xi^3 (1 - \xi) \, \frac{\partial^2 W(\xi; q)}{\partial \xi^2} + \gamma \, \xi^2 \, [4(1 - \xi) - d] \, \frac{\partial W(\xi; q)}{\partial \xi}$$

$$- \Lambda(q) W(\xi; q) + W^3(\xi; q). \tag{40}$$

We construct the zeroth-order deformation equation

$$(1 - q)\mathcal{L}[W(\xi; q) - w_0(\xi)] = q \, \hbar \, \mathcal{N}[W(\xi; q), \Lambda(q)], \quad q \in [0, 1], \tag{41}$$

subject to the boundary conditions

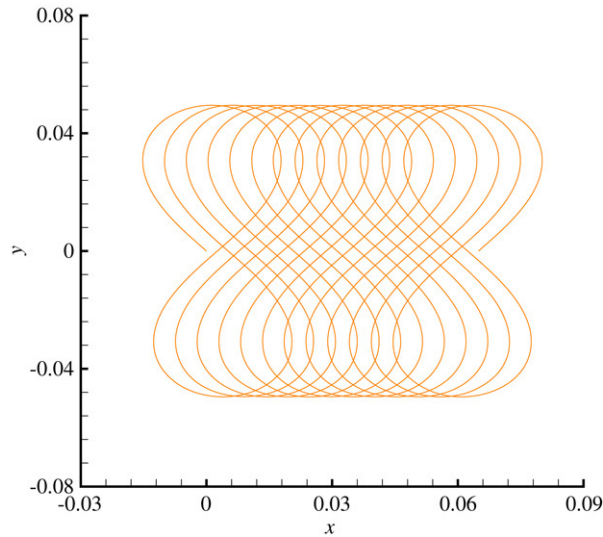$$W(0; q) = 0, \qquad W(1; q) = 1, \quad q \in [0, 1]. \tag{42}$$

Fig. 8. The deformation of the beam when $\alpha = 2.25$ and $\kappa = 25$ with $\lambda/\kappa^2 = 2.126220$.

When $q = 0$, because $w_0(\xi)$ satisfies the boundary conditions, it holds obviously

$$W(\xi; 0) = w_0(\xi). \tag{43}$$

When $q = 1$, since $\hbar \neq 0$, Eqs. (41) and (42) are equivalent to the original equation (37), provided

$$W(\xi; 1) = w(\xi), \qquad \Lambda(1) = \lambda. \tag{44}$$

Thus, as the embedding parameter q increases from 0 to 1, $W(\xi; q)$ varies (or deform) from the initial guess $w_0(\xi)$ to the eigenfunction $w(\xi)$, so does $\Lambda(q)$ from the initial guess $\lambda_0 = \Lambda(0)$ to the eigenvalue λ. The deformations of $W(\xi; q)$ and $\Lambda(q)$ are completely determined by Eqs. (41) and (42).

Using (43) and by means of Taylor series, we expand $W(\xi; q)$ and $\Lambda(q)$ into the following homotopy-series in the embedding parameter q:

$$W(\xi; q) = w_0(\xi) + \sum_{m=1}^{+\infty} w_m(\xi)\, q^m, \tag{45}$$

$$\Lambda(q) = \lambda_0 + \sum_{m=1}^{+\infty} \lambda_m\, q^m, \tag{46}$$

where

$$w_m(\xi) = D_m\left[W(\xi; q)\right], \qquad \lambda_m = D_m\left[\Lambda(q)\right]$$

with the definition D_m by (18). Note that the zeroth-order deformation equation (41) contains the auxiliary parameter $\hbar$. Thus, the homotopy-series (45) and (46) are also dependent upon $\hbar$. Besides, we have large freedom to choose the initial guess $w_0(\xi)$ and the auxiliary linear operator $\mathcal{L}$. Assuming that all of $w_0(\xi)$, $\hbar$ and $\mathcal{L}$ are properly chosen so that the homotopy-series (45) and (46) are convergent at $q = 1$, we have, using (43) and $\Lambda(0) = \lambda_0$, the homotopy-series solutions

$$w(\xi) = w_0(\xi) + \sum_{m=1}^{+\infty} w_m(\xi), \tag{47}$$

$$\lambda = \lambda_0 + \sum_{m=1}^{+\infty} \lambda_m. \tag{48}$$

According to the fundamental theorems in calculus, the coefficient of a Taylor series is unique. Thus, both $w_m(\xi)$ and λ_m are unique, and are governed by unique equations, respectively. Directly substituting the homotopy-series (45) and (46) into the zeroth-order deformation equations (41) and (42), then comparing the coefficients of q^m, we have the mth-order deformation equation

$$\mathcal{L}[w_m(\xi) - \chi_m\, w_{m-1}(\xi)] = \hbar\, R_m(\xi), \tag{49}$$

subject to the boundary conditions

$$w_m(0) = 0, \qquad u_m(1) = 0, \tag{50}$$

where χ_m is defined by (24), and

$$R_m(\xi) = D_{m-1} \mathcal{N}[W(\xi; q), \Lambda(q)]$$

$$= 2\gamma \xi^3 (1-\xi) w''_{m-1} + \gamma \, \xi^2 \, [4(1-\xi) - d] \, w'_{m-1} - \sum_{k=0}^{m-1} \lambda_{m-1-k} w_k + \sum_{k=0}^{m-1} w_{m-1-k} \sum_{j=0}^{k} w_{k-j} w_j. \tag{51}$$

For the details of the deduction, please refer to Liao [17]. Note that, differentiating the zeroth-order deformation equations (41) and (42) m times with respect to q, then setting $q = 0$, and finally dividing by $m!$, one can obtain the same mth-order deformation equation as those listed above, as proved in [22].

Note that we have large freedom to choose the initial guess $w_0(\xi)$ and the auxiliary linear operator $\mathcal{L}$. Both of them should be chosen under the so-called *Rule of Solution Expression*: the high-order deformation equations (49) and (50) should have a unique solution $w_m(\xi)$ that agrees with the solution expression (39). According to the Solution Expression (39) and the boundary conditions $w(0) = 0$, $w(1) = 1$, it is obvious to choose the initial guess of eigenfunction:

$$w_0(\xi) = \xi + \beta(\xi - \xi^2), \tag{52}$$

where β is a parameter to be determined later. Besides, according to the above-mentioned Rule of Solution Expression, we simply choose the auxiliary linear operator

$$\mathcal{L}u = u'' \tag{53}$$

which has the property

$$\mathcal{L}(C_1 + C_2 \, \xi) = 0, \tag{54}$$

for any real coefficients C_1 and C_2. Let $\hat{w}_m(\xi)$ denote a special solution of Eqs. (49) and (50). Then, according to (54), we have the general solution

$$w_m(\xi) = \hat{w}_m(\xi) + \chi_m \, u_{m-1}(\xi) + C_1 + C_2 \xi, \tag{55}$$

where C_1 and C_2 are determined by (50), i.e.

$$C_1 = 0, \qquad C_2 = -\hat{w}_m(1) - \chi_m \, w_{m-1}(1).$$

In this way, we get $u_m(\xi)$ and λ_m (as mentioned below) one by one in the order $m = 1, 2, 3, \ldots$. Note that the high-order deformation equations (49) and (50) are linear. Thus, it is easy to get approximations at high enough order, especially by means of the symbolic computation software such as Mathematica, Maple, MathLab and so on.

Because the eigenfunction $w(\xi)$ is non-periodic, we cannot use the Solution Expression (39) to obtain an additional algebraic equation of λ_m ($m \geq 1$) in a similar way to that described in Section 2. In other words, the idea of avoiding the appearance of secular terms is useless for eigenvalue problems with non-periodic eigenfunctions. So, we must provide a new way to find the eigenvalue. According to Eq. (38), we construct the following zeroth-order deformation equation

$$\Lambda(q) = 1 - (\gamma \, d) \left. \frac{\partial W(\xi; q)}{\partial \xi} \right|_{\xi=1} . \tag{56}$$

Substituting the series (45) and (46) into the above equation, and comparing the coefficients of q^m, we have

$$\sum_{m=0}^{+\infty} \lambda_m \, q^m = 1 - (\gamma \, d) \sum_{m=0}^{+\infty} w'_m(1) \, q^m,$$

i.e.

$$\lambda_0 - 1 + (\gamma \, d) w'_0(1) + \sum_{m=1}^{+\infty} [\lambda_m + (\gamma \, d) w'_m(1)] q^m = 0.$$

Because the above equation holds for every $q \in [0, 1]$, one has:

$$\lambda_0 = 1 - (\gamma \, d) w'_0(1), \qquad \lambda_m = -(\gamma \, d) \, w'_m(1), \quad m \geq 1. \tag{57}$$

So, using (52), we have

$$\lambda_0 = 1 - (\gamma \, d)(1 - \beta). \tag{58}$$

In this way, it is easy to get λ_m.

Note that the auxiliary parameters γ in (37) and β in (52) are unknown. Let $\mathcal{E}_0$ denote the square of residual error of the initial guess, i.e.

$$\mathcal{E}_0 = \int_0^1 \left\{ 2\gamma\xi^2(1-\xi)w_0'' + \gamma\xi^2\left[4(1-\xi)-d\right]w_0' - \lambda_0 w_0 + w_0^2 \right\}^2 d\xi,$$

where λ_0 is given by (58). The "best" initial guess $w_0(\xi)$ is given by the minimum value of $\mathcal{E}_0$, i.e.

$$\frac{\partial \mathcal{E}_0}{\partial \beta} = 0 \qquad \frac{\partial \mathcal{E}_0}{\partial \gamma} = 0,$$

which give

$$\gamma = 0.261, \qquad \beta = -0.895, \quad \text{when } d = 1;$$
$$\gamma = 0.240, \qquad \beta = -0.635, \quad \text{when } d = 2;$$
$$\gamma = 0.226, \qquad \beta = -0.362, \quad \text{when } d = 3.$$

By means of the command **FindMinimum** of Mathematica, it is easy to get the above results. Note that, using (58), we have the corresponding initial guess of the eigenvalue $\lambda_0 = 0.5054, 0.2152, 0.0766$ for $d = 1, 2, 3$, respectively. It is interesting that, they are good enough initial guesses to Silberberg's [31] numerical results $\lambda = 0.5, 0.2055$ and 0.05316 for $d = 1, 2, 3$, respectively.

The procedure is as follows. First, for a given dimension-number d, determine the "best" values of γ and β by the minimum of the square residual error $\mathcal{E}_0$. Then, the initial approximations of eigenvalue $w_0(\xi)$ and eigenvalue λ_0 are obtained by (52) and (58), respectively. Second, because $w_0(\xi)$ and λ_0 are known, it is straightforward to get $R_1(\xi)$ by means of (51). Then, it is easy to get $w_1(\xi)$ by solving the linear ODEs (49) and (50), and then λ_1 by means of (57). Similarly, one can get $w_2(\xi), \lambda_2, w_3(\xi), \lambda_3$, and so on. All of these are easy to do by means of the symbolic computation software such as Mathematica, MathLab, Maple and so on.

It is found that $w_m(\xi)$ is expressed by

$$w_m(\xi) = \sum_{n=1}^{6m+2} a_{m,n}\, \xi^n, \tag{59}$$

where $a_{m,n}$ are coefficients. Substituting (59) into the mth-order deformation equations (49) and (50), one can obtain the recurrence formulas for the coefficient $a_{m,n}$.

Note that the governing equation (37) contains the linear term

$$2\gamma(1-\xi)\xi^3 w'' + \gamma\xi^2[4(1-\xi)-d]\,w' - \lambda\,w,$$

which corresponds to a complicated linear operator

$$\mathcal{L}w = 2\gamma(1-\xi)\xi^3 w'' + \gamma\xi^2[4(1-\xi)-d]\,w' - \lambda\,w.$$

However, the linear differential equation

$$2\gamma(1-\xi)\xi^3 w'' + \gamma\xi^2[4(1-\xi)-d]\,w' - \lambda\,w = 0$$

has no simple, closed-form solution. So, if it is used as the auxiliary linear operator, it is rather hard to get high approximations. Fortunately, unlike all other analytic approaches, our approach provides us with great freedom to choose the auxiliary linear operator $\mathcal{L}$, as mentioned by Liao [16]. It is due to this kind of freedom that we can simply choose $\mathcal{L}w = w''$ as the auxiliary linear operator so as to express the corresponding eigenfunction by the power series of ξ.

3.2. Solution series

Note that the solution series (47) and (48) contain the auxiliary parameter $\hbar$. In a similar way as described in Section 2, we can choose a proper value of $\hbar$ to ensure the convergence of the solution series (47) and (48). For example, let us first consider the case of $d = 1$, whose exact eigenfunction $u(r) = \text{sech}(r)$ and eigenvalue $\lambda = 1/2$ are well known. There exists such a set $\mathbf{R}_\hbar^\lambda$ so that every value of $\hbar \in \mathbf{R}_\hbar^\lambda$ ensures the convergence of the series solution (48). The set $\mathbf{R}_\hbar^\lambda$ can be approximately obtained by plotting the so-called $\lambda \sim \hbar$ curves at high enough order of approximations, as shown in Fig. 9, which indicates that the series solution (48) is convergent when $-30 \le \hbar < 0$ in the case of $d = 1$. This is indeed true: in the case of $d = 1$, the series solution (48) indeed converges to the exact eigenvalue $\lambda = 1/2$ by means of $\hbar = -20$, as shown in Table 3. Besides, by means of the so-called homotopy-Padè technique [13], we also get the same eigenvalue $\lambda = 1/2$. Similarly, there exists such a set $\mathbf{R}_\hbar^u$ so that every value of $\hbar \in \mathbf{R}_\hbar^u$ ensures the convergence of the series solution (47). It is found that $\mathbf{R}_\hbar^u = \mathbf{R}_\hbar^\lambda$. So, in the case of $d = 1$ and by means of $\hbar = -20$, the series solution (47) indeed converges to the exact eigenfunction $u(r) = \text{sech}(r)$, as shown in Fig. 10.

Similarly, in the cases of $d = 2$ and $d = 3$, one can find the corresponding $\mathbf{R}_\hbar^\lambda$ and $\mathbf{R}_\hbar^u$ approximately by plotting the curve $\lambda \sim \hbar$, as shown in Fig. 9. Then, it is found that the solution series (47) and (48) are convergent by means of $\hbar = -20$ in

Table 3
Approximation of λ of Example 2 when $d = 1$ and $\hbar = -20$

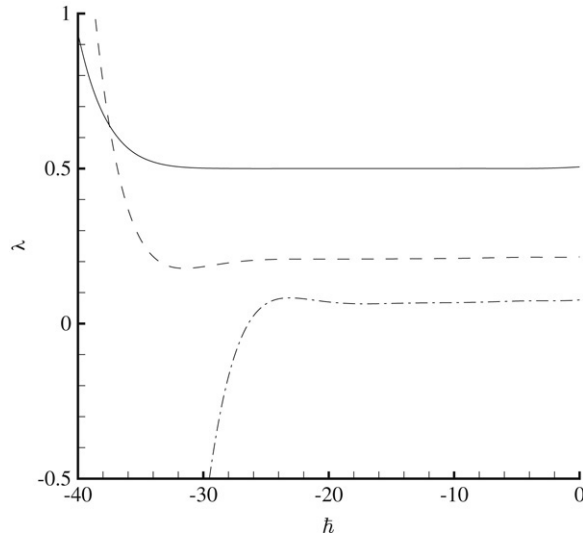| Order of approx.	λ
2	0.5007
4	0.5001
6	0.5000
8	0.5000
10	0.5000
15	0.5000
20	0.5000

Fig. 9. Curve $\lambda \sim \hbar$ at the 10th order of approximation for Example 2. Solid line: $d = 1$; Dashed line: $d = 2$; Dash-dotted line: $d = 3$.
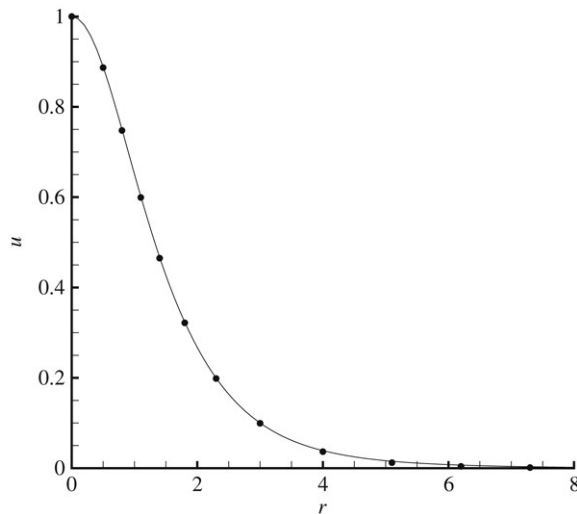

Fig. 10. Comparison of the exact solution $u = \mathrm{sech}(r)$ with the HAM approximation when $d = 1$ by means of $\hbar = -20$ for Example 2. Solid line: 5th-order HAM approximation; Circles: exact solution.

the case of $d = 2$ and $\hbar = -12$ in the case of $d = 3$, respectively, as shown in Fig. 11. The convergent analytic results of the eigenfunction $u(r)$ in the cases of $d = 2$ and $d = 3$ are listed in Table 4. In the cases of $d = 2$ and $d = 3$, the series of eigenvalues converge slowly than that in the case of $d = 1$. Thus, the homotopy-Padé technique [13] is used to accelerate the convergence, which gives $\lambda = 0.2055$ for $d = 2$ and $\lambda = 0.0533$ for $d = 3$, respectively. All of these analytic results agree well with Silberberg's numerical ones [31].

Table 4
Convergent series solution of eigenfunction $u(r)$ of Example 2

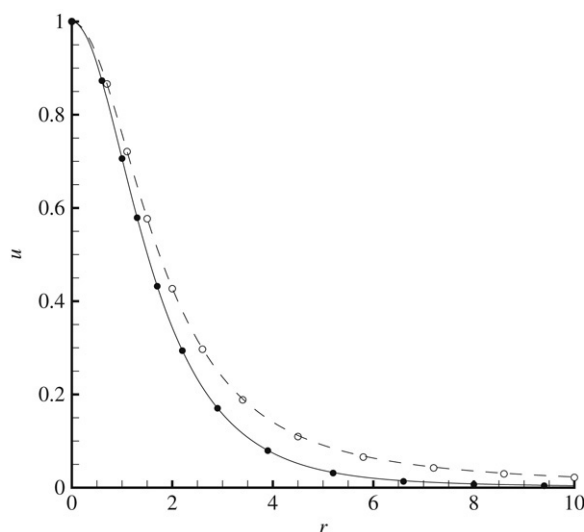| r	$u(r)$ when $d = 2$	$u(r)$ when $d = 3$
0	1	1
0.1	0.9961	0.9969
0.2	0.9843	0.9875
0.3	0.9653	0.9723
0.4	0.9398	0.9518
0.5	0.9087	0.9265
0.6	0.8730	0.8973
0.7	0.8338	0.8649
0.8	0.7923	0.8301
0.9	0.7494	0.7937
1.0	0.7059	0.7563
1.25	0.5994	0.6626
1.50	0.5016	0.5735
1.75	0.4160	0.4929
2.00	0.3433	0.4222
2.25	0.2826	0.3612
2.50	0.2324	0.3092
2.75	0.1912	0.2652
3.00	0.1575	0.2279
3.50	0.1072	0.1698
4	0.0733	0.1278
5	0.0349	0.0745
6	0.0169	0.0453
7	0.0083	0.0282
8	0.0041	0.0180
9	0.0021	0.0117
10	0.0011	0.0078

Fig. 11. The HAM approximation of eigenfunction $u(r)$ when $d = 2$ (by means of $\hbar = -20$) and $d = 3$ (by means of $\hbar = -12$) for Example 2. Solid line: 20th-order approximation when $d = 2$; Filled-circles: numerical approximation when $d = 2$; Dashed line: 20th-order approximation when $d = 3$; Open-circles: numerical approximation when $d = 3$.

In this section, we used an example to show the validity of our approach for nonlinear eigenvalue problems with non-periodic eigenfunctions.

4. Discussions

In this paper a analytic approach to get series solutions of nonlinear eigenvalue problems is described by means of two examples. This analytic approach is valid for nonlinear eigenvalue problems with either periodic or non-periodic eigenfunctions, and thus is rather general. All of our series solutions agree well with exact or numerical results, and this fact shows the validity of our analytic approach.

This analytic approach has some obvious advantages. First of all, unlike perturbation techniques, it is independent of any small physical parameters: it is valid no matter whether or not there exist any small physical parameters in governing equations and/or boundary conditions. Second, different from other traditional techniques, it provides us a simple way to ensure the convergence of series solution of eigenvalue and eigenfunction, so that one can always get accurate enough approximations. Thus, this approach can be applied to solve eigenvalue problems with strong nonlinearity. Third, unlike all other analytic techniques, this approach provides us great freedom to choose an auxiliary linear operator so as to approximate the eigenfunction more effectively by means of better base functions. Therefore, this approach can be widely applied to solve strongly nonlinear eigenvalue problems in science and engineering, no matter whether the corresponding eigenfunction is periodic or not.

Acknowledgements

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