Commun Nonlinear Sci Numer Simulat 14 (2009) 983–997

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Short communication

Notes on the homotopy analysis method: Some definitions and theorems

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ARTICLE INFO

Article history:
Received 18 April 2008
Accepted 18 April 2008
Available online 30 April 2008

PACS:
05.10.−a
04.25.Nx
02.70.Wz

Keywords:
Nonlinear equations
Series solution
Homotopy analysis method

ABSTRACT

We describe, very briefly, the basic ideas and current developments of the homotopy analysis method, an analytic approach to get convergent series solutions of strongly nonlinear problems, which recently attracts interests of more and more researchers. Definitions of some new concepts such as the homotopy-derivative, the convergence-control parameter and so on, are given to redescribe the method more rigorously. Some lemmas and theorems about the homotopy-derivative and the deformation equation are proved. Besides, a few open questions are discussed, and a hypothesis is put forward for future studies.

1. Introduction

Nonlinear equations are difficult to solve, especially analytically. Perturbation techniques [1–12] are widely applied in science and engineering, and do great contribution to help us understand many nonlinear phenomena. However, it is well known that perturbation methods are strongly dependent upon small/large physical parameters, and therefore are valid in principle only for weakly nonlinear problems. The so-called non-perturbation techniques, such as the Lyapunov's artificial small parameter method [13], the δ-expansion method [14,15], Adomian's decomposition method [16–19], and so on, are formally independent of small/large physical parameters. But, all of these traditional non-perturbation methods can *not* ensure the convergence of solution series: they are *in fact* only valid for weakly nonlinear problems, too.

The homotopy analysis method (HAM) [20–27] is a general analytic approach to get series solutions of various types of nonlinear equations, including algebraic equations, ordinary differential equations, partial differential equations, differential-integral equations, differential-difference equation, and coupled equations of them. Unlike perturbation methods, the HAM is independent of small/large physical parameters, and thus is valid no matter whether a nonlinear problem contains small/large physical parameters or not. More importantly, different from all perturbation and traditional non-perturbation methods, the HAM provides us a simple way to ensure the convergence of solution series, and therefore, the HAM is valid even for strongly nonlinear problems. Besides, different from all perturbation and previous non-perturbation methods, the HAM provides us with great freedom to choose proper base functions to approximate a nonlinear problem [21,26]. Since Liao's book [21] for the homotopy analysis method was published in 2003, more and more researchers have been successfully applying this method to various nonlinear problems in science and engineering, such as the viscous flows of non-Newtonian fluids [28–38], the KdV-type equations [39–43], nonlinear heat transfer [44–46], finance problems [47,48], Riemann problems related to nonlinear shallow water equations [49], projectile motion [50], Glauert-jet flow [51], nonlinear water waves [52], groundwater flows

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1007-5704/$ - see front matter
doi:10.1016/j.cnsns.2008.04.013

[53], Burgers–Huxley equation [54], time-dependent Emden–Fowler type equations [55], differential-difference equation [56], Laplace equation with Dirichlet and Neumann boundary conditions [57], thermal–hydraulic networks [58], boundary layer flows over a stretching surface with suction and injection [59], and so on. Especially, some new solutions of a few nonlinear equations are reported [60,61]: these new solutions have *never* been reported by all other previous analytic methods and *even* by numerical methods. This shows the great potential of the HAM for strongly nonlinear problems in science and engineering.

The HAM is based on homotopy, a fundamental concept in topology and differential geometry [62], which can be traced back to Poincaré [63]. Briefly speaking, by means of the HAM, one constructs a continuous mapping of an initial guess approximation to the exact solution of considered equations. An auxiliary linear operator is chosen to construct such kind of continuous mapping, and an auxiliary parameter is used to ensure the convergence of solution series. The method enjoys great freedom in choosing initial approximations and auxiliary linear operators. By means of this kind of freedom, a complicated nonlinear problem can be transferred into an infinite number of simpler, linear sub-problems, as shown by Liao and Tan [26].

For example, let us consider a nonlinear algebraic equation

$$f(x) = 0.$$

First of all, we construct such a homotopy:

$$\mathscr{H}[x;q] = (1-q)[f(x) - f(x_0)] + qf(x),$$

where x_0 is an initial guess of x, and $q \in [0,1]$ is called homotopy-parameter.[1] Obviously, at $q = 0$ and $q = 1$, one has

$$\mathscr{H}[x;0] = f(x) - f(x_0), \quad \mathscr{H}[x;1] = f(x),$$

respectively. Thus, as q increases from 0 to 1, $\mathscr{H}[x;q]$ varies continuously from $f(x) - f(x_0)$ to $f(x)$. Such kind of continuous variation is called deformation in topology [62]. Now, enforcing $\mathscr{H}[x;q] = 0$, i.e.

$$(1-q)[f(x) - f(x_0)] + qf(x) = 0,$$

we have now a family of algebraic equations. Obviously, the solution of the above family of algebraic equations is dependent upon the homotopy-parameter q. So, the family of equations can be rewritten as

$$(1-q)\{f[\phi(q)] - f(x_0)\} + qf[\phi(q)] = 0. \tag{1}$$

At $q = 0$, it gives

$$f[\phi(q)] - f(x_0) = 0, \quad \text{when } q = 0,$$

whose solution is obviously

$$\phi|_{q=0} = \phi(0) = x_0.$$

At $q = 1$, one has

$$f[\phi(q)] = 0, \quad \text{when } q = 1.$$

It is exactly the same as the original algebraic equation $f(x) = 0$, thus

$$\phi|_{q=1} = \phi(1) = x.$$

Therefore, as the homotopy-parameter q increases from 0 to 1, $\phi(q)$ varies (or deforms) from the initial guess x_0 to the solution x of $f(x) = 0$. We call the family of equations like (1) *the zeroth-order deformation equation* (the more rigorous definition will be given in the following section).

Because $\phi(q)$ is now a function of the homotopy-parameter q, we can expand it into Maclaurin series

$$\phi(q) = x_0 + \sum_{k=1}^{+\infty} x_k q^k, \tag{2}$$

where $\phi(0) = x_0$ is employed, and

$$x_k = \frac{1}{k!} \frac{\partial^k \phi(q)}{\partial q^k}\bigg|_{q=0} = D_k(\phi). \tag{3}$$

Here, the series (2) is called *homotopy-series*, $D_k(\phi)$ is called *the kth-order homotopy-derivative* of ϕ (more rigorous definitions and some related theorems will be given in the following section). If the homotopy-series (2) is convergent at $q = 1$, then using the relationship $\phi(1) = x$, one has the so-called *homotopy-series solution*

$$x = x_0 + \sum_{k=1}^{+\infty} x_k. \tag{4}$$

Unfortunately, convergence radii of many Maclaurin series of functions are less than 1. So, here, we had to *assume* that the homotopy-series is convergent at $q = 1$. This restriction can be overcome by introducing an auxiliary parameter, as shown later.

[1] In the theory of topology, q is called the embedding parameter.

According to the fundamental theorem of calculus about Taylor series, the coefficient x_k of the homotopy-series (2) is unique. Therefore, the governing equation of x_k is unique, too, and can be deduced directly from the zeroth-order deformation equation (1). Taking the 1st-order homotopy-derivative on both sides of the zeroth-order deformation equation (1) gives the so-called *1st-order deformation equation*:

$$x_1 f'(x_0) + f(x_0) = 0,$$

whose solution is

$$x_1 = -\frac{f(x_0)}{f'(x_0)}.$$

Taking the 2nd-order homotopy-derivative[2] on both sides of (1) gives the *2nd-order deformation equation*:

$$x_2 f'(x_0) + \frac{1}{2} x_1^2 f''(x_0) = 0,$$

whose solution is

$$x_2 = -\frac{x_1^2 f''(x_0)}{2f'(x_0)} = -\frac{f^2(x_0) f''(x_0)}{2[f'(x_0)]^3}.$$

In this way, one obtains x_k one by one in the order $k = 1, 2, 3, \ldots$. Here, we emphasize that all of these high-order deformation equations are *linear*, and therefore are easy to solve. Then, we have the *1st-order homotopy-series approximation*

$$x \approx= x_0 + x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{5}$$

and the *2nd-order homotopy-series approximation*

$$x \approx x_0 + x_1 + x_2 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f^2(x_0) f''(x_0)}{2[f'(x_0)]^3}. \tag{6}$$

Note that (5) is exactly the same as the famous Newton's iteration formula, and thus (6) can be regarded as the 2nd-order Newton's iteration formula. In fact, one can give a family of Newton's iteration formula in a similar way.

The above analytic approach is independent of any physical parameters at all: no matter whether a nonlinear problem contains small/large physical parameters or not, one can always introduce the homotopy-parameter $q \in [0, 1]$ to construct a zeroth-order deformation equation and then to get the homotopy-series solution. Unfortunately, the homotopy-series like (2) is not always convergent at $q = 1$, therefore the corresponding homotopy-series solution (4) might be divergent. For example, it is well-known that Newton's 1st-order iteration formula (5) often gives divergent results. This is mainly because the above approach is based on such a assumption that the homotopy-series like (2) is convergent at $q = 1$, but this assumption does *not* hold in general, especially for nonlinear problems with strong nonlinearity. To overcome this restriction of the early HAM, Liao [22] introduced an auxiliary parameter $\hbar \neq 0$ to construct such a kind of zeroth-order deformation equation

$$(1 - q)\{f[\phi(q)] - f(x_0)\} = q\hbar f[\phi(q)]. \tag{7}$$

Since $\hbar \neq 0$, the above equation at $q = 1$ becomes

$$\hbar f[\phi(q)] = 0, \quad \text{when } q = 1,$$

which is equivalent to the original equation $f(x) = 0$, provided $x = \phi(1)$. All other formulas like (2) and (4) are the same, except the high-order deformation equation. Similarly, taking the 1st-order homotopy-derivative on both sides of (7), we have the corresponding 1st-order deformation equation

$$x_1 f'(x_0) - \hbar f(x_0) = 0,$$

whose solution is

$$x_1 = \hbar \frac{f(x_0)}{f'(x_0)}.$$

Taking the 2nd-order homotopy-derivative on both sides of (7) gives the 2nd-order deformation equation:

$$x_2 f'(x_0) - (1 + \hbar) x_1 f'(x_0) + \frac{1}{2} x_1^2 f''(x_0) = 0,$$

whose solution is

$$x_2 = (1 + \hbar) x_1 f'(x_0) - \frac{x_1^2 f''(x_0)}{2f'(x_0)} = \hbar(1 + \hbar) f(x_0) - \frac{f^2(x_0) f''(x_0)}{2[f'(x_0)]^3}.$$

[2] According to the definition (3), we here differentiates the zeroth-order deformation equation (1) twice with respect to q, then divides by $2!$ and finally sets $q = 0$.

Similarly, one has the corresponding first-order homotopy-series approximation

$$x \approx= x_0 + x_1 = x_0 + \hbar \frac{f(x_0)}{f'(x_0)}, \tag{8}$$

and the corresponding 2nd-order homotopy-series approximation

$$x \approx x_0 + x_1 + x_2 = (1 + \hbar + \hbar^2)x_0 + \hbar \frac{f(x_0)}{f'(x_0)} - \frac{f^2(x_0)f''(x_0)}{2[f'(x_0)]^3}. \tag{9}$$

Obviously, (5) and (6) are special cases of (8) and (9) when $\hbar = -1$, respectively. The auxiliary-parameter $\hbar$ in (5) can be regarded as a iteration factor that is widely used in numerical computations. It is well known that a properly chosen iteration factor can ensure the convergence of iteration. Similarly, it is found that the convergence of the homotopy-series like (2) is dependent upon the value of $\hbar$: one can ensure the convergence of the homotopy-series solution simply by means of choosing a proper value of $\hbar$, as shown by Liao [21–23,25,26,60] and others [28]-[59]. In fact, it is the auxiliary parameter $\hbar$ that provides us, for the first time, a simple way to ensure the convergence of series solution. Due to this reason, it seems reasonable to rename $\hbar$ *the convergence-control parameter*, which was suggested by Dr. Pradeep Siddheshwar in our private discussions.

It should be emphasized that, without the use of the convergence-parameter $\hbar$, one had to *assume* that the homotopy-series like (2) is convergent. However, with the use of the convergence-parameter $\hbar$, such an assumption is *unnecessary*, because it seems that one can always choose a proper value of $\hbar$ to obtain convergent homotopy-series solution. So, the use of the convergence-parameter $\hbar$ in the zeroth-order deformation equation greatly modifies the early homotopy analysis method. Since then, the homotopy analysis method have been developing greatly, and more generalized zeroth-order deformation equations are suggested by Liao [21,23,24,26]. Currently, some developments [64–66] of the HAM are reported.

At the current stage of the HAM, it is urgently necessary to redescribe this method in a more rigorous way. So, in this paper, definitions of some new concepts such as the homotopy-derivative, the convergence-control parameter, the convergence-control vector, and so on, are given so as to redescribe the method more rigorously. Besides, some lemmas and theorems about the homotopy-derivative and the deformation equation are proved. Furthermore, a few open questions are discussed, and a hypothesis is put forward for future studies. All of these can help the HAM users easily understand this method, and simplify the applications of the HAM for new, complicated nonlinear problems in science and engineering.

2. Properties of homotopy-derivative

As mentioned in Section 1 the so-called *homotopy-derivative* is used to deduce the high-order deformation equation. Here, we first give rigorous definitions and then prove some properties of the homotopy-derivative. These properties are useful to deduce the high-order deformation equations.

Definition 2.1. Let ϕ be a function of the homotopy-parameter q, then

$$D_m(\phi) = \frac{1}{m!} \left. \frac{\partial^m \phi}{\partial q^m} \right|_{q=0} \tag{10}$$

is called the mth-order homotopy-derivative of ϕ, where $m \geqslant 0$ is an integer.

Definition 2.2. Let $\mathcal{N}[u] = 0$ denote a nonlinear equation, ϕ be a function of the homotopy-parameter $q \in [0,1]$, whose Maclaurin series is

$$\phi = \sum_{k=0}^{+\infty} u_k q^k. \tag{11}$$

The family of equations

$$\Pi[\phi, q] = 0, \quad q \in [0,1]$$

is called the *zeroth-order deformation equation* of $\mathcal{N}[u] = 0$, if, at $q = 1$, it is equivalent to the original equation $\mathcal{N}[u] = 0$ so that

$$u = \phi|_{q=1} = \sum_{k=0}^{+\infty} u_k, \tag{12}$$

and besides, its solution is obvious at $q = 0$. The series (11) is called the *homotopy-series*, the series (12) is called *homotopy-series solution* of $\mathcal{N}[u] = 0$, and the equations governing u_k are called the *kth-order deformation equations*.

Molabahrami and Khani [54] proved the following theorem:

Molabahrami and Khani's Theorem. *For homotopy-series*

$$\phi = \sum_{i=0}^{+\infty} u_i q^i,$$

it holds

$$D_m(\phi^k) = \sum_{r_1=0}^{m} u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \cdots \sum_{r_{k-2}=0}^{r_{k-3}} u_{r_{k-3}-r_{k-2}} \sum_{r_{k-1}=0}^{r_{k-2}} u_{r_{k-2}-r_{k-1}} u_{r_{k-1}},$$

where $m \geqslant 0$ and $k \geqslant 1$ are positive integers.

For details, please refer to Molabahrami and Khani [54]. Here, we prove some other properties of the homotopy-derivative.

Theorem 2.1. *Let f and g be functions independent of the homotopy-parameter q. For homotopy-series*

$$\phi = \sum_{i=0}^{+\infty} u_i q^i, \quad \psi = \sum_{j=0}^{+\infty} v_j q^j,$$

it holds

$$D_m(f\phi + g\psi) = fD_m(\phi) + gD_m(\psi).$$

Proof. Because f and g are independent of q, and besides D_m defined by (10) is a liner operator, it obviously holds

$$D_m(f\phi + g\psi) = D_m(f\phi) + D_m(g\psi) = fD_m(\phi) + gD_m(\psi). \qquad \square$$

Theorem 2.2. *For homotopy-series*

$$\phi = \sum_{i=0}^{+\infty} u_i q^i, \quad \psi = \sum_{j=0}^{+\infty} v_j q^j,$$

it holds

(a) $D_m(\phi) = u_m,$

(b) $D_m(q^k\phi) = D_{m-k}(\phi),$

(c) $D_m(\phi\psi) = \sum_{i=0}^{m} D_i(\phi) D_{m-i}(\psi) = \sum_{i=0}^{m} D_i(\psi) D_{m-i}(\phi),$

(d) $D_m(\phi^n\psi^l) = \sum_{i=0}^{m} D_i(\phi^n) D_{m-i}(\psi^l) = \sum_{i=0}^{m} D_i(\psi^l) D_{m-i}(\phi^n),$

where $m \geqslant 0$, $n \geqslant 0$, $l \geqslant 0$ and $0 \leqslant k \leqslant m$ are integers.

Proof. (A) According to Taylor theorem, the unique coefficient u_m of the Maclaurin series of ϕ is given by

$$u_m = \frac{1}{m!} \frac{\partial^m \phi}{\partial q^m}\bigg|_{q=0},$$

which gives (a) by means of the definition of $D_m(\phi)$.

(B) It holds

$$q^k\phi = q^k \sum_{i=0}^{+\infty} u_i q^i = \sum_{i=0}^{+\infty} u_i q^{i+k} = \sum_{m=k}^{+\infty} u_{m-k} q^m,$$

which gives by means of (a) that

$$D_m(q^k\phi) = u_{m-k} = D_{m-k}(\phi).$$

(C) According to Leibnitz's rule for derivatives of product, it holds

$$\frac{\partial^m(\phi\psi)}{\partial q^m} = \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} \frac{\partial^i \phi}{\partial q^i} \frac{\partial^{m-i}\psi}{\partial q^{m-i}} = \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} \frac{\partial^i \psi}{\partial q^i} \frac{\partial^{m-i}\phi}{\partial q^{m-i}},$$

which gives by the definition (10):

$$D_m(\phi\psi) = \frac{1}{m!} \frac{\partial^m(\phi\psi)}{\partial q^m}\bigg|_{q=0} = \sum_{i=0}^{+\infty} \left(\frac{1}{i!} \frac{\partial^i \phi}{\partial q^i}\bigg|_{q=0} \right) \left(\frac{1}{(m-i)!} \frac{\partial^{m-i}\psi}{\partial q^{m-i}}\bigg|_{q=0} \right) = \sum_{i=0}^{+\infty} D_i(\phi) D_{m-i}(\psi).$$

Similarly, it holds

$$D_m(\phi\psi) = \sum_{i=0}^{+\infty} D_i(\psi) D_{m-i}(\phi).$$

(D) Write $\Phi = \phi^n$ and $\Psi = \psi^l$. According to (c), it holds

$$D_m(\phi^n\psi^l) = D_m(\Phi\Psi) = \sum_{i=0}^{+\infty} D_i(\Phi)D_{m-i}(\Psi) = \sum_{i=0}^{+\infty} D_i(\phi^n)D_{m-i}(\psi^l).$$

Similarly, it holds

$$D_m(\phi^n\psi^l) = \sum_{i=0}^{+\infty} D_i(\psi^l)D_{m-i}(\phi^n),$$

which ends the proof. □

Theorem 2.3. *Let $\mathscr{L}$ be a linear operator independent of the homotopy-parameter q. For homotopy-series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k,$$

it holds

$$D_m(\mathscr{L}\phi) = \mathscr{L}[D_m(\phi)].$$

Proof. Since $\mathscr{L}$ is independent of q, it holds

$$\mathscr{L}\phi = \sum_{k=0}^{+\infty} [\mathscr{L}(u_k)]q^k.$$

Taking mth-order homotopy-derivative on both sides of the above expression and using Theorem 2.1(a), one has $D_m(\mathscr{L}\phi) = \mathscr{L}(u_m)$. On the other side, according to Theorem 2.1(a), it holds obviously $\mathscr{L}[D_m(\phi)] = \mathscr{L}(u_m)$. Thus, $D_m(\mathscr{L}\phi) = \mathscr{L}[D_m(\phi)]$ holds. □

Theorem 2.4. *For homotopy-series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k,$$

it holds the recurrence formulas

$$D_0(e^\phi) = e^{u_0},$$

$$D_m(e^\phi) = \sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)D_k(e^\phi)D_{m-k}(\phi),$$

where $m \geqslant 1$ is integer.

Proof. According to the definition (10) of the operator D_m, it holds obviously

$$D_0(e^\phi) = e^{u_0}.$$

Besides, one has

$$\frac{\partial e^\phi}{\partial q} = e^\phi\frac{\partial \phi}{\partial q}.$$

Thus, according to Leibnitz's rule for derivatives of product, it holds

$$\frac{1}{m!}\frac{\partial^m e^\phi}{\partial q^m} = \frac{1}{m!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left(e^\phi\frac{\partial\phi}{\partial q}\right) = \frac{1}{m}\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}\frac{\partial^k e^\phi}{\partial q^k}\frac{\partial^{m-k}\phi}{\partial q^{m-k}} = \sum_{k=0}^{m-1}\frac{(m-k)}{m}\left[\frac{1}{k!}\frac{\partial^k e^\phi}{\partial q^k}\right]\left[\frac{1}{(m-k)!}\frac{\partial^{m-k}\phi}{\partial q^{m-k}}\right].$$

Setting $q = 0$ in above expression and using the definition (10), one has

$$D_m(e^\phi) = \sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)D_k(e^\phi)D_{m-k}(\phi),$$

where $m \geqslant 1$ is an integer. □

Theorem 2.5. *For homotopy-series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k,$$

it holds the recurrence formulas

$$D_0(\sin\phi) = \sin(u_0), \quad D_0(\cos\phi) = \cos(u_0),$$
$$D_m(\sin\phi) = \sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)D_k(\cos\phi)D_{m-k}(\phi),$$
$$D_m(\cos\phi) = -\sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)D_k(\sin\phi)D_{m-k}(\phi),$$

where $m \geqslant 1$ is an integer.

Proof. According to the definition (10), it holds obviously

$$D_0(\sin\phi) = \sin(u_0), \quad D_0(\cos\phi) = \cos(u_0).$$

Write $\mathrm{i} = \sqrt{-1}$. Using Euler formula and Theorem 2.1, it holds for an integer $m \geqslant 1$ that

$$D_m(\sin\phi) = D_m\left(\frac{e^{\mathrm{i}\phi}-e^{-\mathrm{i}\phi}}{2i}\right) = \frac{1}{2\mathrm{i}}\left[D_m(e^{\mathrm{i}\phi}) - D_m(e^{-\mathrm{i}\phi})\right] \tag{13}$$

and

$$D_m(\cos\phi) = D_m\left(\frac{e^{\mathrm{i}\phi}+e^{-\mathrm{i}\phi}}{2}\right) = \frac{1}{2}\left[D_m(e^{\mathrm{i}\phi}) + D_m(e^{-\mathrm{i}\phi})\right]. \tag{14}$$

Using Theorem 2.4 and then Theorem 2.1, we have

$$D_m(e^{\mathrm{i}\phi}) = \sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)D_k(e^{\mathrm{i}\phi})D_{m-k}(\mathrm{i}\phi) = \mathrm{i}\sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)D_k(e^{\mathrm{i}\phi})D_{m-k}(\phi)$$

and similarly,

$$D_m(e^{-\mathrm{i}\phi}) = -\mathrm{i}\sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)D_k(e^{-\mathrm{i}\phi})D_{m-k}(\phi).$$

Substituting the above two expressions into (13) and (14), then using Theorem 2.1 and Euler formula, we have

$$\begin{aligned} D_m(\sin\phi) &= \frac{1}{2}\sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)D_{m-k}(\phi)\left[D_k(e^{\mathrm{i}\phi}) + D_k(e^{-\mathrm{i}\phi})\right] = \sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)D_{m-k}(\phi)D_k\left(\frac{e^{\mathrm{i}\phi}+e^{-\mathrm{i}\phi}}{2}\right) \\ &= \sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)D_{m-k}(\phi)D_k(\cos\phi), \end{aligned}$$

and similarly

$$\begin{aligned} D_m(\cos\phi) &= \frac{i}{2}\sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)D_{m-k}(\phi)\left[D_k(e^{\mathrm{i}\phi}) - D_k(e^{-\mathrm{i}\phi})\right] = -\sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)D_{m-k}(\phi)D_k\left(\frac{e^{\mathrm{i}\phi}-e^{-\mathrm{i}\phi}}{2i}\right) \\ &= -\sum_{k=0}^{m-1}\left(1-\frac{k}{m}\right)D_{m-k}(\phi)D_k(\sin\phi). \end{aligned}$$

This ends the proof. □

Theorem 2.6. *If the two homotopy-series*

$$\phi = \sum_{i=0}^{+\infty} u_i q^i, \quad \psi = \sum_{j=0}^{+\infty} v_j q^j,$$

satisfy $\phi = \psi$ in a domain $q \in [0, a)$, then $D_m(\phi) = D_m(\psi)$ and $u_m = v_m$ for any integer $m \geqslant 0$ and a real number $a > 0$.

Proof. Since $\phi = \psi$, it holds

$$\sum_{k=0}^{+\infty}(u_k - v_k)q^k = 0.$$

The above expression holds for all points $q \in [0, a)$, if and only if

$$u_m = v_m, \quad m \geqslant 0,$$

which gives, due to Theorem 2.2 (a), that

$$D_m(\phi) = D_m(\psi). \qquad \square$$

Remark 2.1. According to Theorem 2.5, taking the mth-order homotopy-derivative on the two sides of the equation $\phi = \psi$ gives the same results as equating the like-power of q of the equation $\phi = \psi$. Note that, here, it is unnecessary to regard q as small parameter at all. From the above theorem, it is clear that the so-called "homotopy perturbation method" [67,68] (proposed in 1998) is exactly a copy of the early homotopy analysis method (proposed in 1992) and is a special case of the late homotopy analysis method in case of $\hbar = -1$. For details, please refer to Abbasbandy [44], Hayat & Sajid [46,69].

3. Deformation equations

In this section, the properties of homotopy-derivatives proved in Section 2 are employed to deduce the high-order deformation equations for various types of zeroth-order deformation equations.

Lemma 3.1. *Let*

$$\phi = \sum_{m=0}^{+\infty} u_m(\vec{x}, t) q^m$$

denote a homotopy-series, where $q \in [0, 1]$ is the homotopy-parameter, u_m is a function of the spatial variable $\vec{x}$ and the temporal variable t, respectively. Let $\mathscr{L}$ denote an auxiliary linear operator with respect to $\vec{x}$ and t, and u_0 an guess solution. It holds

$$D_m\{(1-q)\mathscr{L}[\phi - u_0]\} = \mathscr{L}[u_m(\vec{x}, t) - \chi_m u_{m-1}(\vec{x}, t)],$$

where the operator D_{m-1} is defined by (10) and χ_m is defined by

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{15}$$

Proof. Since $\mathscr{L}$ is a linear operator independent of q, it holds

$$(1-q)\mathscr{L}[\phi - u_0] = \mathscr{L}[\phi - q\phi + u_0 q - u_0].$$

Using Theorems 2.1, 2.2 and 2.3, we have

$$\begin{aligned} D_m\{(1-q)\mathscr{L}[\phi - u_0]\} &= D_m\{\mathscr{L}[\phi - q\phi + u_0 q - u_0]\} = \mathscr{L}\{D_m[\phi - q\phi + u_0 q - u_0]\} = \mathscr{L}[D_m(\phi) - D_m(q\phi) + u_0 D_m(q)] \\ &= \mathscr{L}[u_m - u_{m-1} + u_0 D_m(q)], \end{aligned}$$

which equals to $\mathscr{L}[u_m]$ when $m = 1$, and $\mathscr{L}[u_m - u_{m-1}]$ when $m > 1$, respectively. Thus, using the definition (15) of χ_m, it holds

$$D_m\{(1-q)\mathscr{L}[\phi - u_0]\} = \mathscr{L}[u_m - \chi_m u_{m-1}]. \qquad \square$$

Theorem 3.1. *Write*

$$\phi = \sum_{m=0}^{+\infty} u_m(\vec{x}, t) q^m,$$

where $q \in [0, 1]$ is the homotopy-parameter. Let $\mathscr{L}$ denote an auxiliary linear operator, $\mathscr{N}$ a nonlinear operator, $u_0(\vec{x}, t)$ a guess solution, $\hbar$ the convergence-control parameter independent of q, and $H(\vec{x}, t)$ an auxiliary function independent of q, respectively. For the zeroth-order deformation equation defined by

$$(1-q)\mathscr{L}[\phi - u_0] = q\hbar H(\vec{x}, t)\mathscr{N}[\phi], \tag{16}$$

the corresponding mth-order deformation equation ($m \geqslant 1$) reads

$$\mathscr{L}[u_m(\vec{x}, t) - \chi_m u_{m-1}(\vec{x}, t)] = \hbar H(\vec{x}, t) D_{m-1}(\mathscr{N}[\phi]), \tag{17}$$

where the operator D_{m-1} is defined by (10) and χ_m is defined by (15).

Proof. Using Theorem 2.6, we have

$$D_m\{(1-q)\mathscr{L}[\phi - u_0]\} = D_m(q\hbar H(\vec{x}, t)\mathscr{N}[\phi]). \tag{18}$$

According to Lemma 3.1, it holds

$$D_m\{(1-q)\mathscr{L}[\phi - u_0]\} = \mathscr{L}[u_m - \chi_m u_{m-1}]. \tag{19}$$

According to Theorems 2.1 and 2.2, one has

$$D_m(q\hbar H(\vec{x}, t)\mathscr{N}[\phi]) = \hbar H(\vec{x}, t) D_{m-1}(\mathscr{N}[\phi]). \tag{20}$$

Substituting (19) and (20) into (18), one has the mth-order deformation equation

$$\mathscr{L}[u_m - \chi_m u_{m-1}] = \hbar H(\vec{x}, t) D_{m-1}(\mathscr{N}[\phi]). \qquad \square$$

Theorem 3.2. *Write*

$$\phi = \sum_{m=0}^{+\infty} u_m(\vec{x},t)q^m, \quad \psi = \sum_{k=1}^{+\infty} \alpha_k q^k,$$

where $q \in [0,1]$ is the homotopy-parameter and α_k is a constant. Let $\mathscr{L}$ denote an auxiliary linear operator, $\mathscr{N}$ a nonlinear operator, $u_0(\vec{x},t)$ a guess solution, and $H(\vec{x},t)$ an auxiliary function independent of q, respectively. For the zeroth-order deformation equation defined by

$$(1-q)\mathscr{L}[\phi - u_0] = H(\vec{x},t)\left(\sum_{k=1}^{+\infty}\alpha_k q^k\right)\mathscr{N}[\phi], \tag{21}$$

the corresponding mth-order deformation equation ($m \geqslant 1$) reads

$$\mathscr{L}[u_m(\vec{x},t) - \chi_m u_{m-1}(\vec{x},t)] = H(\vec{x},t)\sum_{k=1}^{m}\alpha_k D_{m-k}(\mathscr{N}[\phi]), \tag{22}$$

where the operator D_k is defined by (10) and χ_m is defined by (15).

Proof. Using Theorem 2.6, we have

$$D_m\{(1-q)\mathscr{L}[\phi - u_0]\} = D_m(H(\vec{x},t)\psi\mathscr{N}[\phi]). \tag{23}$$

According to Lemma 3.1, it holds

$$D_m\{(1-q)\mathscr{L}[\phi - u_0]\} = \mathscr{L}[u_m - \chi_m u_{m-1}]. \tag{24}$$

Using Theorems 2.1 and 2.2, we have

$$D_m(H(\vec{x},t)\psi\mathscr{N}[\phi]) = H(\vec{x},t)\sum_{k=0}^{m} D_k(\psi)D_{m-k}(\mathscr{N}[\phi]) = H(\vec{x},t)\sum_{k=0}^{m}\alpha_k D_{m-k}(\mathscr{N}[\phi]),$$

which gives, since $\alpha_0 = 0$,

$$D_m(H(\vec{x},t)\psi\mathscr{N}[\phi]) = H(\vec{x},t)\sum_{k=1}^{m}\alpha_k D_{m-k}(\mathscr{N}[\phi]). \tag{25}$$

Substituting (24) and (25) into (23) ends the proof. □

Remark 3.1. The zeroth-order deformation equation (16) is a special case of the zeroth-order deformation equation (21) in case of $\alpha_1 = \hbar$ and $\alpha_k = 0$ for $k > 1$.

Theorem 3.3. *Write*

$$\phi = \sum_{m=0}^{+\infty} u_m(\vec{x},t)q^m, \quad \psi = \sum_{k=1}^{+\infty} \beta_k(\vec{x},t) q^k,$$

where $q \in [0,1]$ is the homotopy-parameter, $\beta_k(\vec{x},t)$ is either equal to zero or a non-zero function, but at least one of them is non-zero. Let $\mathscr{L}$ denote an auxiliary linear operator, $\mathscr{N}$ a nonlinear operator, and $u_0(\vec{x},t)$ a guess solution, respectively. For the zeroth-order deformation equation defined by

$$(1-q)\mathscr{L}[\phi - u_0] = \left(\sum_{k=1}^{+\infty}\beta_k(\vec{x},t) q^k\right)\mathscr{N}[\phi], \tag{26}$$

the corresponding mth-order deformation equation ($m \geqslant 1$) reads

$$\mathscr{L}[u_m(\vec{x},t) - \chi_m u_{m-1}(\vec{x},t)] = \sum_{k=1}^{m}\beta_k(\vec{x},t) D_{m-k}(\mathscr{N}[\phi]), \tag{27}$$

where the operator D_k is defined by (10) and χ_m is defined by (15).

Proof. Using Theorem 2.6, we have

$$D_m\{(1-q)\mathscr{L}[\phi - u_0]\} = D_m(\psi\mathscr{N}[\phi]). \tag{28}$$

According to Lemma 3.1, it holds

$$D_m\{(1-q)\mathscr{L}[\phi - u_0]\} = \mathscr{L}[u_m - \chi_m u_{m-1}]. \tag{29}$$

Using Theorems 2.1 and 2.2, we have

$$D_m(\psi\mathcal{N}[\phi]) = \sum_{k=0}^{m} D_k(\psi)D_{m-k}(\mathcal{N}[\phi]) = \sum_{k=0}^{m}\beta_k(\vec{x},t)D_{m-k}(\mathcal{N}[\phi]),$$

which gives, since $\beta_0(\vec{x},t)=0$, that

$$D_m(\psi\mathcal{N}[\phi]) = \sum_{k=1}^{m}\beta_k(\vec{x},t)D_{m-k}(\mathcal{N}[\phi]). \tag{30}$$

Substituting (29) and (30) into (28) ends the proof. □

Remark 3.2. The zeroth-order deformation equation (21) is a special case of the zeroth-order deformation equation (26) in case of $\beta_k(\vec{x},t)=\alpha_k H(\vec{x},t)$ for $k \geqslant 1$.

Theorem 3.4. *Write*

$$\phi = \sum_{m=0}^{+\infty} u_m(\vec{x},t)q^m, \quad \psi = \sum_{k=1}^{+\infty}\beta_k(\vec{x},t)q^k,$$

where $q \in [0,1]$ is the homotopy-parameter, $\beta_k(\vec{x},t)$ is either equal to zero or a non-zero function, but at least one of them is non-zero. Let $\mathcal{L}$ denote an auxiliary linear operator, $\mathcal{N}$ a nonlinear operator, and $u_0(\vec{x},t)$ a guess solution, respectively. Besides, let $\mathcal{A}[\phi,\vec{x},t,q]$ be a function of $\phi,\vec{x},t$ and q, which satisfies

$$\mathcal{A}[\phi,\vec{x},t,q]=0, \quad \text{when } q=0 \quad \text{and} \quad q=1.$$

For the zeroth-order deformation equation defined by

$$(1-q)\mathcal{L}[\phi-u_0] = \left(\sum_{k=1}^{+\infty}\beta_k(\vec{x},t)q^k\right)\mathcal{N}[\phi] + \mathcal{A}[\phi,\vec{x},t,q], \tag{31}$$

the corresponding mth-order deformation equation ($m \geqslant 1$) reads

$$\mathcal{L}[u_m(\vec{x},t)-\chi_m u_{m-1}(\vec{x},t)] = \sum_{k=1}^{m}\beta_k(\vec{x},t)D_{m-k}(\mathcal{N}[\phi]) + D_m(\mathcal{A}[\phi,\vec{x},t,q]), \tag{32}$$

where the operator D_k is defined by (10) and χ_m is defined by (15).

Proof. Using Theorems 2.6 and 2.1, we have

$$D_m\{(1-q)\mathcal{L}[\phi-u_0]\} = D_m(\psi\mathcal{N}[\phi]) + D_m(\mathcal{A}[\phi,\vec{x},t,q]). \tag{33}$$

According to Lemma 3.1, it holds

$$D_m\{(1-q)\mathcal{L}[\phi-u_0]\} = \mathcal{L}[u_m-\chi_m u_{m-1}]. \tag{34}$$

Using Theorems 2.1 and 2.2, we have

$$D_m(\psi\mathcal{N}[\phi]) = \sum_{k=0}^{m} D_k(\psi)D_{m-k}(\mathcal{N}[\phi]) = \sum_{k=0}^{m}\beta_k(\vec{x},t)D_{m-k}(\mathcal{N}[\phi]),$$

which gives, since $\beta_0(\vec{x},t)=0$, that

$$D_m(\psi\mathcal{N}[\phi]) = \sum_{k=1}^{m}\beta_k(\vec{x},t)D_{m-k}(\mathcal{N}[\phi]). \tag{35}$$

Substituting (39) and (35) into (38) ends the proof. □

Remark 3.3. The zeroth-order deformation equation (26) is a special case of the zeroth-order deformation equation (31) in case of $\mathcal{A}[\phi,\vec{x},t,q]=0$.

Theorem 3.5. *Write*

$$\phi = \sum_{m=0}^{+\infty} u_m(\vec{x},t)q^m,$$

where $q \in [0,1]$ is the homotopy-parameter. Let $\mathcal{L}$ denote an auxiliary linear operator, $\mathcal{N}$ a nonlinear operator, and $u_0(\vec{x},t)$ a guess solution, respectively. Besides, let $\mathcal{B}[\phi,\vec{x},t,q]$ be a function of $\phi,\vec{x},t$ and q, which satisfies

$$\mathcal{B}[\phi,\vec{x},t,q]=0, \quad \text{when } q=0,$$
$$\mathcal{B}[\phi,\vec{x},t,q]=\gamma(\vec{x},t)\mathcal{N}[\phi], \quad \text{when } q=1,$$

where $\gamma(\vec{x},t)\neq 0$ *is a non-zero function. For the zeroth-order deformation equation defined by*

$$(1-q)\mathscr{L}[\phi-u_0]=\mathscr{B}[\phi,\vec{x},t,q], \tag{36}$$

the corresponding mth-order deformation equation ($m \geqslant 1$) *reads*

$$\mathscr{L}[u_m(\vec{x},t)-\chi_m u_{m-1}(\vec{x},t)]=D_m(\mathscr{B}[\phi,\vec{x},t,q]), \tag{37}$$

where the operator D_m *is defined by (10) and* χ_m *is defined by (15).*

Proof. Using Theorem 2.6, we have

$$D_m\{(1-q)\mathscr{L}[\phi-u_0]\}=D_m(\mathscr{B}[\phi,\vec{x},t,q]). \tag{38}$$

According to Lemma 3.1, it holds

$$D_m\{(1-q)\mathscr{L}[\phi-u_0]\}=\mathscr{L}[u_m-\chi_m u_{m-1}]. \tag{39}$$

This ends the proof. □

Remark 3.4. Obviously, the zeroth-order deformation equation (16), (21), (26) and (31) are special cases of the zeroth-order deformation equation (36). However, up to now, it is unknown which kind of zeroth-order deformation equation is better or best, as discussed in Liao's book [21]. In most cases, the zeroth-order deformation equation (16) can give satisfied homotopy-series solution, if the auxiliary linear operator $\mathscr{L}$, the auxiliary function $H(\vec{x},t)$ and convergence-control parameter $\hbar$ are properly chosen, as pointed out by Liao [21–23,25,60,26] and others [28–59].

In most cases, in order to get the high-order deformation equation related to a nonlinear equation $\mathscr{N}[u]=0$, one just needs to calculate the term $D_k(\mathscr{N}[\phi])$. This can be easily done by means of the properties of the homotopy-derivatives given in Section 2.

4. Examples

In this section, some simple examples are used to show how to apply the theorems given in this article to deduce high-order deformation equations of nonlinear problems.

Example 1. Consider the nonlinear heat transfer problem [44]:

$$(1+\epsilon u)u'+u=0,\quad u(0)=1.$$

Choosing $\mathscr{L}u=u'+u$ as the auxiliary linear operator, and defining the nonlinear operator

$$\mathscr{N}[\phi]=(1+\epsilon\phi)\phi'+\phi,$$

where

$$\phi=\sum_{k=0}^{+\infty}u_k(t)q^k$$

is a homotopy-series, we construct such a zeroth-order deformation equation

$$(1-q)\mathscr{L}[\phi-u_0(t)]=q\hbar\mathscr{N}[\phi],$$

subject to the initial condition

$$\phi=1,\quad \text{when } t=0,$$

where $u_0(t)$ is an initial guess satisfying the initial condition. According to Theorem 3.1, the corresponding mth-order deformation equation reads

$$\mathscr{L}[u_m(t)-\chi_m u_{m-1}(t)]=\hbar D_{m-1}(\mathscr{N}[\phi]),$$

subject to the initial guess

$$u_m(0)=0.$$

For this example, using Theorems 2.1, 2.2 and 2.3, one has

$$D_k(\mathscr{N}[\phi])=D_k(\phi')+D_k(\phi)+\epsilon D_k(\phi\phi')=u_k'+u_k+\epsilon\sum_{n=0}^{k}u_{k-n}u_n'.$$

The corresponding homotopy-series solution is given by

$$u(t)=\sum_{k=0}^{+\infty}u_k(t),$$

which is convergent for *any* physical parameter $0 \leqslant \epsilon < +\infty$ if one chooses the convergence-control parameter $\hbar = -(1+\epsilon)^{-1}$. For details, please refer to Abbasbandy [44].

Example 2. Consider the nonlinear oscillation equation [26]:

$$u''(t) + \lambda u(t) + \epsilon u^3(t) = 0, \quad u(0) = 1, \quad u'(0) = 0.$$

Let ω denote the unknown frequency of the solution. Write $\tau = \omega t$, the above equation becomes

$$\omega^2 u''(\tau) + \lambda u(\tau) + \epsilon u^3(\tau) = 0, \quad u(0) = 1, \quad u'(0) = 0.$$

Define

$$\mathcal{N}[\phi, \Omega] = \Omega^2 \phi'' + \lambda \phi + \epsilon \phi^3,$$

where the prime denotes the differentiation with respect to τ, and

$$\phi = \sum_{k=0}^{+\infty} u_k(t) q^k, \quad \Omega = \sum_{k=0}^{+\infty} \omega_k q^k$$

are two homotopy-series. Choosing the auxiliary linear operator

$$\mathcal{L} u = u'' + u,$$

we construct the following zeroth-order deformation equation

$$(1-q)\mathcal{L}[\phi - u_0] = q\hbar \mathcal{N}[\phi, \Omega], \quad q \in [0, 1]$$

subject to the initial conditions

$$\phi = 1, \quad \phi' = 0, \quad \text{at } t = 0,$$

where $u_0(t)$ is an initial guess satisfying the initial conditions. According to Theorem 3.1, the corresponding high-order deformation equation reads

$$\mathcal{L}[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar D_{m-1}(\mathcal{N}[\phi, \Omega]),$$

subject to the initial conditions

$$u_m(0) = 0, \quad u'_m(0) = 0.$$

In this case, using Theorems 2.1–2.3, and Molabahrami and Khani's Theorem, we have

$$D_k(\mathcal{N}[\phi, \Omega]) = D_k(\Omega^2 \phi'') + \lambda D_k(\phi) + \epsilon D_k(\phi^3) = \sum_{i=0}^{k} u''_{k-i} \sum_{j=0}^{i} \omega_{i-j}\omega_j + \lambda u_k + \epsilon \sum_{i=0}^{k} u_{k-i} \sum_{j=0}^{i} u_{i-j} u_j.$$

The corresponding homotopy-series solutions are given by

$$u(t) = \sum_{k=0}^{+\infty} u_k(\omega t), \quad \omega = \sum_{k=0}^{+\infty} \omega_k,$$

which are convergent if the convergence-control parameter $\hbar$ is chosen properly. For example, when $\lambda = 0$, the homotopy-series solutions are convergent for *any* a physical parameter $0 \leqslant \epsilon < +\infty$ by using $\hbar = -(1+\epsilon)^{-1}$. For details, please refer to Liao and Tan [26].

5. Discussions

The introduction of the convergence-control parameter $\hbar$ greatly improves the early homotopy analysis method. It is the convergence-control parameter $\hbar$ that provides us, for the first time, a simple way to ensure the convergence of series solution of nonlinear problems. Different from all previous analytic methods, one can ensure the convergence of series solution of strongly nonlinear problems by means of choosing a proper value of the convergence-control parameter $\hbar$. This is an obvious advantage of the HAM. Besides, unlike all perturbation and previous non-perturbation methods, the HAM provides us with great freedom to choose proper base functions so as to give better approximations of nonlinear problems. Note that (16) can be rewritten as

$$(1-q)\hat{\mathcal{L}}[\phi - u_0] = q\mathcal{N}[\phi], \tag{40}$$

where

$$\hat{\mathcal{L}} = \frac{\mathcal{L}}{\hbar H(\vec{x}, t)}$$

can be regarded as a auxiliary linear operator, too. The above expression opens out that, in the frame of the late HAM, the auxiliary linear operator is chosen in several steps: a basic linear operator is first chosen, then an auxiliary function is determined due to the *rule of solution expression* suggested by Liao [21], and finally the convergence-control parameter $\hbar$ is determined to ensure the convergence of the homotopy-series solution. In fact, the zeroth-order deformation equations (21), (26), (31) and (36) are obtained by generalizing the concept of convergence-control parameter $\hbar$. Define the vector

$$\vec{\alpha} = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}.$$

Obviously, the vector $\vec{\alpha}$ in the zeroth-order deformation equation (21) is a kind of generalization of the convergence-control parameter $\hbar$ in Eq. (16), and thus is called *the convergence-control vector*. Similarly, the vector

$$\vec{\beta} = \{\beta_1(\vec{x}, t), \beta_2(\vec{x}, t), \beta_3(\vec{x}, t), \ldots\}$$

in Eqs. (26) and (31) is a further generalization of the so-called convergence-control vector $\vec{\alpha}$ in Eq. (21). Similarly, by properly choosing the convergence-control vectors $\vec{\alpha}$ or $\vec{\beta}$, one can ensure the convergence of the homotopy-series solutions of Eqs. (26), (31), and (36). For the zeroth-order deformation equation (16), Liao suggested to choose a proper value of $\hbar$ by plotting the so-called $\hbar$-curves. Let $\delta(\vec{x}, t)$ denote the residual error of the mth-order homotopy-series approximation, and $\Delta = \int\int \delta^2(\vec{x}, t) dV\, dt$ denote the integral of the residual error. Plotting the curves of $\Delta \sim \hbar$, it is straightforward to find a region of $\hbar$ in which Δ decreases to zero as the order of approximation increases. Then, a convergent homotopy-series solution is obtained by choosing a value in this region. For the zeroth-order deformation equation (21), Marinca [65] currently proposed an interesting approach to determine the convergence-control vector $\vec{\alpha}$ by minimizing the residual error. Obviously, some rigorous mathematical theorems are urgently needed to find the *best* convergence-control parameter $\hbar$ for (16), the *best* convergence-control vector $\vec{\alpha}$ for (21) and the *best* convergence-control vector $\vec{\beta}$ for (26) and (31), respectively.

Obviously, the various types of the zeroth-order deformation equations such as (16), (21), (26), (31) and (36) provide us great freedom and flexibility to apply the HAM. Besides, for each type of these zeroth-order deformation equations, one has great freedom and flexibility to choose the auxiliary linear operator $\mathscr{L}$: *even* the order of $\mathscr{L}$ can be different from original nonlinear problems, as shown by Liao and Tan [26], who illustrated that a 2nd-order nonlinear PDE can be replaced by an infinite number of 4th or 6th-order linear PDEs. Such kind of freedom and flexibility greatly simplifies the resolving of complicated nonlinear equations. By means of the HAM, a nonlinear ODE is often replaced by an infinite number of linear ODEs, and a nonlinear PDE can be transferred into an infinite number of linear ODEs. Besides, a nonlinear differential equation with variable coefficients can be replaced by an infinite number of linear differential equations with constant coefficients. Certainly, this kind of freedom and flexibility increases the possibility of finding satisfactory series solution of a given nonlinear problem. However, it also enhances the difficulties in applying and learning it. Up to now, it is even unknown whether or not there exists the *best* auxiliary linear operator and the *best* zeroth-order deformation equation for a nonlinear equation which has at least one solution. To simplify the applications of the HAM, Liao [21] suggested some rules, i.e. the rule of solution expression, the rule of solution existence, and the rule of ergodicity for coefficient of homotopy-series solution. Obviously, some rigorous mathematical theorems are urgently needed to choose the auxiliary linear operators.

The freedom and flexibility in the choose of the auxiliary linear operator $\mathscr{L}$ in the zeroth-order deformation equation can be used to develop some new numerical techniques for strongly nonlinear problems. For example, the so-called "general boundary element method" [70–73], which is based on the HAM, gives accurate convergent results of the viscous driven flows (governed by the exact Navier–Stokes equations) in a square cavity with the high Reynolds number $Re = 7500$, as shown by Zhao and Liao [74]. Currently, Wu and Cheung [49] applied the HAM to give an explicit numerical approach for Riemann problems related to nonlinear shallow water equations. All of these illustrate the great potential of the HAM combined with traditional numerical techniques.

The zeroth-order deformation equations (21), (26), (31) and (36) are rather general. Using them, Liao [21] proved that the HAM logically contains other previous non-perturbation methods, such as Lyapunov's artificial small parameter method [13], the δ-expansion method [14,15] and Adomian's decomposition method [16–19]. Thus, the HAM unifies the previous non-perturbation methods. Besides, the so-called "homotopy perturbation method" [67,68] (proposed in 1998) is exactly the same as the early homotopy analysis method (proposed in 1992) and is a special case of the late homotopy analysis method in case of $\hbar = -1$, as illustrated by Abbasbandy [44] and proved by Sajid et al. [46,69] in general. Indeed, Dr. He [67,68] simply copied Liao's idea of the early HAM, and his so-called "homotopy perturbation method" proposed in 1998 (6 year later) "has nothing new except its name", as pointed out by Hayat and Sajid [46,69].

Frankly speaking, the HAM is a method for the time of computer: without high-performance computer and symbolic computation software such as Mathematica, maple and so on, it is impossible to solve high-order deformation equations quickly so as to get approximations at high enough order. Without computer and symbolic computation software, it is also impossible to choose a proper value of the convergence-control parameter $\hbar$ by means of analyzing the high-order approximations. It is true that expressions given by the HAM are often lengthy and thus can be hardly expressed on only one page. However, by means of computer and symbolic computation software, it often needs only a few seconds to calculate these lengthy results! Note that, one needs much more time to calculate a traditional " analytic" expression in a length of half page by means of a traditional computational tool such as a slide rule. So, if we regard keyboard of computer as a pen, hard disk as papers, and CPU as a slide rule, we can calculate lengthy "analytic" expressions given by the HAM in a few seconds by means of a computer ! So, it seems that the traditional concept "analytic solution", which was formed hundreds year ago in the time of slide rule, should be modified in the time of computer that has changed our life completely.

Finally, according to our experience, it seems that, as long as a nonlinear equation has at least one solution, then one can always construct a kind of zeroth-order deformation equation to get convergent homotopy-series solution. However, hundreds of successful examples is not better than a rigorous mathematical proof. So, to end this article, we give here such a hypothesis:

Hypothesis 1. If a nonlinear equation has at least one solution, then there exists at least one zeroth-order deformation equation such that its homotopy-series solution converges to the solution of the original nonlinear equation.

Acknowledgements

Thanks to Prof. Pradeep Siddheshwar, Department of Mathematics, Bangalore University, India, for his valuable suggestion of renaming $\hbar$ as the convergence-control parameter. This work is supported by National Natural Science Foundation of China (Approve No. 10572095).

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