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Research paper

A HAM-based wavelet approach for nonlinear partial differential equations: Two dimensional Bratu problem as an application

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ABSTRACT

In this paper, a new analytic approach, namely the wavelet homotopy analysis method (wHAM), is developed for boundary value problems (BVPs) governed by nonlinear partial differential equations (PDEs), which successfully combines the homotopy analysis method (HAM) and the generalized Coiflet-type wavelet. To improve the computational efficiency and accuracy, a section-based wavelet approximation for partial derivatives is proposed. The two-dimensional Bratu equation is used as an example to illustrate its basic ideas of the wHAM. Numerical results verify the validity as well as great advantages of the wHAM. Compared with the normal HAM, the wHAM possesses not only larger freedom to choose the auxiliary linear operator, but also better convergence property and higher computational efficiency. In addition, the iteration approach can greatly accelerate convergence.

1. Introduction

A lot of scientific and engineering problems are described by nonlinear partial differential equations (PDEs) [1–6]. However, except very few special cases, it is extremely difficult and even impossible to obtain the exact solutions of boundary value problems (BVPs) governed by nonlinear PDEs. Therefore, analytical approximation arises as a kind of powerful tool to solve nonlinear PDEs.

The well-known methods to get analytical approximations are perturbation techniques [7,8] and some non-perturbative techniques, such as the artificial small parameter method [9], the δ-expansion method [10] and the Adomian's decomposition method [11], which are widely applied in science and engineering and do great contributions to help us understand many nonlinear phenomena. However, these methods are essentially dependent on small/large physical parameters and cannot guarantee the convergence of solution series. These greatly restrict their applications. The homotopy analysis method (HAM) proposed by Liao [12–18] not only overcomes these limitations, but also provides us large freedom to choose equation types by constructing a homotopy with proper linear operators and base functions. Due to its advantages, the HAM is widely applied to solve scientific and engineering problems governed by nonlinear partial differential equations, such as the two-dimensional Bratu equation [16], two-dimensional viscous flow [19], magnetohydrodynamic flows of non-newtonian fluids [20], non-Newtonian nanofluids [21], heat radiation equation [22], etc.

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During the past two decades, the HAM has been greatly developed and extended both in theories and applications. The systematic theories of the HAM has been built, which are systematically described in the three books [13,16,23] by Liao et al. Based on the normal HAM, some techniques have been developed and the application scopes are greatly extended, such as the truncation technique (called tHAM) [16,24], the iteration approach (called iHAM) [16,24], the spectral homotopy analysis method (sHAM) [5,25,26], the optimal homotopy analysis method (oHAM) [17] and the latest new approach MDDiM [18].

Even though the HAM possesses great advantages in solving nonlinear differential equations, there is still room for improvement. To our knowledge, there are at least two aspects that can be improved:

(1) Higher efficiency. The exponential expansion of the right-hand side term in high-order deformation equation commonly exist in the normal HAM, it is expected that this problem can be overcome in a convenient way so that the HAM can be more efficient;
(2) Larger freedom on the choice of auxiliary linear operators. Although the normal HAM provides great freedom to choose the auxiliary linear operators, the solution expression is strongly dependent upon the auxiliary linear operator, so it is purposed to develop a method that can be more flexible and adaptive to choose the auxiliary linear operator.

Based on such purpose, the wavelet as a powerful mathematical tool is considered here. Due to its good properties, the wavelet is widely applied to solve differential equations. Briefly speaking, there are three kinds of wavelet methods: wavelet finite element method [[27–29], wavelet collocation method [30–33] and wavelet Galerkin method [34–38]. Comparing these wavelet-based methods, it is found that the wavelet type plays a very important role for the convergence property as well as the efficiency. So, how could we choose a "proper wavelet" to solve nonlinear BVPs? Such kind of "proper wavelet" should possess the following properties: orthogonality, compact support, interpolation properties as well as high algebraic accuracy and the ability to represent functions at different levels of resolution [39,40]. The generalized Coiflet-type wavelet possesses almost all of these properties.

Of course, there are some other works that tried to combine the analytical approximations with some numerical methods. In 1999, Damil et al. [41] proposed an iterative method based on perturbation techniques, finite element method as well as Padé approximants. In 2002, Lahmam et al. [42] extended it and developed some high-order predictor–corrector algorithms for nonlinear problems written in a quadratic framework with automatic procedures. However, based on perturbation techniques, the limitations are not overcome.

In 2017, Yang and Liao [43] proposed the wavelet homotopy analysis method (wHAM) for nonlinear ordinary differential equations (ODEs), which combines the HAM [12–18,44] and the generalized Coiflet-type wavelet [45–49]. In this paper, the wavelet homotopy analysis method (wHAM) is further developed to solve nonlinear partial differential equations (PDEs) and the two-dimensional Bratu's problem [50–53] is taken as an example to describe the basic ideas of this method. The two-dimensional Bratu equation is chosen mainly for two reasons. Firstly, it arises in a wide variety of physical applications [40,54]. Secondly, as a simple but typical nonlinear PDE, the two-dimensional Bratu equation is widely used to verify the validity of new methods [52,55,56]. In 1986, Boyd [56] gave a research on the symmetry of the two-dimensional Bratu equation and solved it by pseudospectral methods. In 2006, Odejide et al. [52] constructed a near exact solution of the two-dimensional Bratu equation which satisfies the equation on the midpoint ($x = 1/2, y = 1/2$) and all the boundary conditions, and compared the results to those obtained by finite difference method (FDM) and weighted residual method (WRM), which showed that the weighted residual method gave the best results. In 2011, Liao [16] solved the two-dimensional Bratu equation by homotopy analysis method (HAM) and gave some analytical approximations.

The paper is arranged as follows. In Section 2, using the generalized Coiflet-type wavelet developed by Wang and Zhou et al. [45–49], the base functions for solutions with two variables are constructed. More importantly, a section-based wavelet approximation to approximate partial derivatives is proposed, which greatly improves the efficiency. In Section 3, the basic ideas of wHAM for nonlinear PDEs is described by means of the two-dimensional Bratu's problem as an example. In Section 4, some results are presented. In Section 5, the concluding remarks about the wHAM are given.

2. Wavelet basis and a section-based approximation

According to [45,49], the scaling function $\varphi(x)$ and wavelet function $\psi(x)$ of the generalized Coiflet-type wavelet possess the following properties :

$$(a)\ \varphi(x) = \sum_{l\in\mathbb{Z}} p_k \varphi(2x-l); \tag{1}$$

$$(b)\ \psi(x) = \sum_{l\in\mathbb{Z}} (-1)^k p_{1-k} \psi(2x-l); \tag{2}$$

$$(c)\ M_n = M_1^n, \quad for\ 0 \le n < N; \tag{3}$$

$$(d)\ \int_{-\infty}^{+\infty} x^n \psi(x) dx = 0, \quad for\ 0 \le n < N; \tag{4}$$

$$(e)\ \sum_{l\in\mathbb{Z}} p_k\varphi(x-l) = 1; \tag{5}$$

where p_k are the low-pass filter coefficients that can be found in [40,57], N is the number of vanishing moment, and $M_n = \int_{-\infty}^{+\infty} x^n\varphi(x)dx$ is the nth-order moment of the scaling function. In this paper, the generalized Coiflet-type wavelet with $N=6$ and $M_1 = 7$ is adopted.

Based on the original scaling function $\varphi(x)$, the scaling function for the two-dimensional space is defined by

$$\varphi(x,y) = \varphi(x)\varphi(y). \tag{6}$$

Using the scaling function above, a multi-resolution analysis [45,58] of the $L^2(\mathbb{R}^2)$ can be constructed, which consists of a sequence of nested subspaces $\{0\} \subset \cdots \subset \mathbb{V}_0^2 \subset \mathbb{V}_1^2 \subset \cdots \subset \mathbb{V}_j^2 \subset \mathbb{V}_{j+1}^2 \subset \cdots \subset L^2(\mathbb{R}^2)$. Here, the subspace with resolution level j is

$$\mathbb{V}_j^2 = \mathrm{span}\left\{\varphi(2^jx-k)\varphi(2^jy-l)\right\}_{k,l\in\mathbb{Z}}, \tag{7}$$

in which all the base functions are mutually orthogonal.

In most cases, boundary value problems are defined in a bounded region. Therefore, it is necessary to modify the base functions, so that an arbitrary region bounded L^2 function can be conveniently approximated. For simplicity, the $L^2([0,1]^2)$ space is considered in this paper.

Through the boundary extension technique developed by Wang [45] and considering that the original scaling function $\varphi(x)$ has a support interval [0, 17], the subspace $\mathbb{V}_j^2([0,1]^2)$ for $L^2([0,1]^2)$ can be constructed as

$$\mathbb{V}_j^2([0,1]^2) = \mathrm{span}\left\{\varphi_{j,k}(x)\varphi_{j,l}(y)\right\}_{k,l=0\sim 2^j}, \tag{8}$$

where the modified basis $\varphi_{j,\,\kappa}(\xi)$ is defined by

$$\varphi_{j,\kappa}(\xi) = \begin{cases} \displaystyle\sum_{i=2-3N+M_1}^{-1} T_{0,\kappa}\left(\frac{i}{2^j}\right)\varphi(2^j\xi - i + M_1) + \varphi(2^j\xi-\kappa+M_1), \\ \qquad\qquad\qquad\qquad \kappa\in[0,3], \\ \varphi(2^j\xi-\kappa+M_1), \\ \qquad\qquad\qquad\qquad \kappa\in[4,2^j-4], \\ \displaystyle\sum_{i=2^j+1}^{2^j-1+M_1} T_{1,2^j-\kappa}\left(\frac{i}{2^j}\right)\varphi(2^j\xi - i + M_1) + \varphi(2^j\xi-\kappa+M_1), \\ \qquad\qquad\qquad\qquad \kappa\in[2^j-3,2^j], \end{cases} \tag{9}$$

in which

$$T_{0,\kappa}(\xi) = \sum_{i=0}^{3}\frac{p_{0,i,\kappa}}{i!}\xi^i\ ,\ \ T_{1,2^j-\kappa}(\xi) = \sum_{i=0}^{3}\frac{p_{1,i,\kappa}}{i!}(\xi-1)^i, \tag{10}$$

and coefficients $p_{0,\,i,\,\kappa}$ and $p_{1,\,i,\,\kappa}$ are assigned as

$$\mathbf{P_0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -11/6 & 3 & -3/2 & 1/3 \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{bmatrix}, \tag{11}$$

$$\mathbf{P_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 11/6 & -3 & 3/2 & -1/3 \\ 2 & -5 & 4 & -1 \\ 1 & -3 & 3 & -1 \end{bmatrix} \tag{12}$$

through relations $\mathbf{P_0} = \left\{2^{-ij}p_{0,i,\kappa}\right\}$ and $\mathbf{P_1} = \left\{2^{-ij}p_{1,i,\kappa}\right\}$ with $i,\kappa = 0,1,2,3$.

Furthermore, using the generalized Gaussian integral method developed by Zhou et al. [59], an arbitrary function $f(x,y) \in L^2([0,1]^2)$ can be approximated by

$$f(x,y) \approx P^j f(x,y) = \sum_{k=0}^{2^j}\sum_{l=0}^{2^j} f\left(\frac{k}{2^j},\frac{l}{2^j}\right)\varphi_{j,k}(x)\varphi_{j,l}(y), \tag{13}$$

where P^j is a projection operator on the subspace space $\mathbb{V}_j^2([0,1]^2)$.

Table 1
Comparison on the number of terms between the general method and the section-based method for the approximation of partial derivatives.

Resolution level	General method	Section-based method
$j=3$	81	9
$j=4$	289	17
$j=5$	1089	33
$j=6$	4225	65

Generally, from the approximation (13), the nth-order derivatives with respect to x and y can be obtained by the following formulas

$$\frac{\partial^n f(x,y)}{\partial x^n} \approx \frac{\partial^n P^j f(x,y)}{\partial x^n} = \sum_{k=0}^{2^j}\sum_{l=0}^{2^j} f\left(\frac{k}{2^j},\frac{l}{2^j}\right)\varphi_{j,k}^{(n)}(x)\varphi_{j,l}(y), \tag{14}$$

$$\frac{\partial^n f(x,y)}{\partial y^n} \approx \frac{\partial^n P^j f(x,y)}{\partial y^n} = \sum_{k=0}^{2^j}\sum_{l=0}^{2^j} f\left(\frac{k}{2^j},\frac{l}{2^j}\right)\varphi_{j,k}(x)\varphi_{j,l}^{(n)}(y), \tag{15}$$

where $\varphi_{j,l}^{(n)}(x)$ and $\varphi_{j,l}^{(n)}(y)$ are the nth-order derivatives of the base functions.

However, it is a rather bad choice to calculate the partial derivatives through Eqs. (14) and (15). The extensive summation not only makes the algorithm inefficient, but also causes serious precision loss. In order to avoid these, we propose a section-based wavelet approximation for partial derivatives. To illustrate the basic ideas of this new approximation, let us define the partial projection operators on the subspace space $\mathbb{V}_j^2([0,1]^2)$

$$f(x,y) \approx P_x^j f(x,y) = \sum_{k=0}^{2^j} f\left(\frac{k}{2^j},y\right)\varphi_{j,k}(x), \tag{16}$$

and

$$f(x,y) \approx P_y^j f(x,y) = \sum_{l=0}^{2^j} f\left(x,\frac{l}{2^j}\right)\varphi_{j,l}(y). \tag{17}$$

From the point of geometry, $z=f(x,y)$ is a surface and the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ denote the slope of section curves. More generally, the nth-order partial derivative $\frac{\partial^n f}{\partial x^n}$ or $\frac{\partial^n f}{\partial y^n}$ at an given point is determined by a section curve instead of the whole surface. Therefore, partial derivatives at all points on the surface $z=f(x,y)$ can be calculated through approximations of section curves. Taking the nth-order derivatives of Eqs. (14) and (15), respectively, the derivatives $\frac{\partial^n f}{\partial x^n}$ and $\frac{\partial^n f}{\partial y^n}$ can be obtained. Thus, we have the following formulas

$$\frac{\partial^n f(x,y)}{\partial x^n} \approx \frac{\partial^n P_x^j f(x,y)}{\partial x^n} = \sum_{k=0}^{2^j} f\left(\frac{k}{2^j},y\right)\varphi_{j,k}^{(n)}(x), \tag{18}$$

$$\frac{\partial^n f(x,y)}{\partial y^n} \approx \frac{\partial^n P_y^j f(x,y)}{\partial y^n} = \sum_{l=0}^{2^j} f\left(x,\frac{l}{2^j}\right)\varphi_{j,l}^{(n)}(y). \tag{19}$$

Table 1 shows the comparison on the number of terms between the general method and the section-based method at different resolution levels. Obviously, it is much more efficient to calculate the partial derivatives through the section-based method rather than the general method.

According to [40], the error estimations of the above-mentioned wavelet approximations are

$$\left\| \frac{\partial^n f(x,y)}{\partial x^n} - \frac{\partial^n P^j f(x,y)}{\partial x^n} \right\|_{L^2} \le C_1 2^{-j(N-n)}, \tag{20}$$

and

$$\left\| \frac{\partial^n f(x,y)}{\partial x^n} - \frac{\partial^n P_x^j f(x,y)}{\partial x^n} \right\|_{L^2} \le C_2 2^{-j(N-n)}, \tag{21}$$

where C_1, C_2 are positive constants that depend only on the function $f(x, y)$ and low-pass filter coefficients p_k, N is the number of vanishing moment, and $0 \le n < N$. If $n=0$, Eqs. (20) and (21) give estimations for approximations (13) and (16), respectively. Error estimations for partial derivatives with respect to y can be estimated similarly.

The error estimations show that the resolution level j plays a significant role in accuracy and efficiency control. More importantly, comparing with the general method, it avoids extensive summation, so numerically the section-based approximations can greatly improve the efficiency as well as the accuracy.

3. The wHAM for the two-dimensional Bratu problem

The basic ideas of the wHAM is illustrated here by means of the two-dimensional Bratu's problem [50–53]:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda e^u = 0 \tag{22}$$

subject to the boundary conditions

$$u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0, \tag{23}$$

where λ is a positive number, variables x, $y \in (0, 1)$, respectively.

The Bratu equation arises in a wide variety of physical applications, ranging from chemical reactor theory and radiative heat transfer to the expansion of the universe [40,54]. For simplicity, in this paper we take it as a solid fuel ignition model to discuss the physical meaning of this equation. In this model, u represents a dimensionless temperature, $\lambda > 0$ is known as the Frank–Kamenetskii parameter [50], x and y are location variables, and the term λe^u represents the combustion heat. Therefore, the two-dimensional Bratu equation describes such a physical problem: Can this system reach a steady state? And how is the temperature distributed at steady state when a combustible medium is placed in a square vessel whose walls are maintained at a fixed temperature?

According to [50,51,53], there exist two known bifurcated solutions for $0 < \lambda < \lambda_c$, no solutions for $\lambda > \lambda_c$ and a unique solution when $\lambda = \lambda_c$, where the widely accepted value of λ_c is 6.808124423. For simplicity, the lower branch of the solution is discussed in this work.

Although the HAM provides a recurrence formula [15,16] to calculate the right-hand side term deduced from the transcendental nonlinear term e^u in the Bratu Eq. (22) and the "algorithmic differentiation" [60] also gives a similar formula to directly deal with such terms, it is not an efficient way to calculate the right-hand side term by this recurrence formula. To avoid the inefficient recursion during the calculation of the right-hand side term, we introduce the following transformation

$$V(x, y) = \exp\left[-\frac{1}{2}u(x, y)\right]. \tag{24}$$

Then, the two-dimensional Bratu's problem can be rewritten as

$$V(x, y)\nabla^2 V(x, y) - \nabla V(x, y) \cdot \nabla V(x, y) - \frac{\lambda}{2} = 0 \tag{25}$$

subject to the boundary conditions

$$V(0, y) = V(1, y) = V(x, 0) = V(x, 1) = 1, \tag{26}$$

where ∇ denotes the gradient operator, and ∇^2 is the Laplacian operator, respectively.

In this form, the right-hand side term can be calculated much more efficiently. By means of wHAM, $V(x, y)$ can be obtained. Then, we have the solution of the original Eq. (22) by the inverse transformation

$$u(x, y) = -2\ln V(x, y). \tag{27}$$

3.1. The normal HAM

Based on the basic concept of homotopy, the HAM describes a continuous deformation or variation from a topological space to another. Therefore, the most important step is to construct a homotopy between the initial guess $V_0(x, y)$ and the final solution $V(x, y)$. Let $q \in [0, 1]$ denote an embedding parameter for homotopy, c_0 the convergence-control parameter, $\mathscr{L}$ an auxiliary linear operator and $\mathscr{N}$ the nonlinear operator, respectively. Then, a homotopy (or continuous variation) from the initial guess $V_0(x, y)$ to the solution $V(x, y)$ can be built by means of the so-called zeroth-order deformation equation

$$(1 - q)\mathscr{L}[\phi(x, y; q) - V_0(x, y)] = c_0 q \mathscr{N}[\phi(x, y; q)], \tag{28}$$

subject to the boundary conditions

$$\phi(0, y; q) = \phi(1, y; q) = \phi(x, 0; q) = \phi(x, 1; q) = 1, \tag{29}$$

where

$$\mathscr{N}[\phi(x, y; q)] = \phi(x, y; q)\nabla^2\phi(x, y; q) - \nabla\phi(x, y; q) \cdot \nabla\phi(x, y; q) - \frac{\lambda}{2}. \tag{30}$$

Note that the initial guess should satisfy the given boundary conditions. According to (26), the initial guess $V_0(x, y)$ is chosen in the simplest form

$$V_0(x, y) = 1. \tag{31}$$

According to the zeroth-order deformation Eq. (28), it is obvious that $\phi(x, y; 0) = V_0(x, y)$ when $q = 0$ and $\phi(x, y; 1) = V(x, y)$ when $q = 1$, respectively. In other words, $\phi(x, y; q)$ denotes a continuous variation from the initial guess $V_0(x, y)$ to the solution $V(x, y)$, as q increases from 0 to 1. Then, $\phi(x, y; q)$ can be expanded in Maclaurin series with respect to the embedding parameter q, i.e.

$$\phi(x, y; q) = V_0(x, y) + \sum_{k=1}^{+\infty} V_k(x, y) q^k, \tag{32}$$

where

$$V_k(x, y) = \mathscr{D}_k[\phi(x, y; q)] = \frac{1}{k!} \frac{\partial^k \phi(x, y; q)}{\partial q^k}\bigg|_{q=0}, \tag{33}$$

in which $\mathscr{D}_k[\phi]$ is called the mth-order homotopy-derivative operator [15].

Assuming that the linear operator $\mathscr{L}$ and the convergence-control parameter c_0 is properly chosen so that the Maclaurin series (32) converges at $q = 1$, we gain the series solution

$$V(x, y) = V_0(x, y) + \sum_{k=1}^{+\infty} V_k(x, y). \tag{34}$$

Thus, the corresponding mth-order approximation is given by

$$V(x, y) \approx \tilde{V}_m(x, y) = V_0(x, y) + \sum_{k=1}^{m} V_k(x, y). \tag{35}$$

Taking the mth-order homotopy-derivative operator on both sides of the zeroth-order deformation Eq. (29) as well as the boundary conditions (37), we have the mth-order deformation equation

$$\mathscr{L}[V_m(x, y) - \chi_m V_{m-1}(x, y)] = c_0 R_m(x, y) \tag{36}$$

subject to the boundary conditions

$$V_m(0, y) = V_m(1, y) = V_m(x, 0) = V_m(x, 1) = 0, \tag{37}$$

where

$$\begin{aligned} R_m(x, y) &= \mathscr{D}_{m-1}\{\mathscr{N}[\phi(x, y; q)]\} \\ &= \sum_{s=0}^{m-1} V_s(x, y) \nabla^2 V_{m-1-s}(x, y) \\ &\quad - \sum_{s=0}^{m-1} \nabla V_s(x, y) \cdot \nabla V_{m-1-s}(x, y) - \frac{\lambda}{2}(1 - \chi_m) \end{aligned} \tag{38}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{39}$$

As for the choice of the auxiliary linear operator $\mathscr{L}$, the normal HAM gives three rules to choose a proper linear operator: the rule of solution expression, the rule of coefficient ergodicity, the rule of solution existence, which are discussed systematically in [13,16]. These rules are of great reference value. However, as the wHAM gives larger freedom to choose the auxiliary linear operator, it might be more easy to choose a proper linear operator in the frame of the wHAM. Generally, there are just two main considerations:

(a) The rule of solution existence. It is necessary to choose an auxiliary linear operator that can guarantee the existence of solution for the high-order deformation equation;
(b) The order of the original equation. It is a prior choice to try an linear operator with the same order as the original equation.

How can we guarantee the convergence as well as the efficiency of the solution? First of all, the wHAM is not so sensitive to the choice of the linear operator, as illustrated later via examples. On the other hand, the convergence-control parameter c_0 gives us a convenient way to guarantee the convergence of solution series. Of course, there are some simple ways to choose a proper value of the convergence-control parameter c_0. Generally, the valid interval of the convergence-control parameter c_0 is $[-1,0)$ if the coefficient of the highest order term in the linear operator $\mathscr{L}$ and the original equation are the same. Also, the normal HAM gives an optimal technique to find the "optimal c_0 [16]. In general, a proper value in a subinterval of $[-1,0)$ is acceptable, as illustrated later by some examples.

Considering that the governing Eq. (25) contains second order partial derivatives and is symmetric with respect to x and y, we choose the Laplacian operator as an auxiliary linear operator, i.e.

$$\mathscr{L}[\phi(x, y; q)] = \nabla^2 \phi(x, y; q), \tag{40}$$

which can always ensure the existence of the solution for the BVPs governed by the high-order deformation equations. The more generalized form of the auxiliary linear operator will also be discussed later.

It should be emphasized that the Laplacian operator is not suitable for this problem in the normal HAM, because the Laplacian equation with the boundaries $-1 \le x \le 1, -1 \le y \le 1$ has a very complicated common solution [16,61]:

$$\begin{aligned} V(x, y) = &\sum_{k=0}^{+\infty} [C_{k,1} \sinh(\alpha kx) + C_{k,2} \cosh(\alpha kx)][C_{k,3} \sin(\alpha ky) + C_{k,4} \cos(\alpha ky)] \\ &+ \sum_{k=0}^{+\infty} [C_{k,5} \sinh(\beta ky) + C_{k,6} \cosh(\beta ky)][C_{k,7} \sin(\beta kx) + C_{k,8} \cos(\beta kx)], \end{aligned} \tag{41}$$

where the coefficients α, β and $C_{k,\,j}$ are determined by the boundary conditions. Obviously, in the frame of the normal HAM, it is impossible to satisfy all the boundary conditions and the rule of solution expression at the same time.

Thanks to the wavelet approximation, all the boundary conditions and the rule of solution-expression can be easily satisfied in the frame of the wHAM. Therefore, by the wHAM, it is convenient to use linear differential operators with any kind of common solutions as auxiliary linear operators. Without any doubt, this is a great advantage of the wHAM.

3.2. High-order wavelet-Galerkin equation

Considering the boundary conditions (37) and using the wavelet approximation (13), $V_m(x, y)$ and the right-hand side term $R_m(x, y)$ in (36) can be expressed by

$$V_m(x, y) \approx P^j V_m(x, y) = \sum_{k=1}^{2^j-1} \sum_{l=1}^{2^j-1} V\left(\frac{k}{2^j}, \frac{l}{2^j}\right) \varphi_{j,k}(x) \varphi_{j,l}(y) \tag{42}$$

and

$$R_m(x, y) \approx P^j R_m(x, y) = \sum_{k=0}^{2^j} \sum_{l=0}^{2^j} R_m\left(\frac{k}{2^j}, \frac{l}{2^j}\right) \varphi_{j,k}(x) \varphi_{j,l}(y), \tag{43}$$

where

$$\begin{aligned} R_m\left(\frac{k}{2^j}, \frac{l}{2^j}\right) = &\sum_{s=0}^{m-1} V_s\left(\frac{k}{2^j}, \frac{l}{2^j}\right) \nabla^2 V_{m-1-s}\left(\frac{k}{2^j}, \frac{l}{2^j}\right) \\ &- \sum_{s=0}^{m-1} \nabla V_s\left(\frac{k}{2^j}, \frac{l}{2^j}\right) \cdot \nabla V_{m-1-s}\left(\frac{k}{2^j}, \frac{l}{2^j}\right) - \frac{\lambda}{2}(1 - \chi_m) \end{aligned} \tag{44}$$

in which all the partial derivatives can be approximated by section-based wavelet approximations (18) and (19). Note that the boundary conditions (37) are embedded in (42).

Substituting Eqs. (42) and (43) into (36), we have the mth-order wavelet equation

$$\begin{aligned} &\sum_{k=1}^{2^j-1} \sum_{l=1}^{2^j-1} \left[V_m\left(\frac{k}{2^j}, \frac{l}{2^j}\right) - \chi_m V_{m-1}\left(\frac{k}{2^j}, \frac{l}{2^j}\right) \right] \mathscr{L}\left[\varphi_{j,k}(x) \varphi_{j,l}(y)\right] \\ &= c_0 \sum_{k=0}^{2^j} \sum_{l=0}^{2^j} R_m\left(\frac{k}{2^j}, \frac{l}{2^j}\right) \varphi_{j,k}(x) \varphi_{j,l}(y). \end{aligned} \tag{45}$$

Different from the normal HAM, no additional boundary condition is needed for Eq. (45), since the wavelet approximation (42) satisfies all the boundary conditions (37).

To solve the mth-order wavelet Eq. (45), the Galerkin method is applied, in which the base functions $\varphi_{j,\,k}(x)\varphi_{j,\,l}(y)$ $(k, l = 1, 2, \ldots, 2^j - 1)$ are taken as weight functions. Multiplying both sides of Eq. (45) by $\varphi_{j,\,k}(x)\varphi_{j,\,l}(y)$ $(k, l = 1, 2, \ldots, 2^j - 1)$, respectively, and integrating over the region [0, 1] × [0, 1], we have the mth-order wavelet-Galerlin equation

$$\mathbf{A}\ \mathrm{vec}(\mathbf{V_m} - \chi_m \mathbf{V_{m-1}}) = c_0 \mathbf{B}\ \mathrm{vec}(\mathbf{R_m}), \tag{46}$$

where vec(·) is a vectorization operator which pulls a matrix into a vector row by row, and $\mathbf{V_m}$ and $\mathbf{V_{m-1}}$ are $(2^j-1)\times(2^j-1)$ matrixes defined by

$$\begin{cases} \mathbf{V_m} = \left\{ v_{k,l} = V_m\left(\frac{k}{2^j}, \frac{l}{2^j}\right) \right\}_{k,l=1}^{2^j-1} \\ \mathbf{V_{m-1}} = \left\{ v_{k,l} = V_{m-1}\left(\frac{k}{2^j}, \frac{l}{2^j}\right) \right\}_{k,l=1}^{2^j-1}, \end{cases} \tag{47}$$

$\mathbf{R_m}$ is a $(2^j+1)\times(2^j+1)$ matrix defined by

$$\mathbf{R_m} = \left\{ r_{k,l} = R_m\left(\frac{k}{2^j}, \frac{l}{2^j}\right) \right\}_{k,\ l=0}^{2^j}. \tag{48}$$

Besides, the coefficient matrix **A** is determined by the auxiliary linear operator, and **B** is a constant coefficient matrix. Since Laplacian operator is chosen here as the auxiliary linear operator, the coefficient matrixes can be obtained through

$$\mathbf{A} = \mathbf{A_0}\otimes\mathbf{A_2} + \mathbf{A_2}\otimes\mathbf{A_0} \tag{49}$$

and

$$\mathbf{B} = \mathbf{B_0}\otimes\mathbf{B_0}, \tag{50}$$

where $\otimes$ denotes the operator of Kronecker tensor product, $\mathbf{A_0}$, $\mathbf{A_2}$ are $(2^j-1)\times(2^j-1)$ matrixes, and $\mathbf{B_0}$ is a $(2^j-1)\times(2^j+1)$ matrix, i.e.

$$\begin{cases} \mathbf{A_0} = \left\{ a_{k,l} = \Gamma_{l,k}^{j,0} \right\}_{k,l=1}^{2^j-1} \\ \mathbf{A_2} = \left\{ a_{k,l} = \Gamma_{l,k}^{j,2} \right\}_{k,l=1}^{2^j-1} \\ \mathbf{B_0} = \left\{ b_{k,l} = \Gamma_{l,k}^{j,0} \right\}_{k=1,\quad l=0}^{k=2^j-1,\ l=2^j}, \end{cases} \tag{51}$$

in which $\Gamma_{l,k}^{j,n}$ is the so-called generalized connection coefficients [40,62] defined by

$$\Gamma_{l,k}^{j,n} = \int_0^1 \frac{d^n\varphi_{j,l}(\xi)}{d\xi^n}\varphi_{j,k}(\xi)d\xi. \tag{52}$$

For more details about these generalized connection coefficients, please refer to [40].

By solving mth-order wavelet-Galerlin equations for $m = 1, 2, 3, \dots,$ and substituting the results into Eq. (35), the wavelet approximation are obtained.

4. Results and analysis

All computations mentioned below are carried on the same desktop, i.e. DELL Inspiron 3847, Intel(R) Core (TM) i5-4460 CPU@ 3.20GHz, 8GB memory (SAMSUNG DDR3 1600MHz).

4.1. Convergence analysis

Like the normal HAM, the convergent solution can be obtained by adjusting the convergence-control parameter c_0 in the frame of the wHAM. Without loss of generality, let us consider the case of $\lambda = 1$. Here, $c_0 = -0.4$ is used for the results mentioned below.

Fig. 1 shows the section curves at $y = 1/2$ of the 1st, 5th, 10th, 20th and 50th-order wHAM solutions with the resolution level $j = 5$. It shows that the section curve quickly tends to the convergent solution with the increase of the approximation order. Especially, the 20th-order wHAM solution agrees quite well with the 50th-order solution.

Define the residual error

$$ResErr_m = \frac{1}{(2^j+1)^2}\sum_{k=0}^{2^j}\sum_{l=0}^{2^j}\left[\tilde{u}_m\left(\frac{k}{2^j}, \frac{l}{2^j}\right) - \tilde{u}_{m-1}\left(\frac{k}{2^j}, \frac{l}{2^j}\right)\right]^2, \tag{53}$$

where $\tilde{u}_m(x, y)$ and $\tilde{u}_{m-1}(x, y)$ are the mth-order and the (m−1)th-order wHAM solutions in case of $\lambda = 1$, respectively. As shown in Fig. 2, the residual error decreases sharply with the increase of the approximation order. It indicates that the wHAM solution for the two-dimensional Bratu's problem converges quickly.

Table 2 shows the comparison of the maximum values $\max u(x, y) = u(1/2, 1/2)$ for different values of λ obtained by the weighted residual method (WRM) [52], the optimal HAM (oHAM) and the wHAM (iterative wHAM as well). In the optimal HAM [16], we choose the polynomial as base functions, $\mathscr{L}[\phi(x, y; q)] = \frac{c_2}{xy}\frac{\partial^2\phi(x,y;q)}{\partial x\partial y} + c_4\frac{\partial^4\phi(x,y;q)}{\partial x^2\partial y^2}$ as the auxiliary

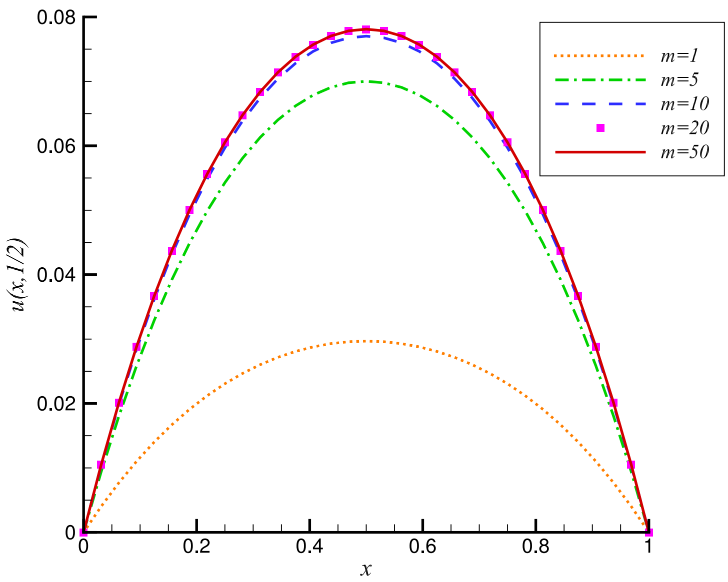

Fig. 1. Section curves at the different orders of approximation, given by the wHAM with the resolution level $j = 5$ and the convergence-control parameter $c_0 = -0.4$. dotted line: $m = 1$; dash-dotted line: $m = 5$; dashed line: $m = 10$; square symbol: $m = 20$; solid line: $m = 50$.

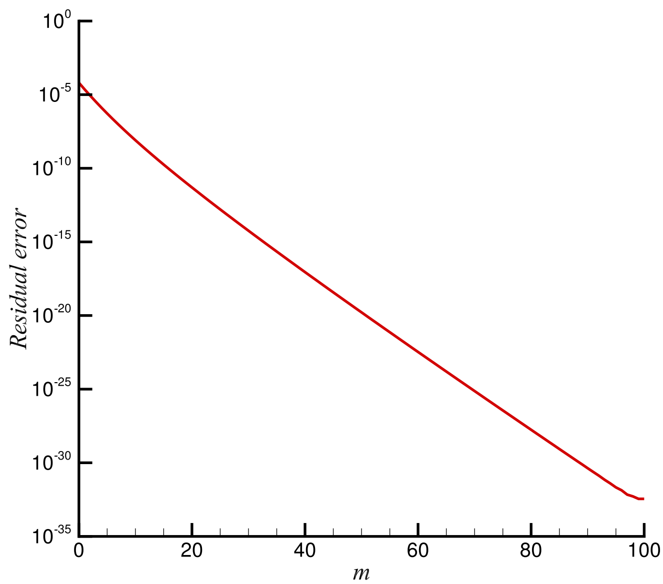

Fig. 2. Residual error of the solution of Bratu's problem in the case of $\lambda = 1$ given by wHAM at the resolution level $j = 5$ by means of the convergence-control parameter $c_0 = -0.4$.

Table 2
The maximum value of the solution by the WRM [52], the optimal HAM (oHAM), the wHAM ($j = 5, c_0 = -0.4$) and the iterative wHAM ($j = 5, c_0 = -0.4$) for different values of λ.

λ	$\max u(x, y)$			
	WRM [52]	oHAM	wHAM	iterative wHAM
1.0	0.07795	0.07810	0.07810	0.07810
2.0	0.16659	0.16689	0.16690	0.16690
3.0	0.26985	0.27036	0.27037	0.27037
4.0	0.39453	0.39552	0.39554	0.39554
5.0	0.55439	0.55696	0.55697	0.55697
6.0	0.78711	0.79711	0.79680	0.79710

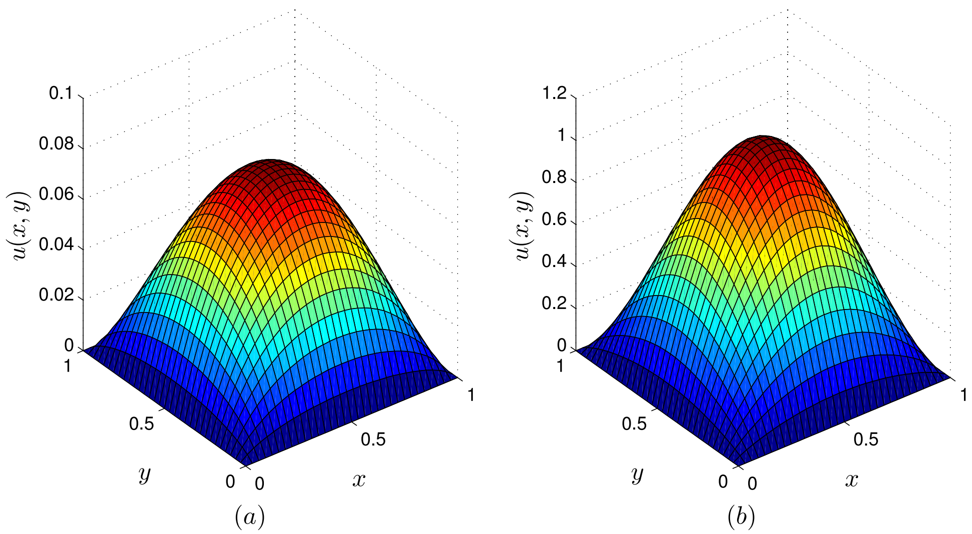

Fig. 3. The 50th-order wavelet solutions of the two-dimensional Bratu's problem in case of $\lambda = 1$ and $\lambda = 6.8 \approx \lambda_c$ by wHAM with resolution level $j = 5$ and convergence control parameter $c_0 = -0.4$. (a) $\lambda = 1$; (b) $\lambda = 6.8$.

Table 3
The 50th-order approximation of $u(1/2, 1/2)$, the corresponding relative error and the used CPU time in case of $\lambda = 1$ by the optimal HAM (oHAM, with $c_0 = -1$, $c_2 = 0$ and $c_4 = -1/2$), the wHAM ($c_0 = -0.4$) and the iteration wHAM ($c_0 = -0.4$).

	Resolution level	$u(1/2, 1/2)$	Err_{50}	CPU time/sec
oHAM	–	0.07810	1.2×10^{-10}	1950.0
wHAM	$j = 4$	0.07815	2.3×10^{-09}	0.56
wHAM	$j = 5$	0.07810	2.5×10^{-11}	1.78
wHAM	$j = 6$	0.07810	1.0×10^{-18}	42.56
Iterative wHAM	$j = 4$	0.07815	2.3×10^{-09}	0.47
Iterative wHAM	$j = 5$	0.07810	2.5×10^{-11}	1.71
Iterative wHAM	$j = 6$	0.07810	1.5×10^{-23}	39.15

linear operator. Through optimizing the parameters c_0, c_2 and c_4, we finally used $c_0 = -1$, $c_2 = 0$ and $c_4 = -1/2$ (as c_0 is correlated to the other two parameter, it is set as $c_0 = -1$). For more details about this method, please refer to Liao [16]. Obviously, the results obtained by the wHAM as well as the iterative wHAM agree well with the others.

Convergent solutions can be obtained by wHAM in case of $\lambda \in (0, \lambda_c]$. Fig. 3 gives the 50th-order wHAM solutions in case of $\lambda = 1$ and $\lambda = 6.8 \approx \lambda_c$. For a solid fuel ignition model, it presents the distribution of the temperature at the steady state, which shows that the temperature decrease continuously from the midpoint to the boundaries. The results indicate that when $\lambda \le \lambda_c$, the system can reach a steady state, which means that the competing forces of cooling (due to diffusion from the boundary) and heating (due to the positive reaction term) are in balance. However, when $\lambda > \lambda_c$, no convergent solution could be obtained. From the physical point of view, for large values of λ, the reaction term will dominate and drive the temperature to infinity (explosion), so the system cannot reach a steady state.

4.2. High efficiency of the wHAM

Note that the two-dimensional Bratu's problem was solved by the optimal HAM [16] by using the polynomial as base functions and $\mathscr{L}[\phi(x, y; q)] = \frac{c_2}{xy}\frac{\partial^2 \phi(x,y;q)}{\partial x \partial y} + c_4 \frac{\partial^4 \phi(x,y;q)}{\partial x^2 \partial y^2}$ as the auxiliary linear operator.

In order to compare the results among the optimal HAM (oHAM), the wHAM and the iteration wHAM, the 120th-order wHAM solution with resolution level $j = 6$ is used as a reference solution $u_{ref}(x, y)$, and the relative error is defined by

$$Err_m = \frac{1}{(2^j+1)^2} \sum_{k=0}^{2^j} \sum_{l=0}^{2^j} \left[\tilde{u}_m\left(\frac{k}{2^j}, \frac{l}{2^j}\right) - u_{ref}\left(\frac{k}{2^j}, \frac{l}{2^j}\right) \right]^2, \tag{54}$$

where $\tilde{u}_m(x, y)$ denotes the mth-order HAM solution and j is the resolution level of wavelet.

Table 3 presents the maximum value $u(1/2, 1/2)$ of 50th-order approximation, the corresponding relative error Err_{50} and used CPU time, respectively. Here, we only give results of the 1st-order iteration of wHAM. Obviously, the wHAM needs much less CPU time than the optimal HAM. The wHAM needs less than two seconds at the resolution level $j \le 5$ and less

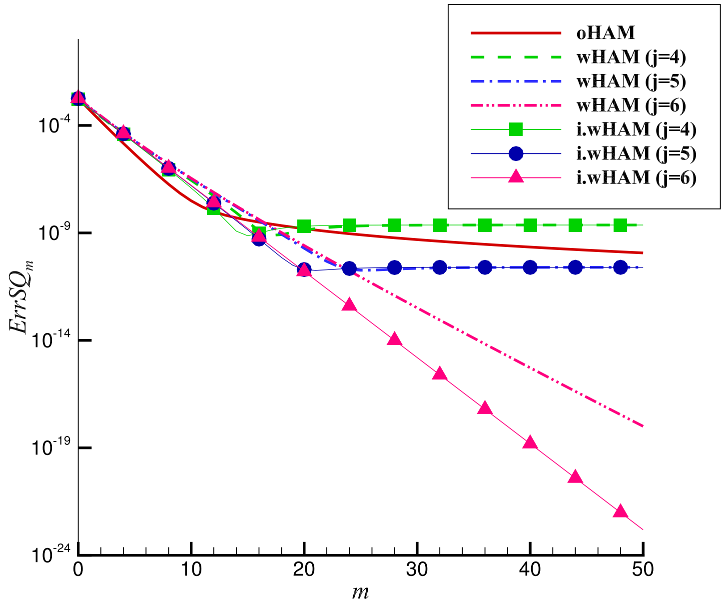

Fig. 4. Comparison of the relative error versus the approximation order or the iteration time for the two-dimensional Bratu's problem in the case of $\lambda = 1$ by means of the optimal HAM ($c_0 = -1$, $c_2 = 0$ and $c_4 = -1/2$), the wHAM ($c_0 = -0.4$) and 1st-order iterative wHAM ($c_0 = -0.4$) at the different resolution levels. Solid line: oHAM; dashed line: wHAM with $j = 4$; dash-dotted line: wHAM with $j = 5$; dash-dot-dotted line: wHAM with $j = 6$; line with square symbol: iterative wHAM with $j = 4$; line with circle symbol: iterative wHAM with $j = 5$; line with delta symbol: iterative wHAM with $j = 6$.

than a minute at $j = 6$. However, by the optimal HAM, it takes more than half an hour, which is 40 more times longer. Besides, at $j \geq 5$, the 50th-order approximation given by the wHAM is more accurate than the optimal HAM. So, the wHAM is more efficient than the optimal HAM. Fig. 4 gives a clear comparison among the optimal HAM, the wHAM and the iterative HAM. It clearly shows that the wHAM is more efficient than the optimal HAM and that iteration can greatly accelerate the convergence of the wHAM. Theoretically the error of the normal HAM can infinitely get near to zero with the increase of the approximation order while at a given resolution level the wHAM will reach a constant level (due to the limitation of the significance digit). However, the normal HAM costs much more time than the wHAM to reach the same level. Obviously, the wHAM possesses much higher computational efficiency than the normal HAM. This is an advantage of the wHAM.

4.3. Influence of the auxiliary linear operator

To further reveal the influence of the auxiliary linear operator and the convergence-control parameter c_0, a generalized form of the auxiliary linear operator is tested. Considering that Eq. (25) is a second-order PDE and has symmetry with respect to x and y, we consider such a generalized form of the second-order symmetric auxiliary linear operator

$$\begin{aligned}\mathscr{L}[\phi(x,y;q)] = &\frac{\partial^2\phi(x,y;q)}{\partial x^2} + \frac{\partial^2\phi(x,y;q)}{\partial y^2}\\ &+\kappa_1\left[\frac{\partial\phi(x,y;q)}{\partial x} + \frac{\partial\phi(x,y;q)}{\partial y}\right] + \kappa_0\phi(x,y;q),\end{aligned} \tag{55}$$

where κ_0 and κ_1 are two constants. Obviously, different values of κ_0 and κ_1 correspond to rather different auxiliary linear operators that lead to quite different base functions for the solution expression. We solve the two-dimensional Bratu's equation in the frame of the wHAM using the generalized auxiiary linear operator (55) with different values of $\kappa_0 \in [0, 5]$ and $\kappa_1 \in [0, 5]$ in case of $\lambda = 1$ by means of $c_0 = -0.4$ and the resolution level $j = 5$. The relative errors and used CPU times for the 50th-order approximation are presented in Table 4. It is very interesting that the convergence of solution series and especially even the computational efficiency are *not* sensitive to the choice of the linear operator: the approximations with the *same* accuracy are always obtained in a few seconds, using almost the *same* CPU times ! Indeed, compared to other analytic methods, such as perturbation techniques and so on, the normal HAM provides us large freedom to choose the auxiliary linear operator. However, Table 4 clearly indicates that the wHAM possesses even larger freedom on the choice of the auxiliary linear operator than the normal HAM. So, in the frame of the wHAM, it becomes easier to choose a proper auxiliary linear operator for a given problem.

Table 4
Relative error and used CPU time for 50th-order approximation given by means of the wHAM with different auxiliary linear operators in case of $\lambda = 1$, $c_0 = -0.4$ and $j = 5$.

κ_1	κ_0	Err_{50}	CPU time/sec
0	0	2.5×10^{-11}	1.78
0	1	2.5×10^{-11}	1.89
0	5	2.5×10^{-11}	1.88
1	0	2.5×10^{-11}	1.95
5	0	2.5×10^{-11}	1.96
1	1	2.5×10^{-11}	2.02
5	5	2.5×10^{-11}	2.02

Table 5
Relative error and used CPU time for 50th-order approximation given by means of the wHAM using different convergence-control parameters in case of $\lambda = 1$, $\kappa_0 = \kappa_1 = 0$ and $j = 5$.

c_0	Err_{50}	CPU time/sec
−0.10	1.9×10^{-07}	1.77
−0.15	2.3×10^{-09}	1.78
−0.20	3.0×10^{-11}	1.79
−0.25	2.2×10^{-11}	1.79
−0.30	2.5×10^{-11}	1.78
−0.35	2.5×10^{-11}	1.77
−0.40	2.5×10^{-11}	1.78

4.4. Influence of the convergence-control parameter

The convergence-control parameter c_0 plays an important role in the frame of the HAM [16,17]. To illustrate the influence of c_0, some different convergence-control parameters are tested by means of the wHAM using the Laplacian operator as the auxiliary linear operator, i.e. $\kappa_0 = \kappa_1 = 0$. The relative error and used CPU time for 50th-order approximation are listed in Table 5. Note that, for $c_0 \in [-0.4, -0.2]$, we always obtain the convergent approximation with the same accuracy almost in the same CPU times! In case of $c_0 \in (-0.2, 0)$, we can still gain the convergent results, but with a smaller convergence rate. Thus, in the frame of the wHAM, the computational efficiency is not very sensitive to the convergence-control parameter c_0. This is another advantage of the wHAM.

In the normal HAM, Liao [17] suggested to use the so-called optimal convergence-control parameter, determined by the minimum of the residual error square of considered equations. Similarly, one can also choose such an optimal value of the convergence-control parameter in the frame of the wHAM. In this way, we can always guarantee the convergence of solution series given by the wHAM.

5. Concluding remarks and discussions

In this paper, we propose a new analytic approach for nonlinear partial differential equations, namely, the wavelet homotopy analysis method (wHAM). Based on the Coiflet-type wavelet, a multi-resolution analysis for $L^2([0, 1]^2)$ is constructed and arbitrary L^2 function can be conveniently approximated. However, if the traditional method is used to calculate the partial derivatives, it leads to serious loss of computational efficiency and approximation accuracy. To solve these, a section-based method is proposed to approximate the derivatives, which not only successfully avoids serious accuracy loss, but also significantly improves the computational efficiency. In this way, the wHAM can be widely used to solve high-dimensional PDEs with high accuracy and computational efficiency.

As an example, the two-dimensional Bratu's boundary value problem is used to illustrate the basic ideas as well as the validity of the wHAM for nonlinear PDEs. By means of the Laplacian operator as an auxiliary linear operator and adjusting the convergence-control parameter c_0, convergent results are obtained, which agree well with those obtained by other methods.

Compared to the normal HAM, the wHAM has some obvious advantages. Firstly, the wHAM provides us larger freedom to choose the auxiliary linear operator than the normal HAM. Secondly, the wHAM possesses much higher computational efficiency than the normal HAM. In addition, it is found that the iteration approach of the wHAM can greatly accelerate the convergence. Therefore, the iteration wHAM is strongly suggested in practice.

Moreover, it is found that the convergence of solution series and the computational efficiency is not sensitive to the choice of the auxiliary linear operator, which makes it easy for users to use the wHAM. On the other hand, like the normal HAM, the convergence-control parameter c_0 also provides us a convenient way to guarantee the convergence of solution series.

It should be emphasized that, the wHAM is still a kind of analytic approximation approach. So, like perturbation methods and the normal HAM, the wHAM is suitable mainly for equations defined in regular domains. Besides, since the wHAM is not a numerical one, it is unnecessary to be automatic.

Without doubt, the wHAM can be used to solve high-dimensional nonlinear boundary value problems. Its great freedom on the choice of the auxiliary linear operator, good convergence property and the high computational efficiency bring great potential for the wHAM to be used for more complicated boundary value problems in science and engineering.

In the frame of the HAM, Liao [13,14] proved such a convergence theorem that the homotopy series must tend to a solution of nonlinear equations under consideration, as long as its residual errors tend to zero. This is the reason why we compare the residual errors of analytic approximations given by the different HAM approaches in this paper. However, it is a pity that there do not exist a generally valid theory on error estimation of approximations given by the different HAM approaches. Obviously, such a theory on error estimation should be of benefit to wider applications of the HAM in general.

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