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An analytical solution for a nonlinear time-delay model in biology

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A B S T R A C T

In this paper, the homotopy analysis method is applied to develop a analytic approach for nonlinear differential equations with time-delay. A nonlinear model in biology is used as an example to show the basic ideas of this analytic approach. Different from other analytic techniques, the homotopy analysis method provides a simple way to ensure the convergence of the solution series, so that one can always get accurate approximations. A new discontinuous function is defined so as to express the piecewise continuous solutions of time-delay differential equations in a way convenient for symbolic computations. It is found that the time-delay has a great influence on the solution of the time-delay nonlinear differential equation. This approach has general meanings and can be applied to solve other nonlinear problems with time-delay.

1. Introduction

Nonlinear delay differential equations (DDE) arises when the evolution of a system not only depends on its present state but also on its history. These nonlinear DDE are more complicated than traditional linear differential equation. Recent studies in various fields such as population dynamics, epidemiology, physiology, immunology, neural networks, economy, the navigational control of ships and air crafts, electrodynamics, etc, have shown that delay differential equations play an important role in explaining many different phenomenon. In particular, they turn out to be fundamental when traditional ODEs-based models fail.

Note that traditional ODEs are only the first approximation to reality while dependence on past is very important in many situations. For example, in dynamics of disease, the past history plays an important role which is described by $-\tau < t < 0$, where τ is the length of the time-delay. Several models with time-delay were analyzed by Hethcote [1] and Cooke [2]. Neeta [3] has discussed some time-delay epidemic models qualitatively. Lenoid [4] established some oscillation and non-oscillation conditions for nonlinear equations arising in population dynamics. Baker et al. [5] has discussed in detail that why in some cases DDEs are more important then ODEs and also provided the numerical way to deal with DDE. To the best of our knowledge, these DDEs are only handled by numerical techniques [6]. According to Brauer et al. [7], "analytic solutions, even for the simplest DDE, are in general hopeless".

The present paper describes a new analytic approach to solve nonlinear time-delay differential equations. This approach is based on the homotopy analysis method (HAM) [8–20], which does not require the assumption of small or large physical parameters. The comparison of the HAM with other perturbation and non-perturbation techniques is discussed in detail by Liao [11]. In this paper, we give the series solution of a nonlinear time-delay model by means of the homotopy analysis method. The model with time-delay is given as follows,

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$$i'(t) = ri(t)\left[1 - \frac{i(t-\tau)}{\kappa}\right], \tag{1}$$

subject to the initial condition

$$i(t) = \alpha, \quad -\tau \leqslant t \leqslant 0, \tag{2}$$

where $r > 0$ and $\kappa > 0$ are given physical parameters and τ is the delay-time, the prime denotes the differentiation with respect to t.

When $\tau = 0$, the solution can be expressed in a closed form

$$i(t) = \frac{\kappa}{1 + (\kappa/\alpha - 1)\exp(-rt)}, \tag{3}$$

which tends to κ as $t \to +\infty$. When $\tau \to +\infty$, it holds $i(t-\tau) = \alpha$, and then the original equation becomes

$$i'(t) = ri(t)\left(1 - \frac{\alpha}{\kappa}\right), \quad i(0) = \alpha,$$

which has the solution

$$i(t) = \alpha \exp\left[-r\left(\frac{\alpha}{\kappa} - 1\right)t\right], \tag{4}$$

with the property

$$i(+\infty) \to \begin{cases} 0, & \text{when } \alpha > \kappa > 0, \\ \alpha, & \text{when } \alpha = \kappa, \\ +\infty, & \text{when } 0 < \alpha < \kappa. \end{cases} \tag{5}$$

Note that, without the time-delay, $i(+\infty)$ is only dependent upon κ and has nothing to do with the initial value $i(0) = \alpha$. However, when the time-delay exists, $i(+\infty)$ depends not only on κ but also on the initial condition. Therefore, the value of the time-delay τ greatly influences the behavior of $i(t)$.

Write $i_\infty = i(+\infty)$. In this paper, we consider the case $\alpha > \kappa$ with the time-delay $\tau > 0$. So, i_∞ have two possible values $i_\infty = \kappa$ or $i_\infty = 0$, dependent upon the value of the time-delay τ.

2. HAM approach for time-delay model

2.1. Continuous variation

The HAM is based on continuous variation from an initial trial to the exact solution. In this problem we construct a continuous mapping $i(t) \to \phi(t,q)$, where $q \in [0,1]$ is an embedding parameter, such that as q increases from 0 to 1, $\phi(t,q)$ varies from the initial trial $i_0(t)$ to the exact solution $i(t)$. Considering the solution (3) for $\tau = 0$ and the solution (4) for $\tau \to +\infty$, it is straightforward that the solution for $\tau > 0$ can be generally expressed in the form

$$i(t) = \sum_{m=1}^{+\infty} a_m e^{-m\beta t}. \tag{6}$$

This provides us the so-called solution expression. According to the solution expression (6) and the initial condition (2), it is straightforward to choose the initial guess

$$i_0(t) \to \begin{cases} \alpha, & \text{when} \quad -\tau \leqslant t < 0, \\ i_\infty + (\alpha - i_\infty)\exp(-\beta t), & \text{when} \quad t \geqslant 0. \end{cases} \tag{7}$$

where i_∞ is either 0 or κ depending upon the value of τ, as discussed earlier. First, we construct the so-called zeroth-order deformation equation,

$$(1-q)\mathscr{L}[\phi(t;q) - i_0(t)] = q\hbar H(t)\mathscr{N}[\phi(t,q)], \tag{8}$$

subject to the initial condition

$$\phi(t;q) = \alpha, \quad -\tau \leqslant t \leqslant 0, \tag{9}$$

where $\hbar \neq 0$ is the so-called convergence-control parameter, $H(t) \neq 0$ is a nonzero auxiliary function, $\mathscr{L}$ is an auxiliary linear differential operator at first order, $\mathscr{N}$ is a nonlinear operator defined by,

$$\mathscr{N}[\phi(t,q)] = \phi'(t,q) - r\phi(t,q)\left[1 - \frac{\phi(t-\tau,q)}{\kappa}\right]. \tag{10}$$

To obey the solution expression (6), we choose the auxiliary linear operator

$$\mathscr{L}u = u'(t) + \beta u(t), \quad \beta > 0, \tag{11}$$

which has the property

$$\mathscr{L}[C_1 e^{-\beta t}] = 0, \tag{12}$$

for any integral constant C_1. When $q = 0$, the solution of Eqs. (8) and (9) is

$$\phi(t;0) = i_0(t). \tag{13}$$

When $q = 1$, since $H(t) \neq 0$ and $\hbar \neq 0$, Eqs. (8) and (9) are equivalent to the original Eqs. (1) and (2), so that it holds

$$\phi(t;1) = i(t). \tag{14}$$

Thus, as q increases from 0 to 1, $\phi(t;q)$ varies from the initial guess $i_0(t)$ to the exact solution $i(t)$ of the original Eqs. (1) and (2). Then we expand $\phi(t,q)$ in the Taylor series with respect to q i.e.,

$$i(t) = i_0(t) + \sum_{m=1}^{+\infty} i_m(t) q^m, \tag{15}$$

where

$$i_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t;q)}{\partial q^m}\bigg|_{q=0}. \tag{16}$$

Note that the above series is dependent upon $\hbar, \kappa, \beta, \alpha$ and i_∞. Assuming that all parameters are chosen properly so that the above series is convergent for $q = 1$, we have the solution series

$$i(t) = i_0(t) + \sum_{m=1}^{+\infty} i_m(t). \tag{17}$$

The above expression provides a relationship between $i_0(t)$ and the exact solution $i(t)$ by means of the unknown terms $i_m(t)$.

2.2. Successive approximations

Differentiating the zeroth-order deformation Eq. (8) m times with respect to the embedding parameter q, then setting $q = 0$ and finally dividing by $m!$, we have the so-called mth-order deformation equation,

$$\mathscr{L}[i_m(t) - \chi_m i_{m-1}(t)] = \hbar H(t) R_m(t), \tag{18}$$

subject to the initial condition

$$i_m(t) = 0, \quad -\tau \leqslant t \leqslant 0, \tag{19}$$

where

$$R_m(t) = i'_{m-1}(t) - r i_{m-1}(t) + \frac{r}{\kappa}\left[\sum_{k=0}^{m-1} i_k(t) i_{m-1-k}(t-\tau)\right], \tag{20}$$

with the definition

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{21}$$

The solution of the linear differential Eqs. (18) and (19) is given by

$$i_m(t) = \chi_m i_{m-1}(t) + \hbar e^{-\beta t} \int_0^t e^{\beta\xi} H(\xi) R_m(\xi) d\xi. \tag{22}$$

Note that we have great freedom to choose the auxiliary function $H(t)$. It is found that the term $te^{-\beta t}$ appears if we choose $H(t) = 1$. This disobeys the solution expression (6). To avoid this, we choose $H(t) = e^{-\beta t}$.

At the first order of approximation, we have the solution

$$i_1(t) = \hbar\left[\alpha\left(\frac{r+\beta}{\beta}\right)e^{-\beta t} - \frac{r\alpha^2}{\kappa\beta}e^{-\beta t} - \alpha\left(\frac{r+\beta}{\beta}\right) + \frac{r\alpha^2}{\kappa\beta}\right]e^{-\beta t} \tag{23}$$

for $0 < t \leqslant \tau$, and

$$i_1(t) = \hbar\left[\alpha\left(\frac{r+\beta}{\beta}\right)e^{-\beta t} - \frac{r\alpha^2}{2\kappa\beta}e^{-\beta\tau} - \frac{r\alpha^2}{2\kappa\beta}e^{-\beta(2t-\tau)} + \frac{r\alpha^2}{\kappa\beta} - \alpha\left(\frac{r+\beta}{\beta}\right)\right]e^{-\beta t} \tag{24}$$

for $t \geqslant \tau$. At the 2nd-order of approximation, we have

$$i_2(t) = \hbar^2 e^{-\beta t}\left(\frac{\alpha r}{2\beta} + \frac{\alpha r^2}{2\beta^2} - \frac{\alpha^2 r^2}{\kappa\beta^2} - \frac{\alpha^2 r}{2\kappa\beta} + \frac{\alpha^3 r^2}{2\kappa^2\beta^2}\right) + \hbar^2 e^{-2\beta t}\left(-\alpha - \frac{2\alpha r}{\beta} + \frac{2\alpha^2 r}{\kappa\beta} - \frac{\alpha r^2}{\beta^2} + \frac{2\alpha^2 r^2}{\kappa\beta^2} - \frac{\alpha^3 r^2}{\kappa^2\beta^2}\right)$$
$$+ \hbar^2 e^{-3\beta t}\left(\alpha + \frac{3\alpha r}{2\beta} - \frac{3\alpha^2 r}{2\kappa\beta} + \frac{\alpha r^2}{2\beta^2} - \frac{\alpha^2 r^2}{\kappa\beta^2} + \frac{\alpha^3 r^2}{2\kappa^2\beta^2}\right) \tag{25}$$

for $0 \leqslant t \leqslant \tau$,

$$i_2(t) = \hbar^2 e^{-\beta t}\left[\frac{\alpha r}{\beta}\left(\frac{1}{2} + \frac{r}{2\beta} - \frac{\alpha r}{2\kappa} + \frac{\alpha^2 r}{2\kappa^2\beta} - \frac{\alpha r}{\kappa\beta}\right) - \frac{\alpha^2 r}{3\kappa\beta}\left(\frac{\alpha r}{\kappa\beta} - 1 - \frac{r}{\beta}\right)e^{-\beta\tau} - \frac{\alpha^2 r}{2\kappa\beta}\left(\frac{1}{3} - \frac{\alpha r}{4\kappa\beta}\right)e^{-2\beta\tau}\right]$$
$$+ \hbar^2 e^{-2\beta t}\left[-\alpha + \frac{\alpha^2 r}{\kappa\beta} - \frac{2\alpha r}{\beta} - \frac{\alpha r^2}{\beta^2} + \frac{\alpha^2 r^2}{\kappa\beta^2} + e^{-\beta\tau}\left(-\frac{\alpha^2 r}{2\kappa\beta} - \frac{\alpha^2 r^2}{2\kappa\beta^2}\right)\right]$$
$$+ \hbar^2 e^{-3\beta t}\left[\alpha + \frac{3\alpha r}{2\beta} + \frac{\alpha r^2}{2\beta^2} + \frac{\alpha^3 r^2}{4\kappa^2\beta^2} + \left(\frac{\alpha^2 r}{\kappa\beta} - \frac{\alpha^3 r^2}{\kappa^2\beta^2} + \frac{\alpha^2 r^2}{\kappa\beta^2}\right)e^{\beta\tau}\right]$$
$$+ \hbar^2 e^{-4\beta t}\left[\left(-\frac{5\alpha^2 r}{6\kappa\beta} - \frac{\alpha^2 r^2}{2\kappa\beta^2}\right)e^{\beta\tau} + \left(\frac{\alpha^3 r^2}{3\kappa^2\beta^2} - \frac{\alpha^2 r}{3\kappa\beta} - \frac{\alpha^2 r^2}{3\kappa\beta^2}\right)e^{2\beta\tau}\right] + \hbar^2 e^{-5\beta t}\left(\frac{\alpha^3 r^2}{8\kappa^2\beta^2}e^{2\beta\tau}\right), \tag{26}$$

for $\tau \leqslant t \leqslant 2\tau$ and

$$i_2(t) = \hbar^2 e^{-\beta t}\left[\frac{\alpha r}{\beta}\left(\frac{1}{2} + \frac{r}{2\beta} - \frac{\alpha r}{2\kappa} + \frac{\alpha^2 r}{2\kappa^2\beta} - \frac{\alpha r}{\kappa\beta}\right) - \frac{\alpha^2 r}{3\kappa\beta}\left(\frac{\alpha r}{\kappa\beta} - 1 - \frac{r}{\beta}\right)e^{-\beta\tau} - \frac{\alpha^2 r}{2\kappa\beta}\left(\frac{1}{3} - \frac{\alpha r}{4\kappa\beta}\right)e^{-2\beta\tau} - e^{-4\beta\tau}\frac{\alpha^3 r^2}{24\kappa^2\beta^2}\right]$$
$$+ \hbar^2 e^{-2\beta t}\left[-\alpha + \frac{\alpha^2 r}{\kappa\beta} - \frac{2\alpha r}{\beta} - \frac{\alpha r^2}{\beta^2} + \frac{\alpha^2 r^2}{\kappa\beta^2} + e^{-\beta\tau}\left(-\frac{\alpha^2 r}{2\kappa\beta} - \frac{\alpha^2 r^2}{2\kappa\beta^2}\right)\right]$$
$$+ \hbar^2 e^{-3\beta t}\left[\alpha + \frac{3\alpha r}{2\beta} + \frac{\alpha r^2}{2\beta^2} + \frac{\alpha^3 r^2}{2\kappa^2\beta^2} + \left(\frac{\alpha^2 r}{\kappa\beta} - \frac{\alpha^3 r^2}{\kappa^2\beta^2} + \frac{\alpha^2 r^2}{\kappa\beta^2}\right)e^{\beta\tau}\right]$$
$$+ \hbar^2 e^{-4\beta t}\left[\left(-\frac{5\alpha^2 r}{6\kappa\beta} - \frac{\alpha^2 r^2}{2\kappa\beta^2}\right)e^{\beta\tau} + \left(-\frac{\alpha^2 r}{3\kappa\beta} - \frac{\alpha^2 r^2}{3\kappa\beta^2}\right)e^{2\beta\tau}\right] + \hbar^2 e^{-5\beta t}\left(\frac{\alpha^3 r^2}{8\kappa^2\beta^2}\right)(e^{2\beta\tau} + e^{4\beta\tau}) \tag{27}$$

for $t \geqslant 2\tau$.

Obviously, it is difficult to continue this for higher order approximations manually. To get high-order approximations, we introduce a new function $\delta^*(t)$ defined by

$$\delta^*(t) = \begin{cases} 1, & \text{when} \quad t > 0, \\ 1/2, & \text{when} \quad t = 0, \\ 0, & \text{when} \quad t < 0, \end{cases} \tag{28}$$

which has the following properties:

$$\frac{d(\delta^*(t)f(t))}{dt} = \delta^*(t)f'(t), \tag{29}$$

and

$$\int_0^t \delta^*(t-a)f(t)dt = \delta^*(t-a)\int_a^t f(t)dt, \tag{30}$$
$$\int_0^t \delta^*(-t+a)f(t)dt = \delta^*(-t+a)\int_0^t f(t)dt + \delta^*(t-a)\int_0^a f(t)dt, \tag{31}$$
$$\int_0^t \delta^*(t-a)\delta^*(-t+b)f(t)dt = \delta^*(t-a)\delta^*(-t+b)\int_a^t f(t)dt + \delta^*(t-b)\int_a^b f(t)dt. \tag{32}$$

Besides, according to the definition (28), the function $\delta^*(t)$ has the following properties:

$$f(t)\delta^*(t+a)\delta^*(-t+b) = 0, \quad \text{if} \quad b < -a, \tag{33}$$
$$\delta^*(t+a)\delta^*(t+b) = \delta^*(t+\min\{a,b\}), \tag{34}$$
$$\delta^*(-t+a)\delta^*(-t+b) = \delta^*(-t+\min\{a,b\}), \tag{35}$$
$$[\delta^*(t+a)]^m = \delta^*(t+a), \tag{36}$$

where $f(t)$ is a piecewise continuous real function and $m > 1$ is an integer.

Then, the initial guess $i_0(t)$ satisfying the initial condition (2) can be expressed in such a uniform formula:

$$i_0(t) = \alpha\delta^*(-t) + \delta^*(t)i_0^*(t), \tag{37}$$

where $i_0^*(t)$ is,

$$i_0^*(t) = i_\infty + (\alpha - i_\infty)\exp(-\beta t), \quad t \geqslant 0. \tag{38}$$

Using such kind of $\delta^*(t)$ function, we can get high-order approximations by means of symbolic computation software such as Mathematica. It is found that the solution of the time-delay problem is a piece-wise continuous function in the form

$$i_m(t) = \begin{cases} \alpha & -\tau \leqslant t \leqslant 0, \\ i_{m,1} & 0 \leqslant t \leqslant \tau, \\ i_{m,2} & \tau \leqslant t \leqslant 2\tau, \\ \vdots & \\ i_{m,(m+1)} & m\tau \leqslant t. \end{cases} \tag{39}$$

3. Result analysis

Eq. (1) has two fixed points at infinity: $i(+\infty) = \kappa$ and $i(+\infty) = 0$. According to (3), in case of $\tau = 0$, the solution of Eq. (1) tends to the constant κ. According to (4), in case of $\tau = +\infty$ and $\alpha > \kappa > 0$, the solution tends to zero as $t \to +\infty$. So, for a given finite value of time-delay τ, the value of $i_\infty = i(+\infty)$ is dependent upon the time-delay τ.

Without loss of generality, we consider here such a case: $\kappa = 1/2$, $r = 2$, $\alpha = 1$ with some different values of time-delay τ. We use this case as an example to show how to get convergent series for given physical values of κ, r, α and τ, and to investigate the influence of the time-delay τ on the solution.

Note that our series solution (17) contains two auxiliary parameters β and $\hbar$. For the sake of simplicity, we choose $\beta = 1$. Then, our mth-order approximation contains the auxiliary parameter $\hbar$, and thus the residual error $\delta_m(x, \hbar)$ of the governing Eq. (1) is dependent upon not only x but also $\hbar$. Write

$$\Delta_m = \int_0^{+\infty} \delta_m^2(x, \hbar)dx.$$

Obviously, Δ_m is a function of $\hbar$. For a convergent series solution (17) with a given $\hbar$, the sequence of

$$\Delta_0, \Delta_1, \Delta_2, \Delta_3, \ldots$$

converges to zero. There exists such a region **s** of $\hbar$ that each series solution (17) by means of any a value of $\hbar \in$ **s** is convergent. Such kind of region of $\hbar$ can be found by plotting the curve Δ_m versus $\hbar$, as shown in Fig. 1. Note that, as mentioned before, the unknown fixed point should be either $i_\infty = \kappa$ or $i_\infty = 0$. We choose $i_\infty = 0$ and $i_\infty = \kappa$, separately, and then compare the corresponding curves of $\Delta \sim \hbar$ at the same order of approximation. Obviously, for a correct choice of i_∞, the residual error Δ should decrease more quickly. For example, in case of $\tau = 1/10$, the residual error Δ given by the approximations with $i_\infty = \kappa$ decreases more quickly than that by $i_\infty = 0$, as shown in Fig. 1 and Table 1, so that one can get convergent series solu-

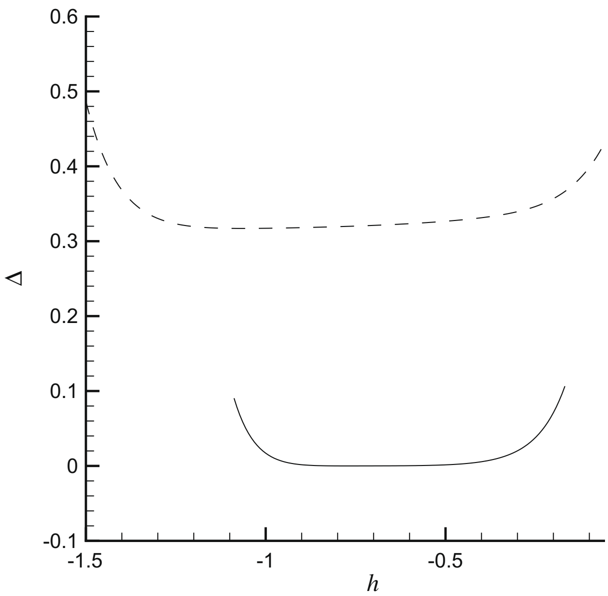

Fig. 1. The curve $\Delta \sim \hbar$ in case of $\kappa = 1/2$, $r = 2$, $\alpha = 1$ and $\tau = 1/10$ by means of different values of i_∞. Solid line: 4th-order result for $i_\infty = \kappa$; dashed line: 4th-order result for $i_\infty = 0$.

Table 1
Residual error Δ for different i_∞ in case of $\kappa = 1/2$, $r = 2$, $\alpha = 1$ and $\tau = 1/10$ by means of $\hbar = -3/4$ and $\beta = 1$.

Order of approximation	$i_\infty = 0$	$i_\infty = \kappa$
1	0.347	0.048
3	0.323	5.4×10^{-4}
5	0.318	2.6×10^{-5}
7	0.3176	2.8×10^{-6}

tion by means of $i_\infty = \kappa$ and $\hbar = -3/4$, as shown in Fig. 2. In this way, we can choose a correct value of i_∞ and also a proper value of $\hbar$ to ensure the convergence of solution series.

Similarly, in case of $\tau = 4$, it is found that the residual error decreases more quickly by $i_\infty = 0$, as shown in Table 2. And our analytic approximation given by $i_\infty = 0$ and $\hbar = -3/4$ agrees well with numerical ones, as shown in Fig. 3. In this way, we can get convergent series solution for different values of time-delay, as shown in Table 3.

From Table 3, it is obvious that there exists a criterion value of the time-delay, remarked by τ^*, so that $i_\infty = \kappa$ when $\tau < \tau^*$ and $i_\infty = 0$ when $\tau > \tau^*$. In case of $\kappa = 1/2$, $r = 2$ and $\alpha = 1$, the criterion time-delay is in the region $0.85 < \tau^* < 0.87$, and its more accurate value should be given by much higher-order approximations. Generally speaking, the criterion time-delay τ^* is dependent upon the physical parameters κ, r and the initial condition. This indicates that the time-delay τ has indeed a great influence on the global dynamics of the system.

4. Conclusion

The significance and importance of time-delay differential equations have attracted us to solve this problem by means of the HAM. Mathematically speaking, it is difficult to solve nonlinear time-delay differential equations, especially analytically. As Brauer et al. [7] has mentioned in his book, "analytic solutions, even for the simplest DDE, are in general hopeless". In this paper, the homotopy analysis method is successfully applied to give a analytic approach to solve nonlinear DDEs by means of a model in biology as an example. Generally speaking, solutions of time-delay differential equations are piecewise continuous. To overcome this difficulty in symbolic computation, we define a function $\delta^*(t)$ by (28) with the properties (29) to (36) so as to express these piecewise continuous functions effectively. In this way, one can get high-order approximations by means of symbolic computation software such as Mathematica, Maple and so on.

To the best of our knowledge, this is the first time that the homotopy analysis method is successfully applied to a nonlinear time-delay differential equation. Different from other analytic techniques, the homotopy analysis method (HAM) provides us with a simple way to ensure the convergence of the solution series. As shown in this paper, we can always find a proper value of the convergence-control parameter $\hbar$ to ensure the convergent series solution, and our analytic results agree well with numerical ones. This example illustrates that the analytic approach based on the HAM is indeed valid for nonlinear time-delay differential equations. Besides, this approach has general meanings and can be applied to solve some other types of nonlinear time-delay differential equations in a similar way.

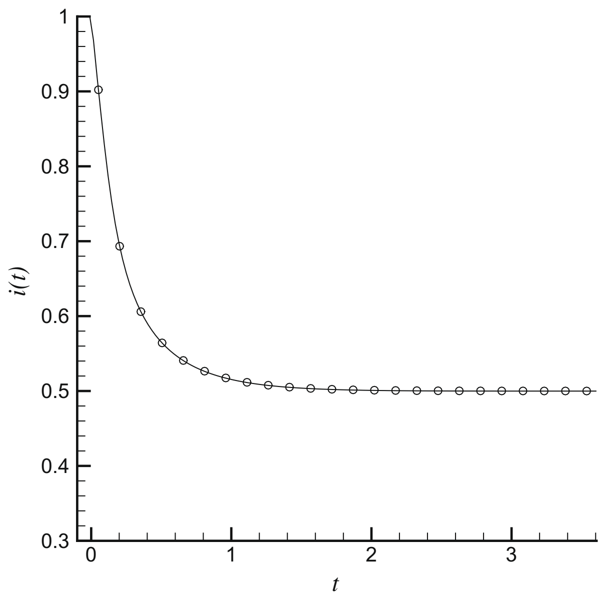

Fig. 2. Comparison of the numerical result with the 8th-order HAM solution of $i(t)$ in case of $\kappa = 1/2$, $r = 2$, $\alpha = 1$ and $\tau = 1/10$. Symbols: numerical solution; solid line: the 8th-order HAM solution by means of $\beta = 1$ and $\hbar = -3/4$.

Table 2
Residual error Δ for different i_∞ in case of $\kappa = 1/2$, $r = 2$, $\alpha = 1$ and $\tau = 4$ by means of $\hbar = -3/4$ and $\beta = 1$.

Order of approximation	$i_\infty = 0$	$i_\infty = \kappa$
1	0.031	2.360
3	1.2×10^{-4}	2.013
5	7.2×10^{-7}	2.009
7	2.4×10^{-7}	2.038

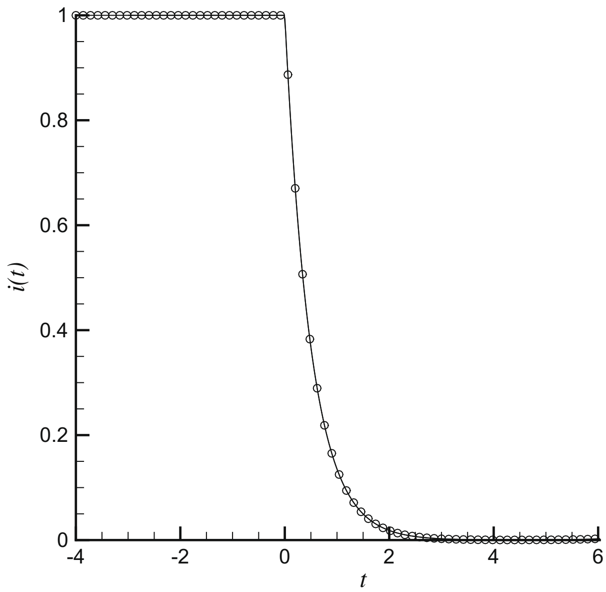

Fig. 3. Comparison of the numerical result with the 8th-order HAM solution of $i(t)$ in case of $\kappa = 1/2$, $r = 2$, $\alpha = 1$ and $\tau = 4$. Symbols: numerical solution; solid line: the 8th-order HAM solution by means of $\beta = 1$ and $\hbar = -3/4$.

Table 3
Influence of the time-delay τ on i_∞ in case of $\kappa = 1/2$, $r = 2$, $\alpha = 1$ by means of $\hbar = -3/4$ and $\beta = 1$.

τ	i_∞
0	κ
0.1	κ
0.5	κ
0.7	κ
0.8	κ
0.85	κ
0.87	0
0.9	0
1.0	0
2.0	0
4.0	0
10	0
$+\infty$	0

Using a special case as an example, we study the influence of the time-delay τ on the global property of the biology model under investigate. Our calculations indicate that there exists a criterion value τ^* dependent upon the physical parameters κ, r and α, so that $i_\infty = \kappa$ when $\tau < \tau^*$ but $i_\infty = 0$ when $\tau > \tau^*$. This indicates that the time-delay indeed has a great influence on the property of the nonlinear dynamic system.

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