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Explicit series solution of travelling waves with a front of Fisher equation

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Abstract

In this paper, an analytic technique, namely the homotopy analysis method, is employed to solve the Fisher equation, which describes a family of travelling waves with a front. The explicit series solution for all possible wave speeds $0 < c < +\infty$ is given. Such kind of explicit series solution has never been reported, to the best of author's knowledge. Our series solution indicates that the solution contains an oscillation part when $0 < c < 2$. The proposed analytic approach is general, and can be applied to solve other similar nonlinear travelling wave problems.
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1. Introduction

The nonlinear reaction–diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + vu(1-u), \tag{1}$$

was first introduced by Fisher as a model for the propagation of a mutant gene [1]. It has wide application in the fields of logistic population growth [2,3], flame propagation [4], neurophysiology [5], autocatalytic chemical reactions [6], branching Brownian motion processes [7], and nuclear reactor theory [8].

In chemical media the function $u(x,t)$ is the concentration of the reactant. D represents its diffusion coefficient, and the positive constant v specifies the rate of chemical reaction. In media of other natures, u might be temperature or electric potential, D might be the thermal conductivity or specific electrical conductivity. The medium described by Eq. (1) is often referred to as a bistable medium because it has two homogeneous stationary states, $u = 0$ and $u = 1$. A kink-like travelling wave solution of Eq. (1) describes a constant-velocity front of transition from one homogeneous state to another.

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Several analytic methods have been developed for obtaining travelling wave solutions: the inverse scattering transfer [9], the Hirota method [10], Lamb's ansatz [11,12], Adomian decomposition method [13], etc. [14]. The problem of obtaining solutions for systems including dissipative losses, e.g., reaction–diffusion systems, turned out to be complex. Most of the above-mentioned methods do not work [15]. The usual way of treating the problem is by perturbation theory or numerical investigation.

Writing $t^* = vt$, $x^* = (v/D)^{1/2}x$, and dropping the star, Eq. (1) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2. \tag{2}$$

In the spatial homogeneous situation the steady states are $u = 0$ and $u = 1$, which are respectively unstable and stable. This suggests that we should look for travelling wave front solutions to Eq. (2).

Consider a travelling wave solution

$$u(x,t) = u(x - ct) = f(z), \quad z = x - ct, \tag{3}$$

propagating with a speed c. It obeys the ordinary differential equation

$$f'' + cf' + f - f^2 = 0, \tag{4}$$

subject to the boundary conditions

$$f(-\infty) = 1, \qquad f(+\infty) = 0, \tag{5}$$

where the prime denotes the differentiation with respect to z.

The above wave solutions are invariant to any shift in the origin of the coordinate system, because Eq. (4) and the boundary condition (5) are the same under the transformation $z = z + C$ for any a constant C. Moreover, Fisher [1] found that Eq. (4) has an infinite number of travelling wave solutions for which $0 \leqslant f \leqslant 1$ and $c > c_{\min} = 2$. A closed-form solution of travelling wave, i.e.,

$$f(z) = \frac{1}{\left[1 + \mathrm{e}^{z/\sqrt{6}}\right]^2}, \tag{6}$$

was found by Ablowitz and Zeppetella [16] for a particular wave speed

$$c = 5/\sqrt{6} = 2.04124.$$

By means of shifting the origin, the expression (6) can be rewritten as

$$f(z) = \frac{1}{\left[1 + (\sqrt{2} - 1)\mathrm{e}^{z/\sqrt{6}}\right]^2}, \tag{7}$$

corresponding to $f(0) = 1/2$. This kind of solution was found by many authors (for example, see also [17]). For large wave speed c, there is a small parameter $\varepsilon = 1/c^2 \ll 1$. In this case, Murray [18] applied perturbation method to obtain an asymptotic solution

$$f(z) = \frac{1}{1 + \mathrm{e}^{z/c}} + \mathrm{e}^{z/c}\frac{\ln[4\mathrm{e}^{z/c}(1 + \mathrm{e}^{z/c})^{-2}]}{c^2(1 + \mathrm{e}^{z/c})^2} + \mathrm{O}(c^{-4}), \tag{8}$$

which is however valid only for large wave speed c. Solutions for $c < 2$ are scarcely reported. To the best of our knowledge, analytic solutions for general wave speed c have been never reported.

The homotopy [19] is a basic concept in topology [20]. Based on the homotopy, some numerical techniques such as the continuation method [21] and the homotopy continuation method [22] were developed. There is a suite of FORTRAN subroutines in Netlib for numerically solving nonlinear systems of equations by homotopy methods, called HOMPACK. In 1992, a kind of analytic method, namely the homotopy analysis method (HAM) [23,24], was developed to solve highly nonlinear problems. Different from perturbation techniques [25], the homotopy analysis method does not depend upon any small or large parameters and thus is valid for most of nonlinear problems in science and engineering. Besides, it logically contains other non-perturbation techniques such as Lyapunov's small parameter method [26], the δ-expansion method [27], and Adomian's decomposition method [28], as proved by Liao in his book [24]. Currently, Liao [29] showed that the so-called "homotopy perturbation method" (proposed by Dr. He in 1998) is only a special case of the homotopy analysis method. The homotopy analysis method has been successfully applied to many nonlinear problems [30–39]. In this paper, by means of the homotopy analysis method, the explicit series solution of Fisher equation for all possible wave speed $0 < c < +\infty$ is given.

2. Series solution

Due to the boundary conditions (5), we divide the solution into two parts, $f^-(z)$ for $z \leqslant 0$ and $f^+(z)$ for $z \geqslant 0$, where $f^-(0) = f^+(0) = 1/2$.

2.1. Case I: Series expression in the region $(-\infty, 0]$

2.1.1. Asymptotic property

In the region $z \in (-\infty, 0]$, we have the boundary conditions

$$f^-(-\infty) = 1, \qquad f^-(0) = 1/2. \tag{9}$$

Write

$$f^-(z) \sim 1 + \alpha \exp(\gamma z), \tag{10}$$

where γ is an unknown positive constant. Substituting it into Eq. (4) and balancing the dominant terms, we have

$$\gamma = (\sqrt{4 + c^2} - c)/2. \tag{11}$$

Therefore, $f^-(z)$ can be expressed by a set of base functions

$$\{\exp(m\gamma z) | m \geqslant 0\}, \tag{12}$$

in the form

$$f^-(z) = 1 + \sum_{m=1}^{+\infty} a_m \exp(m\gamma z), \tag{13}$$

where a_m is the coefficient to be determined. This provides us with the Rule of Solution Expression for $f^-(z)$.

2.1.2. Zeroth-order deformation equation

According to the Rule of Solution Expression (12) and the boundary conditions (9), it is straightforward to choose the initial solution

$$f_0^-(z) = 1 - \frac{1}{2}\exp(\gamma z), \tag{14}$$

and besides to choose the linear operator

$$\mathscr{L}^-[\phi] = \frac{\partial^2 \phi}{\partial z^2} - \gamma \frac{\partial \phi}{\partial z}, \tag{15}$$

which has the following property:

$$\mathscr{L}^-[C_1 + C_2 \exp(\gamma z)] = 0, \tag{16}$$

where C_1 and C_2 are integral constants.

From Eq. (4), we are led to define a nonlinear operator

$$\mathscr{N}[F^-(z;q)] = \frac{\partial^2 F^-(z;q)}{\partial z^2} + c\frac{\partial F^-(z;q)}{\partial z} + F^-(z;q) - F^-(z;q)^2. \tag{17}$$

Let $\hbar^-$ denotes a nonzero auxiliary parameter and $H^-(z)$ a nonzero auxiliary function. We construct the zero-order deformation equation

$$(1-q)\mathscr{L}^-[F^-(z;q) - f_0^-(z)] = q\hbar^- H^-(z)\mathscr{N}[F^-(z;q)], \tag{18}$$

subject to the corresponding boundary conditions

$$F^-(-\infty;q) = 1, \qquad F^-(0;q) = \frac{1}{2}, \tag{19}$$

where $q \in [0,1]$ is an embedding parameter.

When $q = 0$, we have, from Eqs. (18) and (19), the solution

$$F^-(z;0) = f_0^-(z). \tag{20}$$

Since $\hbar^- \neq 0$ and $H^-(z) \neq 0$, Eq. (18) when $q = 1$ is equivalent to Eq. (4), provided

$$F^-(z;1) = f^-(z). \tag{21}$$

Thus, as q increases from 0 to 1, $F^-(z;q)$ varies from the initial guess $f_0^-(z)$ to the solution $f^-(z)$ of the original equations (4) and (9).

Using (20), we expand $F^-(z;q)$ in the Taylor series with respect to q

$$F^-(z;q) = f_0^-(z) + \sum_{m=1}^{+\infty} f_m^-(z) q^m, \tag{22}$$

where

$$f_m^-(z) = \frac{1}{m!} \frac{\partial^m F^-(z;q)}{\partial q^m}\bigg|_{q=0}. \tag{23}$$

Note that $F^-(z;q)$ is dependent upon the auxiliary parameters $\hbar^-$ and the auxiliary function $H^-(z)$. Assuming that $\hbar^-$ and $H^-(z)$ are properly chosen so that the above series converge at $q = 1$, and using (22), we have the solution series

$$f^-(z) = f_0^-(z) + \sum_{m=1}^{+\infty} f_m^-(z). \tag{24}$$

2.1.3. High-order deformation equation

For brevity, define the vector

$$\vec{f}_m^- = \{f_0^-(z), f_1^-(z), f_2^-(z), \ldots, f_m^-(z)\}.$$

Differentiating the zeroth-order deformation equation (18) m times with respect to the embedding parameter q, then setting $q = 0$, and finally dividing by $m!$, we have the so-called mth-order deformation equation

$$\mathscr{L}^-[f_m^-(z) - \chi_m f_{m-1}^-(z)] = \hbar^- H^-(z) R_m[\vec{f}_{m-1}^-], \tag{25}$$

subject to the boundary conditions

$$f_m^-(-\infty) = 0, \qquad f_m^-(0) = 0, \tag{26}$$

where

$$R_m[\vec{f}_{m-1}^-] = f_{m-1}''^-(z) + c f_{m-1}'^-(z) + f_{m-1}^-(z) - \sum_{i=0}^{m-1} f_i^-(z) f_{m-1-i}^-(z) \tag{27}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{28}$$

According to the Rule of Solution Expression denoted by (13), the auxiliary function $H^-(z)$ can be in the form

$$H^-(z) = \exp(\kappa\gamma z), \tag{29}$$

where κ is an integer. It is found that, when $\kappa > 0$, the solution $f_m^-(z)$ does not contain the term $\exp(2\gamma z)$. This disobeys the so-called *Rule of Coefficient Ergodicity* described by Liao [24]. Besides, when $\kappa < 0$, due to the property (16), the solution $f_m^-(z)$ contains the term $z\exp(\gamma z)$, and this disobeys the so-called *Rule of Solution Expression* suggested by Liao [24]. So, only $\kappa = 0$ is possible, which uniquely determines the auxiliary function

$$H^-(z) = 1. \tag{30}$$

Note that the high-order equations are linear and thus easy to solve, especially by means of symbolic computation software such as Mathematica, Maple and so on. Let $f_m^{*-}(z)$ denote a special solution of Eq. (25). Due to the property (16), its general solution is

$$f_m^-(z) = f_m^{*-}(z) + C_1 \exp(\gamma z) + C_2, \tag{31}$$

and the constants C_1 and C_2 are determined by the conditions (26). It is found that $f_m^-(z)$ can be expressed by

$$f_m^-(z) = a_{0,0}(1 - \chi_{m+1}) + \sum_{k=1}^{m+1} a_{m,k} \exp(k\gamma z), \tag{32}$$

where $a_{m,k}$ are coefficients. Substituting the above expression into the mth-order deformation equations, one has the following recurrence formulae:

$$a_{m,k} = \frac{\hbar}{k(k-1)\gamma^2}(b_{m,k} + d_{m,k}) + \chi_m \lambda_{m-1,k} a_{m-1,k}, \tag{33}$$

where $m \geqslant 1$ and $2 \leqslant k \leqslant m+1$, χ_m is defined by (28), $\lambda_{m,k}$ is defined by

$$\lambda_{m,k} = \begin{cases} 1, & 1 \leqslant k \leqslant m+1, \\ 0, & \text{otherwise}, \end{cases} \tag{34}$$

and

$$b_{m,k} = \lambda_{m-1,k}[(k\gamma)^2 + c(k\gamma) + 1 - 2a_{0,0}]a_{m-1,k}, \tag{35}$$

$$d_{m,k} = -\sum_{i=0}^{m-1} \sum_{j=\max\{1,k-1-i\}}^{\min\{m-i,k-1\}} a_{i,k-j} a_{m-1-i,j}, \tag{36}$$

respectively. Besides, the coefficient $a_{m,1}$ is given by

$$a_{m,1} = -\sum_{k=2}^{m+1} a_{m,k}. \tag{37}$$

Thus, using the foregoing recurrence formulas and the known first two coefficients $a_{0,0} = 1$ and $a_{0,1} = -1/2$, all other coefficients $a_{m,k}$ can be calculated successively. This provides us with an explicit series solution of the Fisher equation in the region $(-\infty, 0]$, i.e.,

$$f^-(z) = a_{0,0} + \sum_{m=0}^{\infty} \sum_{k=1}^{m+1} a_{m,k} \exp(k\gamma z). \tag{38}$$

The corresponding Mth-order approximation of $f^-(z)$ is

$$f^-(z) \approx a_{0,0} + \sum_{m=0}^{M} \sum_{k=1}^{m+1} a_{m,k} \exp(k\gamma z). \tag{39}$$

2.2. Case II: Series expression in the region $[0, +\infty)$

Note that the values of $f(0)$ and $f'(0)$ are known from solutions given in above section. According to Eq. (4), we have

$$f'' = f^2 - cf' - f. \tag{40}$$

Therefore, at the point $z = 0$, it holds

$$f''^-(0) = f''^+(0), \tag{41}$$

since

$$f^-(0) = f^+(0), \qquad f'^-(0) = f'^+(0). \tag{42}$$

So, the boundary conditions of the solution $f^+(z)$ in the region $z \in [0, +\infty)$ become

$$f'^+(0) = f'^-(0), \qquad f^+(0) = f^-(0), \qquad f^+(+\infty) = 0. \tag{43}$$

Obviously, $f^+(z)$ can be expressed by a set of base functions

$$\{z^m \exp(-nz) | m \geqslant 0, n > 0\}, \tag{44}$$

in the following form:

$$f^+(z) = \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} A_{n,j} z^j \exp(-nz), \tag{45}$$

where $A_{n,j}$ is a constant. This provides us with the Rule of Solution Expression.

According to the Rule of Solution Expression (45) and boundary conditions (43), it is straightforward to choose the initial solution

$$f_0^+(z) = [2f^-(0) + f'^-(0)]\exp(-z) + [-f^-(0) - f'^-(0)]\exp(-2z). \tag{46}$$

Similarly, we choose the linear operator

$$\mathscr{L}^+[\phi] = \frac{\partial^2 \phi}{\partial z^2} + 3\frac{\partial \phi}{\partial z} + 2\phi, \tag{47}$$

which have the following property:

$$\mathscr{L}^+[C_3 \exp(-z) + C_4 \exp(-2z)] = 0, \tag{48}$$

where C_3, C_4 are constants. For simplicity, we select here

$$H^+(z) = 1. \tag{49}$$

In the similar way as mentioned in above section, we can construct the zero-order deformation equation

$$(1-q)\mathscr{L}^+[F^+(z;q) - f_0^+(z)] = q\hbar^+ H^+(z)\mathscr{N}[F^+(z;q)], \tag{50}$$

subject to the corresponding boundary conditions

$$F^+(\infty;q) = 0, \qquad F^+(0;q) = \frac{1}{2}, \qquad \left.\frac{\partial F^+(z;q)}{\partial z}\right|_{z=0} = f'^-(0). \tag{51}$$

From above equations, the high-order deformation equations are obtained in the same way as mentioned in above subsection. Similarly, we have the series solution

$$f^+(z) = f_0^+(z) + \sum_{m=1}^{+\infty} f_m^+(z). \tag{52}$$

It is found that $f_m^+(z)$ can be expressed by

$$f_m^+(z) = \sum_{k=1}^{2(m+1)} \sum_{i=0}^{m} \alpha_{m,k}^i z^i \exp(-kz), \tag{53}$$

where

$$\alpha_{m,k}^i = \hbar \sum_{j=\max\{0,i-1\}}^{m-1} \Delta_{m,k}^j \delta_{k,j}^i + \chi_m \mu_{m-1,k}^i \alpha_{m-1,k}^i, \tag{54}$$

for $2 < k \leqslant 2(m+1)$ or $k \leqslant 2$, $i > 0$ and

$$\alpha_{m,1}^0 = \sum_{k=3}^{2(m+1)} (k-2)\alpha_{m,k}^0 - \sum_{k=1}^{2(m+1)} \alpha_{m,k}^1, \tag{55}$$

$$\alpha_{m,2}^0 = \sum_{k=3}^{2(m+1)} (1-k)\alpha_{m,k}^0 + \sum_{k=1}^{2(m+1)} \alpha_{m,k}^1, \tag{56}$$

under the definitions

$$\mu_{m,k}^i = \begin{cases} 1, & 0 \leqslant i \leqslant m \text{ and } 1 \leqslant k \leqslant 2(m+1), \\ 0, & \text{otherelse}, \end{cases} \tag{57}$$

$$\delta_{n,s}^j = \begin{cases} (-1)^{j+1-s}\frac{s!}{j!}, & 0 \leqslant j \leqslant s \text{ and } n = 1, \\ \frac{1}{s+1}, & j = s+1 \text{ and } n = 1, \\ -\frac{s!}{j!}, & 0 \leqslant j \leqslant s \text{ and } n = 2, \\ -\frac{1}{s+1}, & j = s+1 \text{ and } n = 2, \\ \left[\frac{1}{(n-2)^{s+1-j}} - \frac{1}{(n-1)^{s+1-j}}\right]\frac{s!}{j!}, & 0 \leqslant j \leqslant s \text{ and } n > 2, \\ 0, & j = s+1 \text{ and } n > 2, \end{cases} \tag{58}$$

and

$$\Delta^i_{m,k} = \Phi^i_{m,k} + \Psi^i_{m,k}, \tag{59}$$

$$\Phi^i_{m,k} = \Lambda^i_{m-1,k} + c\Gamma^i_{m-1,k} + \mu^i_{m-1,k}\alpha^i_{m-1,k}, \tag{60}$$

$$\Psi^i_{m,k} = -\sum_{n=0}^{m-1} \sum_{q=\max\{1,k-2(n+1)\}}^{\min\{2(m-n),k-1\}} \sum_{j=\max\{0,i-n\}}^{\min\{m-1-n,i\}} \mu^{i-j}_{n,k-q}\mu^{j}_{m-1-n,q}\alpha^{i-j}_{n,k-q}\alpha^{j}_{m-1-n,q}, \tag{61}$$

$$\Gamma^i_{m,k} = (i+1)\mu^{i+1}_{m,k}\alpha^{i+1}_{m,k} - k\mu^i_{m,k}\alpha^i_{m,k}, \tag{62}$$

$$\Lambda^i_{m,k} = (i+2)(i+1)\mu^{i+2}_{m,k}\alpha^{i+2}_{m,k} - 2k(i+1)\mu^{i+1}_{m,k}\alpha^{i+1}_{m,k} + k^2\mu^i_{m,k}\alpha^i_{m,k}. \tag{63}$$

Using above recursive formulae, we can calculate all coefficients $\alpha^i_{m,k}$ by using the first two coefficients

$$\alpha^0_{0,1} = 2f^-(0) + f'^-(0), \qquad \alpha^0_{0,2} = -f^-(0) - f'^-(0).$$

This provides us with the explicit series expression of the Fisher equation in the region $[0,+\infty)$.

Combining the two series expressions given in above sections, we have an explicit series solution valid for any wave speed in the whole region $(-\infty,+\infty)$. Such kind of explicit series solution is obtained for the first time, to the best of our knowledge.

3. Result analysis

Liao [24] proved that, as long as a series solution given by the homotopy analysis method converges, it must be one of exact solutions. So, it is important to ensure that the solution series (24) and (52) are convergent. Note that the solution series (24) and (52) contain the auxiliary parameter $\hbar^-$ and $\hbar^+$ respectively, which we can choose properly by plotting the so-called $\hbar$-curves to ensure solution series converge, as suggested by Liao [24]. The values of $\hbar^-$ and $\hbar^+$ for some wave speeds are listed in Table 1.

When $c = 5/\sqrt{6}$, the series solution is convergent quickly to the exact ones, as shown in Table 2. The convergence rate can be accelerated by means of the so-called homotopy-Padé technique, as shown in Table 3. Note that, our series solution agrees well with the closed-form solution (7) in the whole region, as shown in Fig. 1.

Table 1
Parameters used in our analytic approach

c	$\hbar^-$	$\hbar^+$
1/4	−1	−1
1/2	−1	−1
1	−1/2	−1
2	−1/2	−1
$5/\sqrt{6}$	−1/4	−1
3	−1/4	−1
4	−1/6	−1
5	−1/10	−1

Table 2
The approximations of $f'(-1)$ and $f'(1)$ when $c = 5/\sqrt{6}$

Order	$f'(-1)$	Order	$f'(1)$
5	−0.106846	2	−0.143270
10	−0.108604	4	−0.120628
15	−0.108348	6	−0.119048
20	−0.108390	8	−0.118984
25	−0.108383	10	−0.118982
30	−0.108384	12	−0.118982
35	−0.108384	14	−0.118982
40	−0.108384	16	−0.118982
Exact result	−0.108384		−0.118982

Table 3
The $[m,m]$ homotopy-Padé approximations of $f'(-1)$ and $f'(1)$ when $c = 5/\sqrt{6}$

m	$f'(-1)$	m	$f'(1)$
[2,2]	−0.112053	[2,2]	−0.119727
[4,4]	−0.108317	[3,3]	−0.118977
[6,6]	−0.108385	[4,4]	−0.118982
[8,8]	−0.108384	[5,5]	−0.118982
[10,10]	−0.108384	[6,6]	−0.118982
Exact result	−0.108384		−0.118982

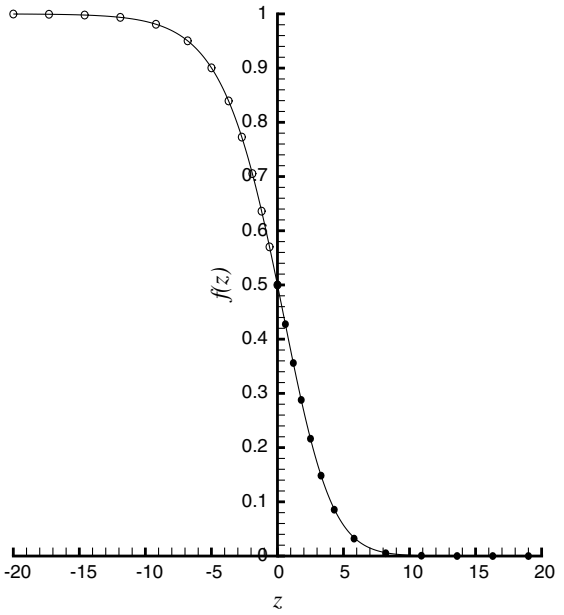

Fig. 1. The comparison of the exact solution (7) with $f^{-}(z)$ 30th-order of approximation and $f^{+}(z)$ 15th-order of approximation when $c = 5/\sqrt{6}$. Solid line: exact solution; open circle: $f^{-}(z)$; filled circle: $f^{+}(z)$.

The travelling wave for different wave speeds are as shown in Figs. 2 and 3. The relationship between the steepness, i.e., the slope of the wavefront, and the wave speed is shown in Fig. 4. Note that, the flatter the wavefront is, the faster the wave moves. This behavior has also been reported by Murray [18].

Mckean [40] showed that solution with any wave speed $c > 2$ is stable. Note that our travelling wave solutions with $c < c_{\min} = 2$ are stable, and the corresponding solution contains an oscillation part, as shown in Fig. 3.

For small parameter $\varepsilon = 1/c^2 \ll 1$, i.e., large wave speed c, perturbation approximations are accurate enough and agrees well with our series solution. However, for small wave speed, for example $c = 1/4$, the perturbation approximation is rather inaccurate (Fig. 5).

4. Conclusion

Travelling waves are an important solution form of nonlinear reaction–diffusion equations, with wide applications to chemistry, biology and medicine. These equations have a family of travelling wave solutions, and each of them has a front with a different wave speed. By means of the homotopy analysis method, we obtain the explicit series solutions. Note that, our series solutions are valid for all possible wave speed $0 < c < +\infty$ in the whole region $z \in (-\infty, +\infty)$. To the best of our knowledge, it is the first time such kind of explicit series solutions of the Fisher equation are reported.

Our series solution agrees well with the closed-form result given by Ablowitz and Zeppetella [16] for a particular wave speed $c = 5/\sqrt{6}$. This indicates that homotopy analysis method is valid for such kind of nonlinear travelling wave

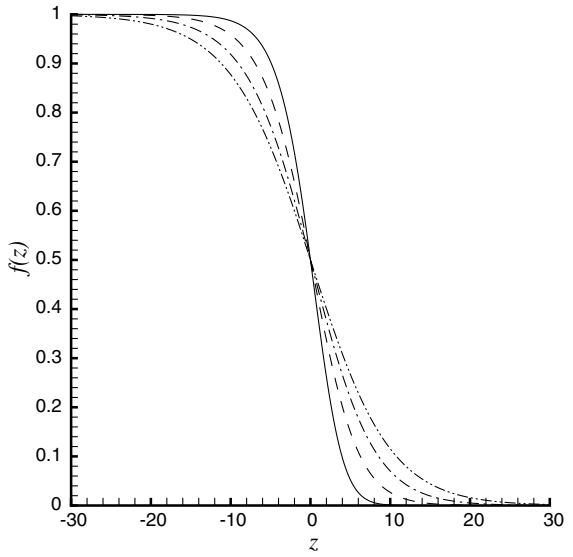

Fig. 2. The comparison of steepness with different wave speed c. Solid line: $c = 2$; dashed line: $c = 3$; dash-dotted line: $c = 4$; dash-dot-dotted line: $c = 5$.

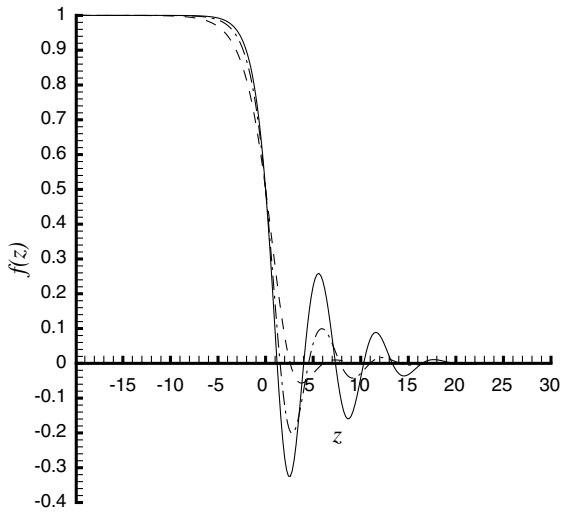

Fig. 3. The travelling wave profile when $c < c_{\min} = 2$. Solid line: $c = 1/4$; dash-dotted line: $c = 1/2$; dashed line: $c = 1$.

problem. Also, the relationship between the steepness of the wavefront and the wave speed has been studied. And our series solution indicates that, the flatter the wavefront is, the faster the wave moves, and that the travelling waves have an oscillation part when $0 < c < 2 = c_{\min}$.

There exist many reaction–diffusion equations in a similar form

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \nu G(u),$$

where $G(u)$ is the function of u. Most of these equations have travelling wave solutions, and can be solved in the similar way without difficulties. So, this article provides us with a general analytic approach to solve such kind of nonlinear travelling wave problems.

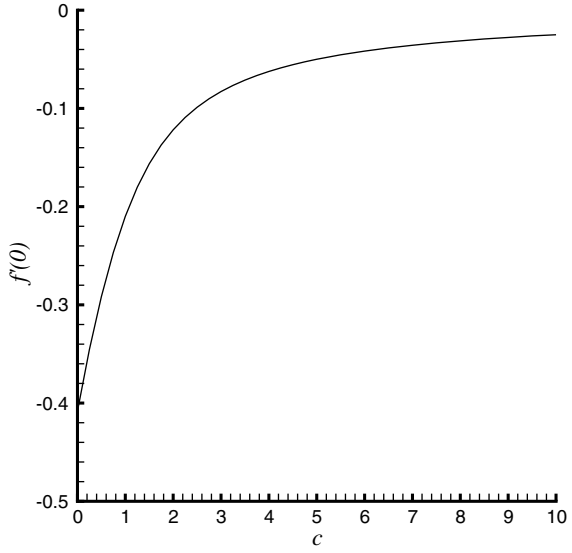

Fig. 4. The steepness at different wave speed c.

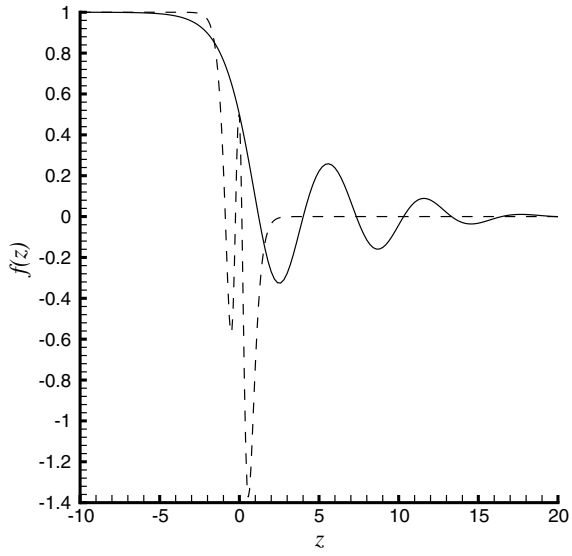

Fig. 5. The comparison of the series solution with the perturbation result when $c = 1/4$. Solid line: series solution given by the HAM; dashed line: perturbation approximation.

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