

Series Solution of Three-Dimensional Unsteady Laminar Viscous Flow Due to a Stretching Surface in a Rotating Fluid

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An analytic technique, namely the homotopy analysis method, is applied to solve the Navier–Stokes equations governing unsteady viscous flows due to a suddenly stretching surface in a rotating fluid. Unlike perturbation methods, the current approach does not depend upon any small parameters at all. Besides contrary to all other analytic techniques, it provides us with a simple way to ensure the convergence of solution series. In contrast to perturbation approximations which have about 40% average errors for the considered problem, our series solutions agree well with numerical results in the whole time region $0 \leq t < +\infty$. Explicit analytic expressions of the skin friction coefficients are given, which agree well with numerical results in the whole time region $0 \leq t < +\infty$. This analytic approach can be applied to solve some complicated three-dimensional unsteady viscous flows governed by the Navier–Stokes equations. [DOI: 10.1115/1.2723816]

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1 Introduction

The flow of a rotating fluid past a stretching surface is encountered in many technical and industrial applications which include the cooling of an infinite metallic plate in a cooling bath, the boundary layer along material handling conveyers, the aerodynamic extrusion of plastic sheets, the boundary layer along a liquid film and condensation processes, the cooling and drying of paper and textiles, the glass fiber production, etc. In particular, in extrusion of a polymer in a melt-spinning process, the extrusion from the die is generally drawn and simultaneously stretched into a sheet, which is then solidified throughout quenching or gradual cooling by direct contact with water.

Wang [1] first considered the two-dimensional stretching of a surface in a rotating fluid. However, relatively little work has been done on unsteady boundary layer flow due to impulsive starting from rest of a stretching sheet in a viscous fluid [2,3]. Nazar et al. [4] solved the unsteady boundary layer flows due to a stretching surface in a rotating fluid by means of both the Keller-box numerical method [5] and the perturbation technique. However, the perturbation approximations given by Nazar et al. [4] have about 40% average error, and are *not* accurate enough in the *whole* time region, as shown in Figs. 4 and 5. Besides, it becomes more and more difficult to get higher-order perturbation approximations. This is mainly because perturbation techniques depend too strongly upon small parameters.

An analytical method for strongly nonlinear problems, namely the homotopy analysis method (HAM) [6–11], has been developed since 1992. In contrast to perturbation techniques, the homotopy analysis method is *independent* of any small parameters at all. Besides, it provides us with a simple way to ensure the convergence of the solution series, so that we can always get accurate enough approximations. Furthermore, it provides us with freedom to choose better basis functions to approximate nonlinear problems. Finally, as proved by Liao [7,9], the homotopy analysis

method logically contains the so-called nonperturbation methods such as Lyapunov's artificial small parameter method, the δ -expansion method, and Adomian's decomposition method. Using the relationship between the homotopy analysis method and Adomian's decomposition method, Allan [12] investigated the accuracy of approximations given by the Adomian's decomposition method. Currently, Hayat et al. [13], Sajid et al. [14], and Abbasbandy [15,16] pointed out that the so-called "homotopy perturbation method" [17] proposed in 1999 is also a special case of the homotopy analysis method [6,7] propounded in 1992. Thus, the homotopy analysis method is rather general. The homotopy analysis method has been successfully applied to many nonlinear problems in science and engineering, such as the similarity boundary-layer flows [9,18–21], nonlinear heat transfer [15], nonlinear evaluation equations [22], nonlinear waves [16], viscous flows of non-Newtonian fluid [13,14,23,24], Thomas–Fermi atom model [7], Volterra's population model [7], etc. It has been applied in many fields of research. For example, Zhu [25,26] applied the HAM to give, for the first time, an explicit series solution of the famous Black–Scholes type equation in finance for American put option, which is a system of nonlinear partial differential equations (PDEs) with an unknown moving boundary. Besides, the HAM has been successfully applied to solve some PDEs in fluid mechanics and heat transfer, such as the unsteady boundary-layer viscous flows [10], the unsteady nonlinear heat transfer problem [27], etc. In this paper, we further employ it to give much more *accurate* analytic approximations (with less than 0.5% error in the *whole* time region) of the unsteady nonlinear problem at hand.

2 Mathematical Description

Let (u, v, w) be the velocity components in the direction of Cartesian axes (x, y, z) , respectively, with the axes rotating at an angular velocity Ω in the z direction. Consider the unsteady boundary-layer flows caused by a stretching surface at $z=0$ in a rotating fluid. When $t < 0$, the surface rotates at an angular velocity Ω in the z direction so that the fluid is at rest relative to the surface. At time $t=0$, the surface at $z=0$ is impulsively stretched

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in the x direction. Due to the Coriolis force, the fluid motion is three dimensional and is governed by the continuity equation and the unsteady Navier–Stokes equations [4]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (1a)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - 2\Omega v = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u \quad (1b)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + 2\Omega u = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v \quad (1c)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 w \quad (1d)$$

where p denotes the pressure; ρ the density; ν the kinematic viscosity; and ∇^2 the three-dimensional Laplacian, respectively. The initial and boundary conditions are

$$t < 0: \quad u = v = w = 0 \quad \text{for any } x, y, z \quad (1e)$$

$$t \geq 0: \quad u = ax, \quad v = w = 0 \quad \text{at } z = 0 \quad (1f)$$

$$u \rightarrow 0, \quad v \rightarrow 0, \quad w \rightarrow 0 \quad \text{as } z \rightarrow \infty \quad (1g)$$

where the constant a ($a > 0$, with the dimension of t^{-1}) represents the stretching rate.

Using Williams and Rhyne's similarity transformation [28]

$$\xi = 1 - e^{-\tau}, \quad \tau = at \quad (2)$$

and introducing the following similarity variables

$$\eta = \sqrt{\frac{a}{\nu \xi}} z, \quad u = axf'(\xi, \eta), \quad v = axg(\xi, \eta), \quad w = -\sqrt{a\nu \xi} f(\xi, \eta) \quad (3)$$

Equations (1a)–(1d) become a system of two coupled nonlinear differential equations

$$f''' + \frac{1}{2}(1 - \xi)\eta f'' + \xi(f'f'' - f'^2 + 2\lambda g) - \xi(1 - \xi)\frac{\partial f'}{\partial \xi} = 0 \quad (4a)$$

$$g'' + \frac{1}{2}(1 - \xi)\eta g' + \xi(fg' - f'g - 2\lambda f') - \xi(1 - \xi)\frac{\partial g}{\partial \xi} = 0 \quad (4b)$$

where the prime denotes the derivation with respect to the similarity variable η ; and $\lambda = \Omega/a$ is a dimensionless parameter. The corresponding boundary conditions read

$$f(\xi, 0) = 0, \quad f'(\xi, 0) = 1, \quad f'(\xi, +\infty) = 0, \quad g(\xi, 0) = 0, \quad g(\xi, +\infty) = 0 \quad (4c)$$

Note that, as $\tau \rightarrow 0$, we have $\xi \rightarrow \tau$. Thus, $\eta = \sqrt{a/\nu \xi} z$ is exactly the same as the traditional definition given in Ref. [4], which has meanings as $\tau \rightarrow 0$.

According to the definitions (2) and (3), we have the dimensionless skin friction coefficient in the x and y directions

$$C_{f_x}^x = (\xi \text{Re}_x)^{-1/2} f''(\xi, 0), \quad C_{f_y}^y = (\xi \text{Re}_x)^{-1/2} g'(\xi, 0) \quad (5)$$

where $\text{Re}_x = ax^2/\nu$ is the local Reynolds number.

3 Perturbation Approximations

3.1 Initial Solution as $\xi \rightarrow 0$. As $\xi \rightarrow 0$, corresponding to the initial flow, Eqs. (4a) and (4b) become

$$f''' + \frac{1}{2}\eta f'' = 0 \quad (6a)$$

$$g'' + \frac{1}{2}\eta g' = 0 \quad (6b)$$

subject to the boundary/initial conditions

$$f(0, 0) = 0, \quad f'(0, 0) = 1, \quad f'(0, \infty) = 0,$$

$$g(0, 0) = 0, \quad g(0, \infty) = 0 \quad (6c)$$

Its solution is given by

$$f(0, \eta) = \eta \operatorname{erfc}(\eta/2) + \frac{2}{\sqrt{\pi}}(1 - e^{-\eta^2/4}) \quad (7)$$

$$g(0, \eta) = 0 \quad (8)$$

where $\operatorname{erfc}(\eta)$ is the so-called complementary error function

$$\operatorname{erfc}(\eta) = \frac{2}{\sqrt{\pi}} \int_{\eta}^{\infty} e^{-s^2} ds \quad (9)$$

Note that Eqs. (6a) and (6b) do not contain the parameter λ . Thus, the result as $\xi \rightarrow 0$ is independent of λ .

3.2 Steady Solution at $\xi = 1$. At $\xi = 1$, corresponding to the steady-state flow, Eqs. (4a) and (4b) read

$$f''' + ff'' - f'^2 + 2\lambda g = 0 \quad (10a)$$

$$g'' + fg' - f'g - 2\lambda f' = 0 \quad (10b)$$

subject to the same boundary and initial conditions as Eq. (6c). The steady-state boundary-layer flows were solved by Wang [1,2].

3.3 Perturbation Approximations for Small ξ . Regarding ξ as a small parameter, one has the perturbation expressions

$$f(\xi, \eta) = \hat{f}_0(\eta) + \hat{f}_1(\eta)\xi + \hat{f}_2(\eta)\xi^2 + \dots \quad (11)$$

$$g(\xi, \eta) = \hat{g}_0(\eta) + \hat{g}_1(\eta)\xi + \hat{g}_2(\eta)\xi^2 + \dots \quad (12)$$

Substituting them into Eqs. (4a)–(4c) and equating the coefficients of the like power of ξ , one has the zero-order perturbation equations

$$\hat{f}_0''' + \frac{1}{2}\eta \hat{f}_0'' = 0 \quad (13a)$$

$$\hat{g}_0'' + \frac{1}{2}\eta \hat{g}_0' = 0 \quad (13b)$$

$$\hat{f}_0(0) = 0, \quad \hat{f}_0'(0) = 1, \quad \hat{f}_0'(\infty) = 0, \quad \hat{g}_0(0) = 0, \quad \hat{g}_0(\infty) = 0 \quad (13c)$$

which are exactly the same as Eqs. (6a)–(6c), respectively. The first-order perturbation equations are given by

$$\hat{f}_1''' + \frac{1}{2}\eta \hat{f}_1'' - \hat{f}_1' = \frac{1}{2}\eta \hat{f}_0'' - \hat{f}_0' \hat{f}_0' + \hat{f}_0'^2 \quad (14a)$$

$$\hat{g}_1'' + \frac{1}{2}\eta \hat{g}_1' - \hat{g}_1 = 2\lambda \hat{f}_0' \quad (14b)$$

$$\hat{f}_1(0) = 0, \quad \hat{f}_1'(0) = 0, \quad \hat{f}_1'(\infty) = 0, \quad \hat{g}_1(0) = 0, \quad \hat{g}_1(\infty) = 0 \quad (14c)$$

Its solution reads

$$\begin{aligned} \hat{f}_1(\eta) = & \left(\frac{1}{2} - \frac{2}{3\pi} \right) \left[\left(1 + \frac{\eta^2}{2} \right) \operatorname{erfc}(\eta/2) - \frac{1}{\sqrt{\pi}} \eta e^{-\eta^2/4} \right] \\ & - \frac{1}{2} \left(1 - \frac{\eta^2}{2} \right) \operatorname{erfc}^2(\eta/2) - \frac{3}{2\sqrt{\pi}} \eta e^{-\eta^2/4} \operatorname{erfc}(\eta/2) + \frac{2}{\pi} e^{-\eta^2/2} \\ & - \frac{1}{\sqrt{\pi}} \left(\frac{1}{4} \eta + \frac{4}{3\sqrt{\pi}} \right) e^{-\eta^2/4} \end{aligned} \quad (15)$$

$$\hat{g}_1(\eta) = \lambda \eta^2 \operatorname{erfc}(\eta/2) - \frac{2}{\sqrt{\pi}} \lambda \eta e^{-\eta^2/4} \quad (16)$$

In general, one can transfer the original nonlinear initial/boundary-value problem into an infinite number of linear boundary-value problems

$$\mathcal{L}_f^p[f_m(\eta)] = R_m^p(\eta), \quad \mathcal{L}_g^p[g_m(\eta)] = G_m^p(\eta) \quad (17)$$

where the two linear operators \mathcal{L}_f^p and \mathcal{L}_g^p are defined by

$$\mathcal{L}_f^p \phi = \frac{\partial^3 \phi}{\partial \eta^3} + \frac{\eta \partial^2 \phi}{2 \partial \eta^2} - \frac{\partial \phi}{\partial \eta} \quad (18)$$

$$\mathcal{L}_g^p \psi = \frac{\partial^2 \psi}{\partial \eta^2} + \frac{\eta \partial \psi}{2 \partial \eta} - \psi \quad (19)$$

The common solutions of the linear equations

$$\mathcal{L}_f^p \phi = 0, \quad \mathcal{L}_g^p \psi = 0$$

are, respectively

$$\begin{aligned} \phi = & C_1 + C_2 \left(\eta + \frac{\eta^3}{6} \right) + C_3 \left[\frac{4}{3} \exp(-\eta^2/4) \left(1 + \frac{\eta^2}{4} \right) + \sqrt{\pi} \operatorname{erf}(\eta/2) \right. \\ & \left. \times \left(\eta - \frac{\eta^3}{6} \right) \right] \end{aligned} \quad (20)$$

and

$$\psi = C_1 \left(1 + \frac{\eta^2}{2} \right) + C_2 \left[\eta \exp(-\eta^2/4) + \sqrt{\pi} \operatorname{erf}(\eta/2) \left(1 + \frac{\eta^2}{2} \right) \right] \quad (21)$$

where C_1 , C_2 , and C_3 are integral constants. Note that the above common solutions contain the error function $\operatorname{erf}(\eta/2)$ and are rather complicated. Owing to this reason, it becomes more and more difficult to get higher-order perturbation approximations. It should be emphasized that these two linear operators \mathcal{L}_f^p and \mathcal{L}_g^p come directly from the original governing equations. Thus, the perturbation method does not provide us with any freedom to choose the linear operators of the linear subproblems.

The first-order perturbation approximation of the skin friction coefficients reads

$$C_f^x \operatorname{Re}_x^{1/2} \approx \xi^{-1/2} [\hat{f}_0'(0) + \xi \hat{f}_1'(0)] = \frac{1}{\sqrt{\pi}} \left[-\xi^{-1/2} + \left(-\frac{7}{4} + \frac{4}{3\pi} \right) \xi^{1/2} \right] \quad (22)$$

$$C_g^y \operatorname{Re}_x^{1/2} \approx \xi^{-1/2} [\hat{g}_0'(0) + \xi \hat{g}_1'(0)] = -\frac{2}{\sqrt{\pi}} \lambda \xi^{1/2} \quad (23)$$

For details, please refer to Nazar et al. [4]. Note that, due to the appearance of the error function $\operatorname{erfc}(\eta/2)$, it becomes more and more difficult to get higher-order perturbation approximations, as mentioned above. Note that the perturbation approximation of $C_f^x \operatorname{Re}_x^{1/2}$ is independent of λ . However, for large λ , these perturbation approximations are not accurate, and would have about 40% average error, as shown in Figs. 4 and 5.

4 Homotopy Analytic Solution

In this part, we solve Eqs. (4a)–(4c) by means of the homotopy analysis method [7,9]. First of all, it is well known that most of boundary-layer flows decay exponentially at infinity (i.e., $\eta \rightarrow +\infty$). Thus, according to the boundary conditions (4c), f' and g decay to zero at infinity exponentially. Besides, Eqs. (4a) and (4b) explicitly contain the terms ξ and η . Thus, $f(\xi, \eta)$ and $g(\xi, \eta)$ should be expressed by such a set of basis functions

$$\{\xi^k \eta^m \exp(-n\eta) | k \geq 0, m \geq 0, n \geq 0\} \quad (24)$$

in the following forms

$$f(\xi, \eta) = a_0^{0,0} + \sum_{k=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{n=1}^{+\infty} a_k^{m,n} \xi^k \eta^m \exp(-n\eta) \quad (25)$$

$$g(\xi, \eta) = \sum_{k=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{n=1}^{+\infty} b_k^{m,n} \xi^k \eta^m \exp(-n\eta) \quad (26)$$

where $a_k^{m,n}$ and $b_k^{m,n}$ are coefficients. Note that the solutions expressed in the above forms decay exponentially as $\eta \rightarrow +\infty$. As shown later, the above expressions are important in the frame of the homotopy analysis method for the choice of the initial guesses and the auxiliary linear operators. The initial guesses and auxiliary linear operators should be chosen in such a way that the approximations must be expressed by the above two expressions: this is so important that it is called *the rule of solution expressions* [7,9].

According to the solution expressions (25) and (26) and the boundary conditions (4c), it is straightforward to choose the initial guesses

$$f_0(\xi, \eta) = 1 - \exp(-\eta) \quad (27)$$

$$g_0(\xi, \eta) = 0 \quad (28)$$

Note that the above initial guesses satisfy the initial/boundary conditions (4c). Let \mathcal{L}_f and \mathcal{L}_g denote two auxiliary linear operators, which we will determine later. Here, we note that we have great freedom to choose \mathcal{L}_f and \mathcal{L}_g . Based on Eqs. (4a) and (4b), we define the following two nonlinear operators

$$\begin{aligned} \mathcal{N}_f[F, G] = & \frac{\partial^3 F}{\partial \eta^3} + \frac{1}{2}(1-\xi)\eta \frac{\partial^2 F}{\partial \eta^2} - \xi(1-\xi) \frac{\partial^2 F}{\partial \xi \partial \eta} + \xi \left[F \frac{\partial^2 F}{\partial \eta^2} \right. \\ & \left. - \left(\frac{\partial F}{\partial \eta} \right)^2 + 2\lambda G \right] \end{aligned} \quad (29)$$

and

$$\begin{aligned} \mathcal{N}_g[F, G] = & \frac{\partial^2 G}{\partial \eta^2} + \frac{1}{2}(1-\xi)\eta \frac{\partial G}{\partial \eta} - \xi(1-\xi) \frac{\partial G}{\partial \xi} + \xi \left(F \frac{\partial G}{\partial \eta} - G \frac{\partial F}{\partial \eta} \right. \\ & \left. - 2\lambda \frac{\partial F}{\partial \eta} \right) \end{aligned} \quad (30)$$

Then, we construct the so-called zero-order deformation equations

$$(1-q)\{\mathcal{L}_f[F(\xi, \eta; q)] - \mathcal{L}_f[f_0(\xi, \eta)]\} = q\hbar \mathcal{N}_f[F(\xi, \eta; q), G(\xi, \eta; q)] \quad (31a)$$

$$(1-q)\{\mathcal{L}_g[G(\xi, \eta; q)] - \mathcal{L}_g[g_0(\xi, \eta)]\} = q\hbar \mathcal{N}_g[F(\xi, \eta; q), G(\xi, \eta; q)] \quad (31b)$$

subject to the corresponding boundary/initial conditions

$$F(\xi, 0; q) = 0, \quad \left. \frac{\partial F(\xi, \eta; q)}{\partial \eta} \right|_{\eta=0} = 1, \quad \left. \frac{\partial F(\xi, \eta; q)}{\partial \eta} \right|_{\eta=\infty} = 0 \quad (31c)$$

and

$$G(\xi, 0; q) = 0, \quad G(\xi, \infty; q) = 0 \quad (31d)$$

where $q \in [0, 1]$ is the embedding parameter; \hbar denotes a nonzero auxiliary parameter; and $F(\xi, \eta; q)$ and $G(\xi, \eta; q)$ are unknown functions related to $f(\xi, \eta)$ and $g(\xi, \eta)$, respectively.

When $q=0$, since the initial guesses $f_0(\xi, \eta)$ and $g_0(\xi, \eta)$ satisfy the initial/boundary conditions (4c), the above zero-order deformation equations have the solution

$$F(\xi, \eta; 0) = f_0(\xi, \eta), \quad G(\xi, \eta; 0) = g_0(\xi, \eta) \quad (32)$$

When $q=1$, since $\hbar \neq 0$, the above zero-order deformation equations are equivalent to the original Eqs. (4a)–(4c), provided

$$F(\xi, \eta; 1) = f(\xi, \eta), \quad G(\xi, \eta; 1) = g(\xi, \eta) \quad (33)$$

respectively. Thus, as q increases from 0 to 1, $F(\xi, \eta; q)$ and $G(\xi, \eta; q)$ vary (or *deform*) from the known initial guesses $f_0(\xi, \eta)$ and $g_0(\xi, \eta)$ to the unknown solutions $f(\xi, \eta)$ and $g(\xi, \eta)$ of the original Eq. (4a)–(4c), respectively. This is the reason why Eqs. (31a)–(31d) are called zero-order deformation equations.

By Taylor's theorem and using Eq. (32), we have the power series

$$F(\xi, \eta; q) = f_0(\xi, \eta) + \sum_{n=1}^{+\infty} f_n(\xi, \eta) q^n \quad (34)$$

$$G(\xi, \eta; q) = g_0(\xi, \eta) + \sum_{n=1}^{+\infty} g_n(\xi, \eta) q^n \quad (35)$$

where

$$f_n(\xi, \eta) = \frac{1}{n!} \left. \frac{\partial^n F(\xi, \eta; q)}{\partial q^n} \right|_{q=0}, \quad g_n(\xi, \eta) = \frac{1}{n!} \left. \frac{\partial^n G(\xi, \eta; q)}{\partial q^n} \right|_{q=0} \quad (36)$$

Note that Eqs. (31a) and (31b) contain the auxiliary parameter \hbar . Obviously, the convergence of the series (34) and (35) is dependent not only upon the auxiliary linear operators \mathcal{L}_f and \mathcal{L}_g but also the auxiliary parameter \hbar , and more importantly, we have great freedom to choose all of them. Assuming that the auxiliary parameter \hbar and the auxiliary linear operators \mathcal{L}_f and \mathcal{L}_g are so properly chosen that the series (34) and (35) converge at $q=1$, we have, using Eq. (33), the solution series

$$f(\xi, \eta) = f_0(\xi, \eta) + \sum_{n=1}^{+\infty} f_n(\xi, \eta) \quad (37)$$

$$g(\xi, \eta) = g_0(\xi, \eta) + \sum_{n=1}^{+\infty} g_n(\xi, \eta) \quad (38)$$

The above series relates the initial guesses $f_0(\xi, \eta)$ and $g_0(\xi, \eta)$ with the exact solution $f(\xi, \eta)$ and $g(\xi, \eta)$ by means of the unknown terms $f_n(\xi, \eta)$ and $g_n(\xi, \eta)$, where $n=1, 2, 3, \dots$. According to the fundamental theorem in calculus, the Taylor series (34) and (35) are *unique*, and are completely determined by the zero-order deformation Eqs. (31a)–(31d). Thus, the governing equations and boundary/initial conditions of $f_n(\xi, \eta)$ and $g_n(\xi, \eta)$ can be deduced directly from the zero-order deformation Eqs. (31a)–(31d). Based on the definition (36) and the solution series (37) and (38), Liao [7–11] provided a rather general approach to obtain the equations governing $f_n(\xi, \eta)$ and $g_n(\xi, \eta)$. For the sake of simplicity, define the vectors

$$\vec{f}_m = \{f_0, f_1, f_2, \dots, f_m\} \quad (39)$$

$$\vec{g}_m = \{g_0, g_1, g_2, \dots, g_m\} \quad (40)$$

Differentiating the zero-order deformation Eqs. (31a)–(31d) n times with respect to the embedding parameter q , then setting $q=0$, and finally dividing by $n!$, we have the so-called n th-order deformation equation

$$\mathcal{L}_f[f_n(\xi, \eta) - \chi_n f_{n-1}(\xi, \eta)] = \hbar R_n^f(\vec{f}_{n-1}, \vec{g}_{n-1}, \xi, \eta) \quad (41a)$$

$$\mathcal{L}_g[g_n(\xi, \eta) - \chi_n g_{n-1}(\xi, \eta)] = \hbar R_n^g(\vec{f}_{n-1}, \vec{g}_{n-1}, \xi, \eta) \quad (41b)$$

subject to the boundary/initial conditions

$$f_n(\xi, 0) = 0, \quad \left. \frac{\partial f_n(\xi, \eta)}{\partial \eta} \right|_{\eta=0} = 0, \quad \left. \frac{\partial f_n(\xi, \eta)}{\partial \eta} \right|_{\eta=\infty} = 0 \quad (41c)$$

and

$$g_n(\xi, 0) = 0, \quad g_n(\xi, \infty) = 0 \quad (41d)$$

where

$$R_n^f(\vec{f}_{n-1}, \vec{g}_{n-1}, \xi, \eta) = \frac{\partial^3 f_{n-1}}{\partial \eta^3} + \frac{1}{2}(1-\xi)\eta \frac{\partial^2 f_{n-1}}{\partial \eta^2} - \xi(1-\xi) \frac{\partial^2 f_{n-1}}{\partial \xi \partial \eta} + \xi \left[\sum_{j=0}^{n-1} \left(f_{n-1-j} \frac{\partial^2 f_j}{\partial \eta^2} - \frac{\partial f_{n-1-j}}{\partial \eta} \frac{\partial f_j}{\partial \eta} \right) + 2\lambda g_{n-1} \right] \quad (41e)$$

and

$$R_n^g(\vec{f}_{n-1}, \vec{g}_{n-1}, \xi, \eta) = \frac{\partial^2 g_{n-1}}{\partial \eta^2} + \frac{1}{2}(1-\xi)\eta \frac{\partial g_{n-1}}{\partial \eta} - \xi(1-\xi) \frac{\partial g_{n-1}}{\partial \xi} + \xi \left[\sum_{j=0}^{n-1} \left(f_{n-1-j} \frac{\partial g_j}{\partial \eta} - g_{n-1-j} \frac{\partial f_j}{\partial \eta} \right) - 2\lambda \frac{\partial f_{n-1}}{\partial \eta} \right] \quad (41f)$$

under the definition

$$\chi_k = \begin{cases} 0, & k \leq 1 \\ 1, & k > 1 \end{cases} \quad (42)$$

As proved by Sajid et al. [13] and Hayat et al. [14], directly substituting the series (34) and (35) into zero-order deformation Eqs. (31a)–(31c), and then equating the coefficients of the like power of q , one can obtain exactly the *same* equations as Eqs. (41a)–(41f), no matter whether q is regarded as a small parameter or not. This is mainly because the Taylor series (34) and (35) are *unique*.

The original Eq. (4a) is third order with respect to the similarity variable η and first order with respect to the dimensionless time ξ , and Eq. (4b) is second order with respect to η and first order with respect to ξ , respectively. In general, it is more difficult to solve these kinds of combined initial-boundary-value problems than pure boundary-value ones, even if they are linear. Fortunately, the homotopy analysis method provides us with great freedom to choose the auxiliary linear operators \mathcal{L}_f and \mathcal{L}_g : we can choose \mathcal{L}_f and \mathcal{L}_g in such a way that the high-order deformation equations are pure boundary-value ones. To do so, \mathcal{L}_f may be a third-order linear differential operator with respect to η , and \mathcal{L}_g a second-order linear operator with respect to η , respectively. Therefore, without loss of generality, we write

$$\mathcal{L}_f[\phi(\xi, \eta)] = \frac{\partial^3 \phi}{\partial \eta^3} + A_2(\eta) \frac{\partial^2 \phi}{\partial \eta^2} + A_1(\eta) \frac{\partial \phi}{\partial \eta} + A_0(\eta) \phi \quad (43)$$

and

$$\mathcal{L}_g[\phi(\xi, \eta)] = \frac{\partial^2 \phi}{\partial \eta^2} + B_1(\eta) \frac{\partial \phi}{\partial \eta} + B_0(\eta) \phi \quad (44)$$

where $A_0(\eta)$, $A_1(\eta)$, $A_2(\eta)$, $B_0(\eta)$, and $B_1(\eta)$ are real functions to be determined below. Let $f_n^*(\xi, \eta)$ and $g_n^*(\xi, \eta)$ denote the special solution of Eqs. (41a)–(41d). Its general solutions read

$$f_n(\xi, \eta) = f_n^*(\xi, \eta) + C_1(\xi) e_1^f(\eta) + C_2(\xi) e_2^f(\eta) + C_3(\xi) e_3^f(\eta) \quad (45)$$

$$g_n(\xi, \eta) = g_n^*(\xi, \eta) + C_4(\xi) e_1^g(\eta) + C_5(\xi) e_2^g(\eta) \quad (46)$$

where $C_1(\xi)$, $C_2(\xi)$, $C_3(\xi)$, $C_4(\xi)$, and $C_5(\xi)$ are integral constants determined by the boundary conditions (41c) and (41d), the real functions $e_1^f(\eta)$, $e_2^f(\eta)$, $e_3^f(\eta)$, $e_1^g(\eta)$, and $e_2^g(\eta)$ are the so-called

kernels of the two auxiliary linear operators, satisfying

$$\mathcal{L}_f[e_1^f(\eta)] = \mathcal{L}_f[e_2^f(\eta)] = \mathcal{L}_f[e_3^f(\eta)] = 0, \quad (47)$$

$$\mathcal{L}_g[e_1^g(\eta)] = \mathcal{L}_g[e_2^g(\eta)] = 0$$

According to the solution expressions (25) and (26), the kernels should belong to the basis functions. If we choose $e_1^f(\eta)$, $e_2^f(\eta)$, $e_3^f(\eta)$ as the first three simplest basis functions among (24), i.e.

$$e_1^f(\eta) = 1, \quad e_2^f(\eta) = \exp(-\eta), \quad e_3^f(\eta) = \exp(-2\eta)$$

then, the boundary condition at infinity is automatically satisfied and the coefficient $C_3(\xi)$ cannot be *uniquely* determined. To ensure that the high-order deformation equations have *unique* solutions, we should choose

$$e_1^f(\eta) = 1, \quad e_2^f(\eta) = \exp(-\eta), \quad e_3^f(\eta) = \exp(+\eta)$$

Then, to satisfy the boundary condition $f_n'(\xi, +\infty) = 0$, we have $C_3(\xi) = 0$, and $C_1(\xi)$, $C_2(\xi)$ are determined by the two boundary conditions $f_n(\xi, 0) = 0$ and $f_n'(\xi, 0) = 0$. Substituting the above kernels into Eq. (47), we have

$$A_0(\eta) = 0 \quad (48)$$

$$-[1 + A_1(\eta) - A_2(\eta)]\exp(-\eta) = 0 \quad (49)$$

$$[1 + A_1(\eta) + A_2(\eta)]\exp(+\eta) = 0 \quad (50)$$

which give

$$A_0(\eta) = A_2(\eta) = 0, \quad A_1(\eta) = -1$$

Similarly, choosing the kernels $e_1^g(\eta) = \exp(-\eta)$ and $e_2^g(\eta) = \exp(+\eta)$, we have

$$B_0(\eta) = -1, \quad B_1(\eta) = 0$$

Thus, we have the two auxiliary linear operators

$$\mathcal{L}_f\phi = \frac{\partial^3\phi}{\partial\eta^3} - \frac{\partial\phi}{\partial\eta} \quad (51)$$

$$\mathcal{L}_g\phi = \frac{\partial^2\phi}{\partial\eta^2} - \phi \quad (52)$$

which have the properties

$$\mathcal{L}_f[C_1(\xi)\exp(-\eta) + C_2(\xi)\exp(+\eta) + C_3(\xi)] = 0 \quad (53)$$

$$\mathcal{L}_g[C_4(\xi)\exp(-\eta) + C_5(\xi)\exp(+\eta)] = 0 \quad (54)$$

Note that, in contrast to perturbation approximations, our HAM series solutions do *not* contain the error function $\text{erf}(\eta/2)$. This is mainly because we have great freedom to choose the auxiliary linear operators \mathcal{L}_f and \mathcal{L}_g , which are much simpler than \mathcal{L}_f^p and \mathcal{L}_g^p appeared in the high-order perturbation Eqs. (14a) and (14b), respectively.

The high-order deformation Eqs. (41a)–(41d) are linear boundary-value equations. Thus, according to Eqs. (37) and (38), the original *nonlinear*, combined initial-boundary-value problem is transferred into an infinite number of *linear* boundary-value problems. However, in contrast to perturbation techniques, this kind of transformation does *not* need any small parameters. Besides, in contrast to the perturbation method, the homotopy analysis method provides us with great *freedom* to choose the auxiliary linear operators \mathcal{L}_f and \mathcal{L}_g . Using this kind of freedom, we can obtain results at rather high order of approximations by means of choosing the linear operators (51) and (52), which are simpler than (18) and (19), and whose kernels do *not* contain the error functions $\text{erf}(\eta/2)$.

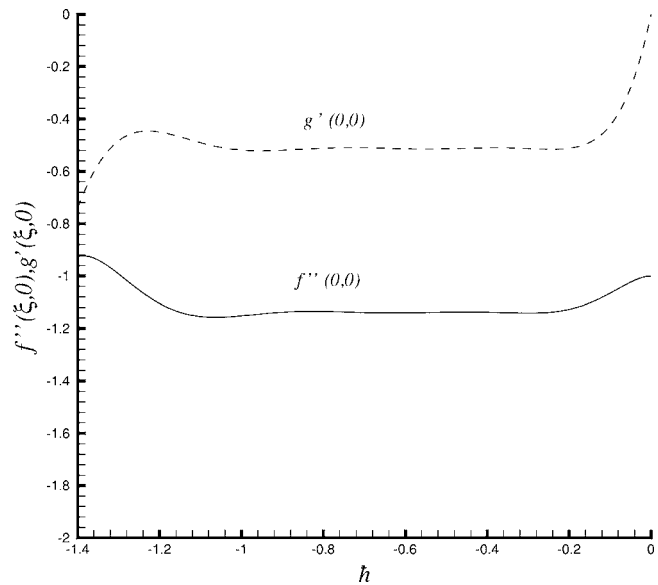


Fig. 1 The 15th-order HAM approximation of $f''(0,0)$ and $g'(0,0)$ in the case of $\lambda=1/2$

5 Result Analysis

Liao [7] proved in general that, as long as a solution series given by the homotopy analysis method converges, it must be one of the solutions of the equation considered. Thus, it is important to ensure that the HAM solution series are convergent. Note that the solution series (37) and (38) contain one auxiliary parameter \hbar . As shown by Liao [7–11] and others [13–16,22], it is the auxiliary parameter \hbar that provides us with a simple way to *adjust* and *control* the convergence region of the solution series. In general, by means of choosing a proper value of the auxiliary parameter \hbar , one can always ensure the convergence of the solution series given by the homotopy analysis method. For example, let us consider the case $\lambda=1/2$ for the nonlinear unsteady problem at hand. First of all, we investigate the convergence of $f''(0,0)$ and $g'(0,0)$ in the case of $\lambda=1/2$ by means of regarding \hbar as a variable. *Physically*, $f''(0,0)$ and $g'(0,0)$ are *independent* of the auxiliary parameter \hbar , and thus are *unique*. This is mainly because the auxiliary parameter \hbar has only mathematical meanings. Thus, mathematically, as long as the series of $f''(0,0)$ and $g'(0,0)$ are convergent, they must be convergent to the *same* values, respectively. As shown in Fig. 1, all points $(\hbar, f''(0,0))$ with different values of \hbar but the same value of $f''(0,0)$ create a horizontal line segment of the curve $f''(0,0) \sim \hbar$ in a region of $-0.8 \leq \hbar \leq -0.2$, as does the curve $g'(0,0) \sim \hbar$. Therefore, if we choose $-0.8 \leq \hbar \leq -0.2$, we can ensure the convergence of the series of $f''(0,0)$ and $g'(0,0)$. In fact, we indeed get convergent results of $f''(0,0)$ and $g'(0,0)$ by choosing $\hbar=-1/2$. In general, by means of plotting such kinds of \hbar curves, we can always choose a proper value of the auxiliary parameter \hbar to get accurate HAM approximations, as suggested by Liao [7–11]. Similarly, one can investigate the convergence of the series solution at $\xi=0$ in the whole region $0 \leq \eta < +\infty$. It is found that, when $\xi=0$ and $\hbar=-1/2$, the solution series of $f(0, \eta)$ and $g(0, \eta)$ converge to the exact initial solutions (7) and (8), as shown in Fig. 2. Furthermore, it is found that when $\hbar=-1/2$, the series solution of $f(\xi, \eta)$ and $g(\xi, \eta)$ are even convergent in the *whole* region $0 \leq \xi \leq 1$ and $0 \leq \eta < +\infty$. When $\lambda=1/2$, the 15th-order approximations of coefficient of skin friction

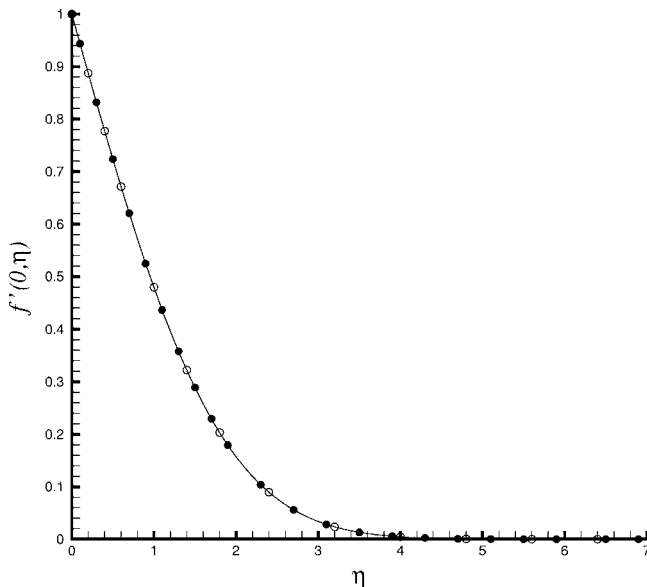


Fig. 2 Comparison of $f'(0, \eta)$ with the exact solution: solid line exact solution; filled circle: 15th-order HAM approximation; open circle: 20th-order HAM approximation

$$\begin{aligned}
 C_f^x \text{Re}_x^{1/2} = & \xi^{-1/2}(-0.5744017005 - 0.4493808859\xi \\
 & - 0.06481391332\xi^2 - 0.01742724136\xi^3 \\
 & - 0.009296051716\xi^4 - 0.005899366598\xi^5 \\
 & - 0.004065430821\xi^6 - 0.002955448193\xi^7 \\
 & - 0.001917447834\xi^8 - 0.009525724822\xi^9 \\
 & + 0.07823876097\xi^{10} - 0.3945931407\xi^{11} \\
 & + 1.034178447\xi^{12} - 1.478649518\xi^{13} + 1.079883330\xi^{14} \\
 & - 0.3178853418\xi^{15})
 \end{aligned} \quad (55)$$

and

$$\begin{aligned}
 C_f^y \text{Re}_x^{1/2} = & \xi^{-1/2}(-0.5524842929\xi + 0.008804549832\xi^2 \\
 & + 0.004367680454\xi^3 + 0.002135276175\xi^4 \\
 & + 0.001719399534\xi^5 + 0.001533164270\xi^6 \\
 & + 0.001399929708\xi^7 + 0.001031699397\xi^8 \\
 & + 0.009003153919\xi^9 - 0.09321414938\xi^{10} \\
 & + 0.5469861363\xi^{11} - 1.668879409\xi^{12} \\
 & + 2.763217719\xi^{13} - 2.335134057\xi^{14} \\
 & + 0.7942135326\xi^{15})
 \end{aligned} \quad (56)$$

agree well with the numerical results for *all* time $\xi \in [0, 1]$, corresponding to $0 \leq \tau < +\infty$, as shown in Fig. 3.

In general, for any given value of λ , we can always choose a proper value of the auxiliary parameter \hbar in a similar way to ensure that the solution series converge for *all* time $0 \leq \tau < +\infty$ in the *whole* spatial region $0 \leq \eta < +\infty$. For example, in cases of $\lambda = 0.5$ and $\lambda = 1$, our HAM results of the coefficient of skin friction $C_f^x \text{Re}_x^{1/2}$ and $C_f^y \text{Re}_x^{1/2}$ agree well with the numerical solutions, as shown in Figs. 4 and 5. As shown in Table 1, the 25th-order HAM approximations have errors less than 0.5%. Note that for a larger value of λ , the perturbation solution becomes worse, and its average error is about 40%. However, by choosing a proper value of the auxiliary parameter \hbar , our HAM approximations always converge to the numerical solutions for any values of λ . Thus, the

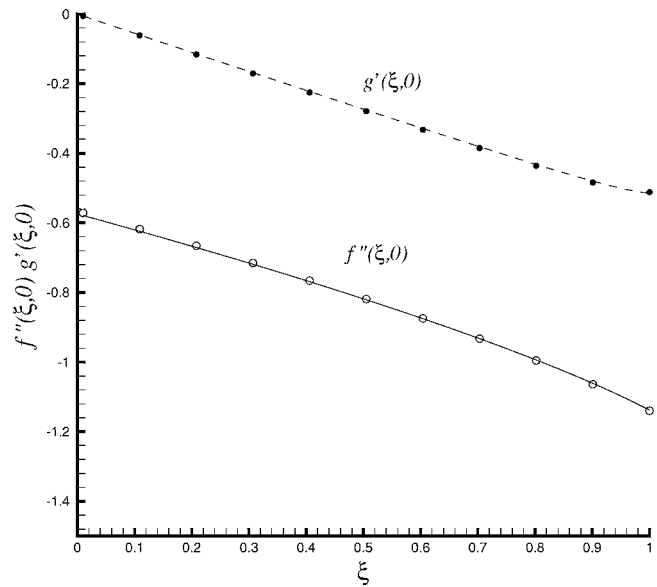


Fig. 3 Comparison of numerical solutions with HAM results of $f'(\xi, 0)$ and $g'(\xi, 0)$ in the case of $\lambda=1/2$ by means of $\hbar=-1/2$; solid line: numerical result of $f'(\xi, 0)$; open circle: 15th-order HAM approximation of $f'(\xi, 0)$; dash line: numerical result of $g'(\xi, 0)$; filled circle: 15th-order HAM approximation of $g'(\xi, 0)$

homotopy analysis method is indeed a powerful analytic tool for nonlinear problems with strong nonlinearity.

Indeed, the homotopy analysis method provides us with a simple way to ensure the convergence of solution series by means of choosing a proper value of the auxiliary parameter \hbar . This is an advantage of the homotopy analysis method over all other analytic techniques. By the way, as proved by Hayat et al. [13] and Sajid et al. [14], and illustrated by Abbasbandy [15,16], the approximations given by the so-called “homotopy perturbation method” [17], which was proposed 7 years later than the homotopy analysis method [6], are only special cases for those given by the homotopy analysis method when $\hbar=-1$. Like other traditional analytic techniques, the “homotopy perturbation method” [17] cannot provide such a simple way to adjust and control the convergence of the solution series [13–16]. For example, in the cases under consideration, the results given by the “homotopy perturbation method” (HPM) are valid only for *small* time, as shown in Figs. 4 and 5.

6 Conclusion

In this paper, the three-dimensional unsteady viscous flows due to the impulsively stretching surface of the incompressible rotating fluid, governed by the Navier–Stokes equations, are solved by means of one analytic technique for strongly nonlinear problems, namely the homotopy analysis method [6–11]. In contrast to the corresponding perturbation approximations which have 40% average errors, our series solutions are uniformly valid for all time $0 \leq \tau < +\infty$ in the whole spatial region $0 \leq \eta < +\infty$, but with only less than 0.5% error, as shown in Table 1. Explicit analytic expressions of coefficients of skin friction are given, which are useful in engineering and for validation of numerical simulations. All of these verify the validity and potential of our approach for complicated viscous flows.

The homotopy analysis method has some advantages over other traditional ones. First, it provides us with great freedom to choose the auxiliary linear operators. Using this kind of freedom, we transfer the original nonlinear, combined initial-boundary-value problem at hand into an infinite number of linear boundary-value subproblems, which are so easy to solve that we can get results at rather high orders of approximations. Second, contrary to all other

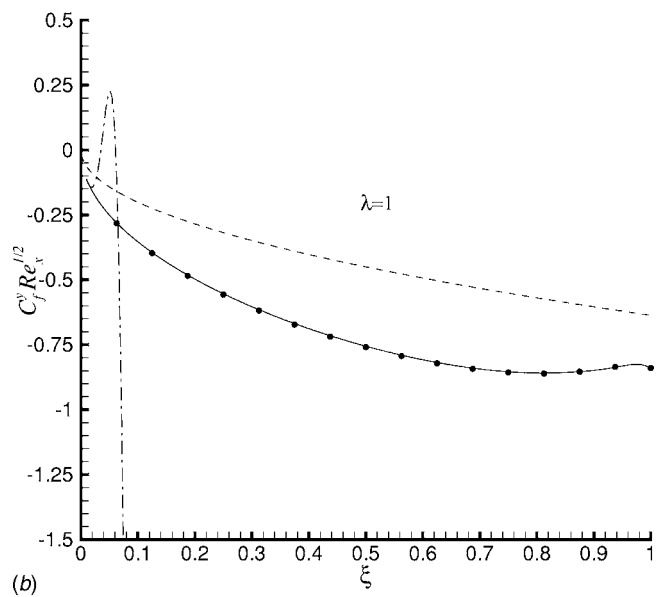
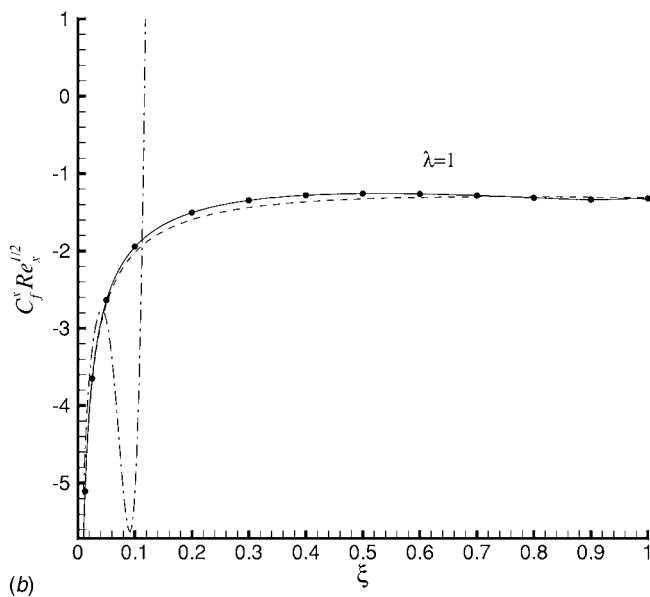
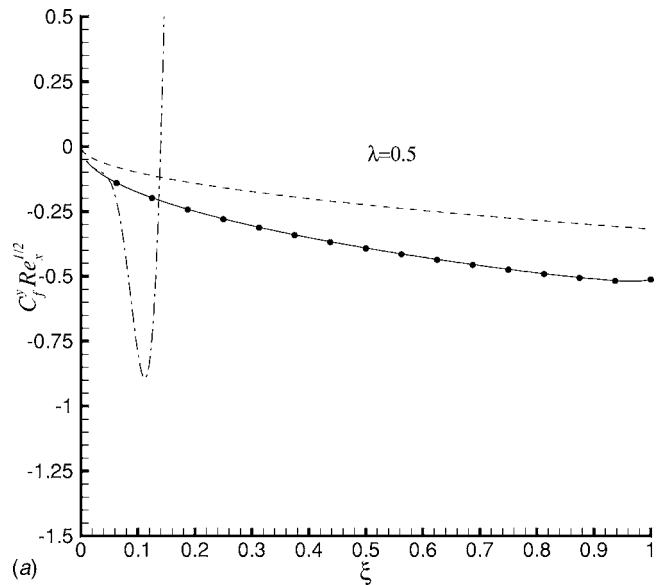
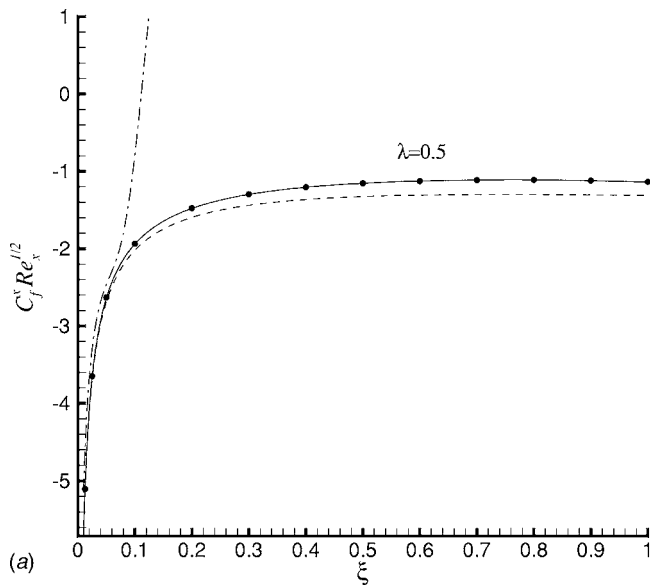


Fig. 4 Comparison of numerical results [4] of $C_f^x \sqrt{Re_x}$ with analytic approximations; symbols: numerical result; solid line: 15th-order HAM approximation: (a) $\hbar = -0.5$; (b) $\hbar = -0.25$; dash line: perturbation approximation given by Nazar et al. [4]; dash-dotted line: 15th-order HPM approximation ($\hbar = -1$)

Fig. 5 Comparison of numerical results [4] of $C_f^y \sqrt{Re_x}$ with analytic approximations; symbols: numerical result; solid line: 15th-order HAM approximation (a) $\hbar = -0.5$; (b) $\hbar = -0.25$; dash line: perturbation approximation given by Nazar et al. [4]; dash-dotted line: 15th-order HPM approximation ($\hbar = -1$)

analytic techniques, the homotopy analysis method provides us with a simple way to ensure convergence of solution series. Thus, we can always get accurate enough approximations by means of the homotopy analysis method. Third, in contrast to perturbation techniques, the homotopy analysis method is independent of small/large parameters. Thus, it is suitable for more nonlinear problems. Finally, the homotopy analysis method logically contains other nonperturbation techniques such as the Lyapunov's small parameter method, the δ -expansion method, and the Adomian's decomposition method, as proved by Liao [7]. Currently, Hayat et al. [13], Sajid et al. [14], and Abbasbandy [15,16] pointed out that the so-called "homotopy perturbation method" [17] proposed in 1999 is also a special case of the homotopy analysis method [6,7] propounded in 1992. Thus, the homotopy analysis method is rather general.

There exist numerous three-dimensional unsteady viscous flows and heat transfer problems, which are often rather complicated. The proposed homotopy analysis method provides us with a new

approach to get accurate and convergent series solutions of unsteady three-dimensional Navier–Stokes equations, which are uniformly valid in the whole time $0 \leq \tau < +\infty$.

Table 1 Comparisons of HAM approximations of $f''(1,0)$ and $g'(1,0)$ with numerical results given by Wang [1] and Nazar et al. [4]

λ	$f''(1,0)$			$g'(1,0)$		
	Wang	Nazar et al.	HAM (25th)	Wang	Nazar et al.	HAM (25th)
0.5	-1.138	-1.138	-1.138	-0.513	-0.513	-0.511
1.0	-1.325	-1.325	-1.323	-0.837	-0.837	-0.830

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Nomenclature

C_f^x	= skin friction coefficient in the x direction
C_f^y	= skin friction coefficient in the y direction
f, g	= real functions
F, G	= real functions
$\mathcal{L}_f, \mathcal{L}_g$	= auxiliary linear operators
$\mathcal{N}_f, \mathcal{N}_g$	= nonlinear operators
p	= pressure
q	= embedding parameter
Re_x	= local Reynolds number, $Re_x = ax^2/\nu$
t	= time
u, v, w	= velocities in the x, y, z direction
x, y, z	= spatial coordinates
η	= similarity variable
λ	= non-dimensional parameter defined by $\lambda = \Omega/a$
ξ	= dimensionless time defined by $\xi = 1 - e^{-\tau}$
ρ	= density
τ	= dimensionless time defined by $\tau = at$
ν	= kinematic viscosity
Ω	= angular velocity in the z direction
∇^2	= Laplace operator
\hbar	= nonzero auxiliary parameter

References

- [1] Wang, C. Y., 1988, "Stretching a Surface in a Rotating Fluid," *ZAMP*, **39**, pp. 177–185.
- [2] Wang, C. Y., Dut, Q., Miklavcic, M., and Chang, C. C., 1997, "Impulsive Stretching of a Surface in a Viscous Fluid," *SIAM J. Appl. Math.*, **57**, pp. 1–14.
- [3] Seshadri, R., Sreeshylan, N., and Nath, G., 2002, "Unsteady Mixed Convection Flow in the Stagnation Region a Heated Vertical Plate due to Impulsive Motion," *Int. J. Heat Mass Transfer*, **45**, pp. 1345–1352.
- [4] Nazar, N., Amin, N., and Pop, I., 2002, "Unsteady Boundary Layer Flow due to Stretching Surface in a Rotating Fluid," *Mech. Res. Commun.*, **31**, pp. 121–128.
- [5] Cebeci, T., and Bradshaw, P., 1984, *Physical and Computational Aspects of Convective Heat Transfer*, Springer, New York.
- [6] Liao, S. J., 1992, "The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems," Ph.D. thesis, Shanghai Jiao Tong University, Shanghai, China.

- [7] Liao, S. J., 2003, *Beyond Perturbation: Introduction to Homotopy Analysis Method*, Chapman & Hall/CRC Press, Boca Raton, FL.
- [8] Liao, S. J., 1999, "An Explicit, Totally Analytic Approximation of Blasius Viscous Flow Problems," *Int. J. Non-Linear Mech.*, **34**(4), pp. 759–778.
- [9] Liao, S. J., 2004, "On the Homotopy Analysis Method for Nonlinear Problems," *Appl. Math. Comput.*, **147**, pp. 499–513.
- [10] Liao, S. J., 2006, "Series Solutions of Unsteady Boundary-Layer Flows Over a Stretching Flat Plate," *Stud. Appl. Math.*, **117**(3), pp. 2529–2539.
- [11] Liao, S. J., and Magyari, E., 2006, "Exponentially Decaying Boundary Layers as Limiting Cases of Families of Algebraically Decaying Ones," *Z. Angew. Math. Phys.*, **57**(5), pp. 777–792.
- [12] Allan, F. M., 2007, "Derivation of the Adomian Decomposition Method Using the Homotopy Analysis Method," *Appl. Math. Comput.*, in press.
- [13] Hayat, T., and Sajid, M., 2007, "On Analytic Solution for Thin Film Flow of a Fourth Grade Fluid Down a Vertical Cylinder," *Phys. Lett. A*, **361**, pp. 316–322.
- [14] Sajid, M., Hayat, T., and Asghar, S., 2007, "Comparison Between the HAM and HPM Solutions of Tin Film Flows of Non-Newtonian Fluids on a Moving Belt," *Nonlinear Dyn.*, in press.
- [15] Abbasbandy, S., 2006, "The Application of the Homotopy Analysis Method to Nonlinear Equations Arising in Heat Transfer," *Phys. Lett. A*, **360**, pp. 109–113.
- [16] Abbasbandy, S., 2007, "The Application of the Homotopy Analysis Method to Solve a Generalized Hirota–Satsuma Coupled KdV Equation," *Phys. Lett. A*, **361**, pp. 478–483.
- [17] He, J. H., 1999, "Homotopy Perturbation Technique," *Comput. Methods Appl. Mech. Eng.*, **178**, pp. 257–262.
- [18] Liao, S. J., 1999, "A Uniformly Valid Analytic Solution of 2D Viscous Flow Past a Semi-infinite Flat Plate," *J. Fluid Mech.*, **385**, pp. 101–128.
- [19] Liao, S. J., and Campo, A., 2002, "Analytic Solutions of the Temperature Distribution in Blasius Viscous Flow Problems," *J. Fluid Mech.*, **453**, pp. 411–425.
- [20] Liao, S. J., 2003, "On the Analytic Solution of Magneto-hydrodynamic Flows of Non-Newtonian Fluids Over a Stretching Sheet," *J. Fluid Mech.*, **488**, pp. 189–212.
- [21] Song, Y., Zheng, L. C., and Zhang, X. X., 2006, "On the Homotopy Analysis Method for Solving the Boundary Layer Flow Problem Over a Stretching Surface With Suction and Injection," *Journal of University of Science and Technology Beijing*, **28**, pp. 782–784, in Chinese.
- [22] Shi, Y. R., Xu, J. X., and Wu, Z. X., (2006), "Application of the Homotopy Analysis Method to Solving Nonlinear Evaluation Equations," *Acta Phys. Sin.*, **55**(4), pp. 1555–1560.
- [23] Asghar, S., Gulzar, M. M., and Ayub, M., 2006, "Effect of Partial Slip on Flow of a Third Grade Fluid," *Acta Mech. Sin.*, **22**, pp. 195–198.
- [24] Hayat, T., and Khan, M., 2005, "Homotopy Solutions for a Generalized Second Grade Fluid Past a Porous Plate," *Nonlinear Dyn.*, **42**, pp. 395–405.
- [25] Zhu, S. P., 2006, "A Closed-Form Analytical Solution for the Valuation of Convertible Bonds With Constant Dividend Yield," *ANZIAM J.*, **47**, pp. 477–494.
- [26] Zhu, S. P., 2006, "An Exact and Explicit Solution for the Valuation of American Put Options," *Quant. Finance*, **6**, pp. 229–242.
- [27] Liao, S. J., Su, J., and Chwang, A. T., 2006, "Series Solutions for a Nonlinear Model of Combined Convective and Radiative Cooling of a Spherical Body," *Int. J. Heat Mass Transfer*, **49**, pp. 2437–2445.
- [28] Williams, J. C., and Rhyne, T. H., 1980, "Boundary Layer Development on a Wedge Impulsively Set into Motion," *SIAM J. Appl. Math.*, **38**, pp. 215–224.