Commun Nonlinear Sci Numer Simulat 15 (2010) 1148–1158

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

An explicit series approximation to the optimal exercise boundary of American put options

Jun Cheng [a,b], Song-Ping Zhu [c,*], Shi-Jun Liao [b]

[a] Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, China
[b] State Key Lab of Ocean Engineering, Shanghai Jiao Tong University, Shanghai 200030, China
[c] School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW 2522, Australia

ARTICLE INFO

Article history:
Received 17 May 2009
Accepted 20 May 2009
Available online 21 June 2009

PACS:
02.30.Jr

Keywords:
American put option
Optimal exercise boundary
Moving boundary problems
Homotopy analysis method

ABSTRACT

This paper derives an explicit series approximation solution for the optimal exercise boundary of an American put option by means of a new analytical method for strongly nonlinear problems, namely the homotopy analysis method (HAM). The Black–Sholes equation subject to the moving boundary conditions for an American put option is transferred into an infinite number of linear sub-problems in a fixed domain through the deformation equations. Different from perturbation/asymptotic approximations, the HAM approximation can be applicable for options with much longer expiry. Accuracy tests are made in comparison with numerical solutions. It is found that the current approximation is as accurate as many numerical methods. Considering its explicit form of expression, it can bring great convenience to the market practitioners.

1. Introduction

The distinctive feature of an American put option is its early exercise privilege. The possibility of the early exercise leads to a moving boundary problem in mathematics. Numerical methods are the most common methods nowadays to seek the approximation solutions. Some prominent methods include the binomial/trinomial methods [1,2], the Monte Carlo simulation [3], the least squares method [4], the variational inequalities [5,6], and the techniques based on solving partial differential equations [7–12]. However, since a majority of market practitioners are not familiar with numerical methods, the duplication of many numerical methods is not an easy job. Therefore, analytical solutions (or approximations) are extremely valuable.

Kim [13] and Carr et al. [14] derived a close-form solution of the American option value. They regarded the optimal exercise boundary as a known function, and hence the option value problem could be solved by means of the regular technique of the linear partial differential equations. From this point of view, the unknown optimal exercise boundary is the key point of an American option. In the past two decades, it has lured the interest of numerous market practitioners and mathematicians.

Qualitative analysis by Van Moerbeke [15] shows that the exercise boundary of an American option has a continuous time derivative except at the maturity, and its value on the expiry date equals the strike price. Blanchet [16] further proved that the exercise boundary of an American option is Hölder continuous with exponent 1/2 in time for all time.

Traditional analytic approaches are based on the perturbation or asymptotic methods, such as Barles [17], Kuske and Keller [18], Alobaidi and Mallier [19], Evans et al. [20], Zhang and Li [21], Knessl [22]. Chen et al. [11] and Chen and Chadam [23] made a careful comparison and analysis of some of the above works. They reported that these approximations were

* Corresponding author.
E-mail addresses: juncheng@shu.edu.cn (J. Cheng), spz@uow.edu.au (S.-P. Zhu), sjliao@sjtu.edu.cn (S.-J. Liao).

1007-5704/$ - see front matter
doi:10.1016/j.cnsns.2009.05.055

valid for very short time prior to expiry, usually on the order of days and weeks. Due to the limitation of the traditional methods, we need some new methods.

In this paper, we employ the homotopy analysis method (HAM) [24–27], which was first proposed and widely applied in the circle of mechanics [28–34]. Different from the perturbation or asymptotic methods, this method is independent of any small parameters, therefore we are able to obtain solutions valid for long time. Second, the HAM is a unified method for some other non-perturbation methods, such as Lyaounv artificial small parameter method, Adomian's decomposition method and δ-expansion method. It is a more general method. Third, the HAM can be well combined with many other mathematical method, such as Padé method, series expansion method, integral transform methods and the numerical methods. Zhu [35] first applied the HAM to the problem of an American put option. He gave a solution in the form of infinite recursive series involving double integrals. Then he performed Simpson's rule for the spatial integration the trapezoidal rule for the temporal integration in each order recursive solution. This is a remarkable contribution to find an analytic formula with no extra parameters involved. With a 30th-order approximation through numerical integration, Zhu [35] numerically demonstrated the convergence of his results, indicating the correctness of his formula.

The aim of this paper is to revisit the American put option problem by means of the HAM. There are two major differences between our paper and Zhu [35]. First, we do not use numerical integration in the recursive procedure, instead, we use the series expand method, therefore, we derive a series approximation solution in the form of power series of $\sqrt{\tau}$, where τ is the time to expiry. It can meet the needs of market practitioners who are not familiar with numerical methods. Second, we do not use the Landau transform [36]: $x = \ln[B(\tau)/S]$, where $B(\tau)$ is the optimal exercise boundary and S is the underlying assert price, to reformulate the moving boundary problem into a fixed boundary problem before it is solve by the HAM. Instead, we directly solve the moving boundary problem in the frame of the HAM. Therefore, for some other American type option problems when the Landau transform (or the equivalent transform) cannot be found, our approach is still applicable.

This paper is organized in five sections. In Section 2, the mathematical description of an American put option is presented. In Section 3, the application of the HAM is described in detail. In Section 4, the validity and accuracy of the explicit formula for the optimal exercise boundary are shown in comparison with some analytical approximations and numerical benchmark values. In Section 5, conclusions and discussions are given.

2. Analytic approach

2.1. Mathematic description

Consider an American put option with strike price X that expires at time T. Let $V(S,t)$ denote the value of an American put option, where S is the price of the underlying asset and t is the time. Let σ be the volatility of the underlying, r the risk-free interest rate. Both are assumed constants for this study for the illustration purpose. At any moment, there exists an optimal exercise boundary $B(t)$ such that it is optimal to exercise the put option when S is at or below $B(t)$. Hence, when $S \leqslant B(t)$ the put option is of value

$$V(S,t) = X - S, \tag{1}$$

where X is the strike price. When $S > B(t)$, $V(S,t)$ satisfies the Black–Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0. \tag{2}$$

The smooth pasting conditions at the exercise boundary $B(\tau)$ are

$$\lim_{S\to B(t)} V(S,t) = X - B(t), \quad \lim_{S\to B(t)} \frac{\partial V}{\partial S}(S,t) = -1, \tag{3}$$

the upper boundary condition is

$$\lim_{S\to +\infty} V(S,t) \to 0. \tag{4}$$

The terminal condition is

$$\lim_{t\to T} V(S,t) = \max\{X - S, 0\}, \tag{5}$$

Usually, the variable $\tau \equiv T - t$ is introduced. When $r > 0$, it is know $B(T) = X$ (see [15]), and hence the terminal condition (5) can be further simplified as

$$\lim_{\tau\to 0} V(S,\tau) = 0. \tag{6}$$

in the range $\Sigma_1 = \{(S,\tau) |\, B(\tau) \leqslant S < +\infty,\ 0 \leqslant \tau \leqslant T\}$.

Eqs. (1), (2), (4) and (5) can be solved by regular linear PDE skills. Kim [13] and Carr et al. [14] derived the following formula for the option price

$$V(S,\tau) = X\exp(-r\tau)N(-d_2) + SN(-d_1) + \int_0^\tau rX\exp(-r\xi)N(-d_{\xi,2})\mathrm{d}\xi, \tag{7}$$

where

$$d_1 = \frac{\ln(S/X) + (r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$
$$d_{\xi,1} = \frac{\ln[S/B(\tau-\xi)] + (r+\sigma^2/2)\xi}{\sigma\sqrt{\xi}}, \quad d_{\xi,2} = d_{\xi,1} - \sigma\sqrt{\xi}$$

and $N(x)$ was a cumulative distribution function for a standardized normal random variable defined by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{w^2}{2}\right) \mathrm{d}w. \tag{8}$$

Hence, according to (7), it is very easy to get the option price $V(S,\tau)$, as long as the optimal exercise boundary $B(\tau)$ is known. So, the optimal exercise boundary $B(\tau)$ is the key point of this problem. In this paper, we give an explicit series expression of the optimal exercise boundary $B(\tau)$ and then calculate the option price by (7).

It is mathematically convenient to adopt dimensionless variables. When $\sigma \neq 0$, we introduce the following dimensionless variables:

$$V' = \frac{V}{X}, \quad S' = \frac{S}{X}, \quad \tau' = \frac{\sigma^2}{2}\tau, \quad \gamma = \frac{2r}{\sigma^2}. \tag{9}$$

With all primes dropped from now on, the dimensionless system becomes

$$\mathscr{L}V \equiv -\frac{\partial V}{\partial \tau} + S^2\frac{\partial^2 V}{\partial S^2} + \gamma S\frac{\partial V}{\partial S} - \gamma V = 0, \tag{10}$$
$$V(B(\tau),\tau) = 1 - B(\tau), \tag{11}$$
$$\frac{\partial V}{\partial S}(B(\tau),\tau) = -1, \tag{12}$$
$$V(S,\tau) \to 0 \quad \text{as } S \to +\infty, \tag{13}$$
$$V(S,0) = 0. \tag{14}$$

3. The analytic approach based on the HAM

The homotopy analysis method (HAM) is based on the concept of continuous variation in algebraic topology. Let us define these (jointly continuous) maps $\Phi(S,\tau;q) \mapsto V(S,\tau)$ and $\Lambda(\tau;q) \mapsto B(\tau)$, where the homotopy parameter $q \in [0,1]$, such that, as q increases from 0 to 1, $\Phi(S,\tau;q)$ and $\Lambda(\tau;q)$ vary from the initial solutions to the exact solutions given by $V(S,\tau)$ and $B(\tau)$, respectively. To ensure this, the zero-order deformation equations are constructed:

$$\mathscr{L}\Phi(S,\tau;q) = 0, \quad (\Sigma_2) \tag{15}$$
$$\Phi'[\Lambda(\tau;q),\tau;q] = -1, \tag{16}$$
$$\Phi(+\infty,\tau;q) \to 0, \tag{17}$$
$$\Phi(S,0;q) = 0, \tag{18}$$

and

$$(1-q)\,[\Lambda(\tau;q) - B_0(\tau)] = -q\,\{\Lambda(\tau;q) + \Phi[\Lambda(\tau;q),\tau;q] - 1\}, \tag{19}$$

where the prime represents differentiation with respect to S hereafter, Σ_2 is a new domain, and $B_0(\tau)$ is an initial guess of $B(\tau)$. Clearly, when $q = 1$, the zero-order deformation Eqs. (15)–(19) give rise to

$$\Phi(S,\tau;1) = V(S,\tau), \quad \Lambda(\tau;1) = B(\tau). \tag{20}$$

When $q = 0$, we have

$$\Lambda(\tau;0) = B_0(\tau). \tag{21}$$

We can also let

$$\Phi(S,\tau;0) = V_0(S,\tau), \tag{22}$$

where $V_0(S,\tau)$ satisfies

$$\mathscr{L}V_0 = 0, \quad (\Sigma_2) \tag{23}$$
$$V_0'(B_0,\tau) = -1, \tag{24}$$
$$V_0(+\infty,\tau) \to 0, \tag{25}$$
$$V_0(S,0) = 0. \tag{26}$$

Thus, $V_0(S,\tau)$ and $V(S,\tau)$, $B_0(\tau)$ and $B(\tau)$ are *homotopic*.

Let us determine the domain Σ_2 and initial guess $B_0(\tau)$ now. Suppose we define $\Sigma_2 = \Sigma_1$. In this case, the problem is still defined in an unknown domain. Since the main purpose of the deformation equations is to fix the unknown domain, such a definition of Σ_2 is certainly unappropriate. We note that the boundary condition (24) is given on $S = B_0(\tau)$, also it is known that $B(\tau) \leqslant B(0)$ for an American put option, therefore, it is reasonable to use a diminished domain $\Sigma_2 = \{(S,\tau) \mid B_0(\tau) < S < +\infty,\ 0 \leqslant \tau \leqslant \tau_{exp}\}$ and a straightforward value $B_0(\tau) = B(0) = 1$. Thus the fixed boundary problem (23)–(26) is well posed.

Expanding $\Phi(S,\tau;q)$ and $\Lambda(\tau;q)$ in Taylor series with respect to the homotopy parameter q, we obtain

$$\Phi(S,\tau;q) = V_0(S,\tau) + \sum_{n=1}^{\infty} V_n(S,\tau)\, q^n, \tag{27}$$

$$\Lambda(\tau;q) = B_0(\tau) + \sum_{n=1}^{\infty} B_n(\tau)\, q^n, \tag{28}$$

where

$$V_n(S,\tau) = \frac{1}{n!}\frac{\partial^n}{\partial q^n}\Phi(S,\tau;0), \quad B_n(\tau) = \frac{1}{n!}\frac{\partial^n}{\partial q^n}\Lambda(\tau;0). \tag{29}$$

Also, we expand $\Phi[\Lambda(\tau;q),\tau;q]$ and $\Phi'[\Lambda(\tau;q),\tau;q]$ in Taylor series with respect to q:

$$\Phi[\Lambda(\tau,q),\tau;q] = V_0(1,\tau) + \sum_{n=1}^{\infty} \frac{q^n}{n!}\left[\frac{\partial}{\partial q} + \frac{\partial \Lambda(\tau;0)}{\partial q}\frac{\partial}{\partial S}\right]^n \Phi(1,\tau;0), \tag{30}$$

$$\Phi'[\Lambda(\tau,q),\tau;q] = V_0'(1,\tau) + \sum_{n=1}^{\infty} \frac{q^n}{n!}\left[\frac{\partial}{\partial q} + \frac{\partial \Lambda(\tau;0)}{\partial q}\frac{\partial}{\partial S}\right]^n \Phi'(1,\tau;0). \tag{31}$$

Assuming that above series are convergent at $q = 1$, we have

$$V(S,\tau) = V_0(S,\tau) + \sum_{n=1}^{\infty} V_n(S,\tau), \tag{32}$$

$$B(\tau) = B_0(\tau) + \sum_{n=1}^{\infty} B_n(\tau), \tag{33}$$

and

$$V[B(\tau),\tau] = V_0(1,\tau) + \sum_{n=1}^{\infty}[V_n(1,\tau) + f_n(\tau)], \tag{34}$$

$$V'[B(\tau),\tau] = V_0'(1,\tau) + \sum_{n=1}^{\infty}[V_n'(1,\tau) + g_n(\tau)]. \tag{35}$$

Eqs. (30), (31), (34) and (35) play the role to transfer the values on the unknown variable to those on the known one. Mathematically, we use the Taylor expansion of a composite function. Financially, Eq. (34) can be interpreted to state that the value of an American option on the optimal exercise boundary can be split into two parts: the at-the-money option value and the "modifying" value of the price, namely $\sum_{n=1}^{\infty} f_n$. Clearly, the "modifying" value of the price is negative for all time before the expiry date. Eq. (35) can be interpreted in the same way. The delta for an American option on the optimal exercise boundary can be split into two parts: the at-the-money option delta and the "modifying" value of the delta. Explicitly, the "modifying" values of the price and delta at each other, namely $f_n(\tau)$ and $g_n(\tau)$ can be expressed by

$$\begin{aligned} f_n(\tau) = \Bigg[& \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial q^{n-1}}\left(\frac{\partial \Lambda}{\partial q}\frac{\partial}{\partial S}\right) + \frac{1}{2!(n-2)!}\frac{\partial^{n-2}}{\partial q^{n-2}}\left(\frac{\partial \Lambda}{\partial q}\frac{\partial}{\partial S}\right)^2 + \cdots \\ & + \frac{1}{(n-1)!}\frac{\partial}{\partial q}\left(\frac{\partial \Lambda}{\partial q}\frac{\partial}{\partial S}\right)^{n-1} + \frac{1}{n!}\left(\frac{\partial \Lambda}{\partial q}\frac{\partial}{\partial S}\right)^n \Bigg]\Bigg|_{q=0} \Phi(1,\tau;0), \end{aligned} \tag{36}$$

$$\begin{aligned} g_n(\tau) = \Bigg[& \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial q^{n-1}}\left(\frac{\partial \Lambda}{\partial q}\frac{\partial}{\partial S}\right) + \frac{1}{2!(n-2)!}\frac{\partial^{n-2}}{\partial q^{n-2}}\left(\frac{\partial \Lambda}{\partial q}\frac{\partial}{\partial S}\right)^2 + \cdots \\ & + \frac{1}{(n-1)!}\frac{\partial}{\partial q}\left(\frac{\partial \Lambda}{\partial q}\frac{\partial}{\partial S}\right)^{n-1} + \frac{1}{n!}\left(\frac{\partial \Lambda}{\partial q}\frac{\partial}{\partial S}\right)^n \Bigg]\Bigg|_{q=0} \Phi'(1,\tau;0). \end{aligned} \tag{37}$$

By means of symbolic computational softwares, such as *Mathematica*, it is easy to express them explicitly. The first several $f_n(\tau)$ and $g_n(\tau)$ are

$$
\begin{aligned}
f_1(\tau) &= B_1\, V_0'(1,\tau),\\
f_2(\tau) &= B_2\, V_0'(1,\tau) + B_1 V_1'(1,\tau) + \frac{1}{2} B_1^2\, V_0''(1,\tau),\\
f_3(\tau) &= B_3\, V_0'(1,\tau) + B_2\, V_1'(1,\tau) + B_1\, V_2'(1,\tau)\\
&\quad + B_1\, B_2\, V_0''(1,\tau) + \frac{1}{2}\, B_1^2\, V_1''(1,\tau) + \frac{1}{6}\, B_1^3\, V_0^{(3)}(1,\tau),\\
\cdots &\qquad \cdots
\end{aligned} \tag{38}
$$

and

$$
\begin{aligned}
g_1(\tau) &= B_1\, V_0''(1,\tau),\\
g_2(\tau) &= B_2\, V_0''(1,\tau) + B_1 V_1''(1,\tau) + \frac{1}{2} B_1^2\, V_0^{(3)}(1,\tau),\\
g_3(\tau) &= B_3\, V_0''(1,\tau) + B_2\, V_1''(1,\tau) + B_1\, V_2''(1,\tau)\\
&\quad + B_1\, B_2\, V_0^{(3)}(1,\tau) + \frac{1}{2}\, B_1^2\, V_1^{(3)}(1,\tau) + \frac{1}{6}\, B_1^3\, V_0^{(4)}(1,\tau),\\
\cdots &\qquad \cdots
\end{aligned} \tag{39}
$$

Differentiating the zero-order deformation Eqs. (15)–(19) n times with respect to the homotopy parameter q, then setting $q = 0$, and finally dividing by $n!$, we obtain the following high-order deformation equations:

$$\mathscr{L} V_n = 0, \tag{40}$$

$$V_n'(1,\tau) = -g_n(\tau), \tag{41}$$

$$V_n(S,\tau) \to 0 \quad \text{as } S \to +\infty, \tag{42}$$

$$V_n(S,0) = 0, \tag{43}$$

and

$$
B_n = \begin{cases} -V_0(1,\tau), & n = 1;\\ -V_{n-1}(1,\tau) - f_{n-1}(\tau), & n > 1. \end{cases} \tag{44}
$$

Adding the definition $f_0(\tau) = 0$ and $g_0(\tau) = 1$, the Eqs. (23)–(26) is thus included in Eqs. (40)–(43). Also, the case of $n = 1$ in Eq. (44) is included in the case $n > 1$. Note that the differential system (40), (40)–(42) can be solved by Laplace Transform in Σ_2. Taking the Laplace transform with respect to τ, and with $\hat{x}$ denoting the Laplace transform of x, we can express the solutions in the Laplace space as

$$\widehat{V}_n(S,p) \equiv \int_0^\infty \exp(-p\tau) V_n(S,\tau)\, d\tau = \widehat{K}(S,p)\, \hat{g}_n(p), \tag{45}$$

and

$$\widehat{B}_n(p) = -\hat{f}_{n-1}(p) - \widehat{K}(1,p)\, \hat{g}_{n-1}(p), \tag{46}$$

where

$$\widehat{K}(S,p) = -\frac{S^{\lambda(p)}}{\lambda(p)}, \quad \lambda(p) = \frac{1 - \gamma - \sqrt{4p + (1+\gamma)^2}}{2}. \tag{47}$$

When $S = 1$ the inverse Laplace transforms, $K(1,\tau)$ and $K^{(n)}(1,\tau)$ are given by

$$
\begin{aligned}
K(1,\tau) &= (1-\gamma)\exp(-\gamma\tau)\, N\!\left(-\frac{\gamma-1}{\sqrt{2}}\sqrt{\tau}\right) + \sqrt{\frac{2}{\tau}}\, N'\!\left(-\frac{\gamma+1}{\sqrt{2}}\sqrt{\tau}\right),\\
K'(1,\tau) &= -\delta(\tau),\\
K''(1,\tau) &= \frac{\gamma+1}{2}\,\delta(\tau) - \frac{1}{\sqrt{2\tau^3}}\, N'\!\left(-\frac{\gamma+1}{\sqrt{2}}\sqrt{\tau}\right),\\
\cdots &\qquad \cdots
\end{aligned} \tag{48}
$$

where $N(x)$ is the cumulative distribution function for a standardized normal random variable, given by (8), $N'(x)$ is its probability density function, and $\delta(\tau)$ is the DiracDelta function.

Thus, the formal solution to the optimal exercise boundary at each order is

$$B_n(\tau) = -f_{n-1}(\tau) - \int_0^\tau K(1,\tau-\eta)\, g_{n-1}(\eta)\, d\eta \quad n \geqslant 1, \tag{49}$$

and the mth derivative $V^{(m)}(1,\tau)$ gives

$$V_n^{(m)}(1,\tau) = \int_0^\tau K^{(m)}(1,\tau-\eta)\, g_n(\eta)\, \mathrm{d}\eta \quad n \geqslant 1. \tag{50}$$

From (33), the exact solution can be expressed by

$$B(\tau) = B_0(\tau) + B_1(\tau) + B_2(\tau) + \cdots + B_n(\tau) + \cdots \tag{51}$$

Note that we use only one assumption here. That is, the series solutions (32) and (33) are convergent. Though we can not prove their convergence theoretically so far, we can show the convergence in §4 through examples. As long as the series solutions are convergent, they must be the exact solutions, as proved by Liao [25].

The formal solution (51) is in a recursive form of expression like Zhu [35]. Zhu [35] employed numerical integral method to approximate each order solutions. In this paper, we use an analytic approximation approach. We note that the square of the volatility σ^2 is small, therefore the dimensionless variable $\tau = (T-t)\,\sigma^2/2$ is small, too. Thus, it is possible to approximate $B(\tau)$ in powers of the time about the expiry date $\tau = 0$ with the first several terms.

Expanding $K(1,\tau)$, the derivatives $K^{(m)}(1,\tau)$ $(m = 1, 2, \cdots)$ and $g_n(\tau)$ in powers of $\sqrt{\tau}$ about $\tau = 0$ to order τ^6, and proceeding to the 19th-order solution, we derive the 19th-order $o(\tau^6)$ approximation solution in the dimensional form:

$$\begin{aligned}
\frac{B(\tau)}{X} \approx{}& 1 - 2.01986\sigma\sqrt{\tau} + (6.12896r + 1.51952\sigma^2)\tau + \left(-13.7479\frac{r^2}{\sigma} - 1.81983r\sigma - 0.39793\sigma^3\right)\tau^{\frac{3}{2}} \\
&+ \left(-12.3587r^2 + 16.8429\frac{r^3}{\sigma^2} - 0.70605r\sigma^2 - 0.08517\sigma^4\right)\tau^2 \\
&+ \left(-0.15721\frac{r^4}{\sigma^3} + 47.5953\frac{r^3}{\sigma} - 4.88089r^2\sigma + 0.60758r\sigma^3 + 0.29608\sigma^5\right)\tau^{\frac{5}{2}} \\
&+ \left(56.8554r^3 - 29.6402\frac{r^5}{\sigma^4} - 45.7453\frac{r^4}{\sigma^2} - 4.56807r^2\sigma^2 - 2.304r\sigma^4 + 0.1162\sigma^6\right)\tau^3 \\
&+ \left(31.0928\frac{r^6}{\sigma^5} - 74.9166\frac{r^5}{\sigma^3} - 128.206\frac{r^4}{\sigma} + 57.2904r^3\sigma + 6.8062r^2\sigma^3 - 5.2708r\sigma^5 - 1.4127\sigma^7\right)\tau^{\frac{7}{2}} \\
&+ \left(-219.119r^4 + 13.936\frac{r^7}{\sigma^6} + 209.41\frac{r^6}{\sigma^4} - 31.4659\frac{r^5}{\sigma^2} + 29.1287r^3\sigma^2 + 43.3385r^2\sigma^4\right. \\
&\left.+9.7450r\sigma^6 - 1.9985\sigma^8\right)\tau^4 + \left(-49.9243\frac{r^8}{\sigma^7} - 88.19\frac{r^7}{\sigma^5} + 529.484\frac{r^6}{\sigma^3} + 195.793\frac{r^5}{\sigma} - 272.421r^4\sigma\right. \\
&\left.-148.624r^3\sigma^3 + 4.9690r^2\sigma^5 + 36.7671r\sigma^7 + 3.0942\sigma^9\right)\tau^{\frac{9}{2}} \\
&+ \left(652.402r^5 + 20.4282\frac{r^9}{\sigma^8} - 249.753\frac{r^8}{\sigma^6} - 675.724\frac{r^7}{\sigma^4} + 668.602\frac{r^6}{\sigma^2}\right. \\
&\left.+ 143.516r^4\sigma^2 - 262.579r^3\sigma^4 - 220.4r^2\sigma^6 + 0.21033r\sigma^8 + 7.1734\sigma^{10}\right)\tau^5 \\
&+ \left(35.0742\frac{r^{10}}{\sigma^9} + 329.439\frac{r^9}{\sigma^7} - 265.481\frac{r^8}{\sigma^5} - 1841.11\frac{r^7}{\sigma^3} - 65.3566\frac{r^6}{\sigma} + 445.224r^5\sigma + 1147.22r^4\sigma^3\right. \\
&\left.+ 530.035r^3\sigma^5 - 254.169r^2\sigma^7 - 94.8303r\sigma^9 - 1.2567\sigma^{11}\right)\tau^{\frac{11}{2}} \\
&+ \left(-1175.8r^6 - 37.5988\frac{r^{11}}{\sigma^{10}} + 40.3366\frac{r^{10}}{\sigma^8} + 1357.33\frac{r^9}{\sigma^6} + 955.274\frac{r^8}{\sigma^4} - 2296.12\frac{r^7}{\sigma^2} - 2378.39r^5\sigma^2\right. \\
&\left.+ 136.586r^4\sigma^4 + 1676.13r^3\sigma^6 + 346.695r^2\sigma^8 - 71.8689r\sigma^{10} - 11.5864\sigma^{12}\right)\tau^6 + \cdots
\end{aligned} \tag{52}$$

where $\tau = T - t$ (year).

It is seen clearly that the optimal exercise boundary (52) is Hölder continuous with exponent $1/2$ in time, as was observed and proved by Blanchet [16]. It should also be remarked that the reason we can obtain an approximation formula for $B(\tau)$ first and then use Eq. (7) to find the option price is because obtaining the approximation formula (52) with the HAM solution procedure only needs the information of $V_n(S,\tau)$ at $S = 1$; a feature that can be clearly seen from Eqs. (33) and (44).

4. Validity of the explicit formula

To show the advantages and accuracy of the HAM approximation solution, we first compare it with some previous explicit approximation solutions. One of the simplest approximation of the optimal exercise boundary is Barles [17]:

$$\frac{B(\tau)}{X} \sim 1 - \sigma\sqrt{-\tau\ln\tau} \quad \text{for } \tau \ll \tau_{exp}, \tag{53}$$

in the dimensional form. Kuske and Keller [18] modified Barles's expression (53) and obtained

$$\frac{B(\tau)}{X} \sim 1 - \sigma\sqrt{-\tau \ln\left(\frac{8\pi r^2 \tau}{\sigma^2}\right)} \quad \text{for } \tau \ll \tau_{exp}. \tag{54}$$

Evans, Kuske and Keller [20] pointed out an incorrect numerical factor in Kuske and Keller [18]. The above expression is the corrected one. Knessl [22] presented the following asymptotic formula:

$$\ln\left[\frac{B(\tau)}{X}\right] = -\sigma\sqrt{-\tau \ln\left(\frac{8\pi r^2 \tau}{\sigma^2}\right)}\left\{1 + \frac{1}{[\log(\frac{8\pi r^2 \tau}{\sigma^2})]^2}\right\} \quad \text{for } \tau \ll \tau_{exp}. \tag{55}$$

It is found that the perturbation/asymptotic approximations contain the transcendental function $\sqrt{a\,\tau \ln(b\,\tau)}$ (where a and b are parameters), but the approximation of $B(\tau)$ in series form (52) we have obtained through the HAM does not contain this term as a result of assuming $g_n(\tau)$ being in powers of $\sqrt{\tau}$. Naturally, a correct asymptotic behavior near $\tau = 0$, as well as near $r = 0$, is not contained in our approximate solution (52). However, this does not mean that our exact solution (32) and (33) in series form is of the same asymptotic behavior near $\tau = 0$, as Eq. (52) is only a direct consequence of assuming $B(\tau)$ being in the powers of $\sqrt{\tau}$. This does not suggest that the $S_f(\tau)$ found in Zhu [35] is of the same asymptotic behavior near $\tau = 0$ as that displayed by (52) either, as the integration at each order was carried out in the former rather than assuming the $B(\tau)$ being in a particular form of τ. On the other hand, the perturbation/asymptotic methods are based on an expansion of the unknown functions in small time τ near $\tau = 0$, but they fail to provide good approximation for intermediate τ values. Our current approximation, on the other hand, can work better for much larger τ values, in comparison with the approximation produced by the perturbation/asymptotic methods, as we shall demonstrate later. This is mainly because the square-root-logarithm term clearly has worse properties globally than our square-root term, in terms of approximating the behavior of $B(\tau)$ for intermediate τ values. Of course, both of them fail to provide correct $B(\tau)$ behavior for very large τ values; the only way to ensure that $B(\tau)$ values are correctly calculated everywhere would be to calculate the analytic series solution without any approximation at all, as did in Zhu [35]. However, that process of calculation would be too time consuming and it is not the objective of this paper. For illustration purposes, we compare the HAM approximation (52) with the above perturbation solutions and the numerical benchmark solutions. In Example 1 which is the case used by Wu and Kwok [8], Carr and Faguet [37] and also Zhu [35], we compare the 19th-order $o(\tau^6)$ HAM approximation (52) with the perturbation approximation and the numerical solutions. We also show the convergence of the HAM through this example for illustration. Example 2 is an option of long term. We use this example to show the validity region of the HAM approximation (52). Also we give a method to extend the validity region. In Example 3, the option values are computed in comparison with Bunch and Johnson [38].

Example 1. This is a sample case discussed in Wu and Kwok [8], Carr and Faguet [37] and also Zhu[35]:

- ◆ strike price $X = \$100$,
- ◆ risk-free interest rate $r = 0.1$,
- ◆ volatility $\sigma = 0.3$,
- ◆ time to expiration $T = 1$ (year).

Fig. 1 shows a comparison of the 19th-order $o(\tau^6)$ HAM approximation (52) with Kuske and Keller's [18] approximation (54), Knessl's [22] approximation (55), and Zhu [35]. It is seen that the 19th-order HAM approximation agrees well with Zhu [35], indicating the accuracy of the HAM approximation. Knessl's singular perturbation solution (55) looks an order higher than Kuske and Keller's approximation (54), but it does not seem better than Kuske and Keller's. Neither of them is valid for more than a couple of weeks prior to expiry. To show the convergence of the HAM approximation in a clearer way. Table 1 shows the optimal exercise prices on the selected time among different order HAM approximations. From Fig. 1 and Table 1, we can see that the HAM approximation is indeed convergent. At the expiration time, $t = T = 1$ (year), the optimal exercise boundary in Wu and Kwok's [8] numerical solution is $B(T) = \$\,76.25$, Zhu's [35] is $B(T) = \$\,76.11$ and the 19th-order $o(\tau^6)$ approximation (52) gives $B(T) = \$\,76.11$, too.

Example 2. The second example (Chen and Chadam et. al [11]) is a long term option with the following characteristics:

- ◆ strike price $X = \$1$,
- ◆ risk-free interest rate $r = 0.08$,
- ◆ volatility $\sigma = 0.4$,
- ◆ time to expiration $T = 3$ (year).

The 19th-order $o(\tau^6)$ approximation (52) keeps the term of τ up to τ^6. But this may not be enough for the solution of options with very long term. One way to solve this problem is to keep more terms of τ. Another method is Padé approximation method [39] which is a more efficient method. Many mathematical tool softwares, such as Mathematica and

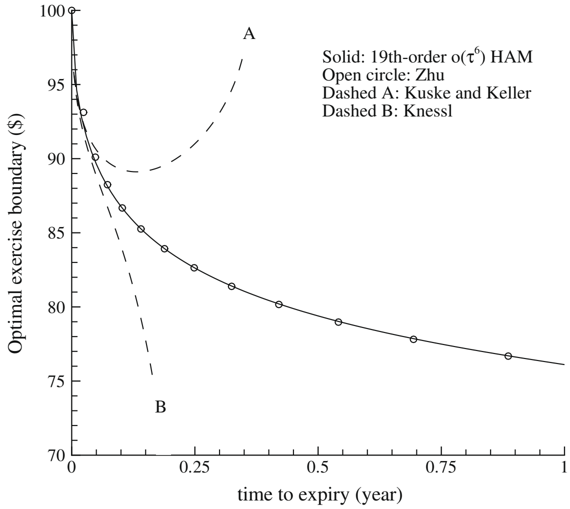

Fig. 1. Optimal exercise boundary for the case in Example 1: $X = \$100$, $r = 0.1$, $\sigma = 0.3$ and $T = 1$ (year). Solid line: the 19th-order $o(\tau^6)$ HAM approximation (52); Circles: Zhu [35]; Dashed line: Kuske and Keller's [18] approximation (54) and Knessl's [22] approximation (55).

Maple, have the standard package of Padé approximation. Regarding $\sqrt{\tau}$ as one variable, we obtain the [6,6] Padé approximation:

$$B(\tau) \approx \frac{1 - 0.48376\sqrt{\tau} + 1.11787\tau - 0.37300\tau^{3/2} + 0.35770\tau^2 - 0.08141\tau^{5/2} + 0.03473\tau^3}{1 + 0.32418\sqrt{\tau} + 0.64635\tau + 0.21512\tau^{3/2} + 0.19230\tau^2 + 0.0698\tau^{5/2} + 0.01659\tau^3}. \quad (56)$$

A comparison is made among the 19th-order $o(\tau^6)$ approximation (52), its Padé approximation (56), Kuske and Keller's approximation (54), Knessl's [22] approximation (55) and Chen et al.'s [11] front tracking and extrapolation solution, as shown in Fig. 2. In the same plot, we also graph the perpetual optimal exercise price, $2Xr/(\sigma^2 + 2r) = \$\,0.5$, which the optimal exercise price is suppose to approach asymptotically when the time to expiry becomes infinite. In other words, this is the exercise price that should never be reached for an American option of finite lifetime. It is seen that 19th-order $o(\tau^6)$ approximation (52) is valid for $T - t$ up to 1 year, while Kuske and Keller's approximation (54) and Knessl's [22] approximation (55) are valid for less than 1 month prior to expiry. Also, it is evident that Padé approximation (56) can indeed enlarge the valid region of τ.

Example 3. As mentioned before, it is easy to get the option price by means of (7) as long as the optimal exercise boundary $B(\tau)$ is known. Here, we obtain the option prices by substitution of the present 19th-order $o(\tau^6)$ approximation (52) into the American put option formula (7). To have a more complete assessment concerning its accuracy, a comparison is made with several numerical solutions. Table 2 (which is from Bunch and Johnson [38]) contains computed prices for a set of 27 American puts. The market parameters are: risk-free interest rate $r = 0.0488$, stock price $S = \$40$, volatility $0.2 \leqslant \sigma \leqslant 0.4$ and time to expiry $1 \leqslant T \leqslant 7$ months. The benchmark for accurate put values is the binomial method with 10,000 steps (reported in Huang et al. [40]) and the second row of each panel of our Table. Rows 1 and 3 contain results form the binomial method with 150 steps and Huang et al. [40] recursive four-point extrapolation values, respectively. Row 4 contains values using our HAM approximation (52). Table 3 reports accuracy for the methods from the Table 2 and for the Geske and Johnson [41]. The root-mean-square error (RMSE) is defined by

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(V_i - \widetilde{V}_i)^2}$$

Table 1
Comparison of the values of different orders of the approximation of the optimal exercise boundary at time $\tau \equiv T - t$ (month) for $X = \$100$, $r = 0.1$, $\sigma = 0.3$ in Example 1.

τ (month)	Optimal exercise boundary $B(\tau)$ ($)				
	11th-order	13th-order	15th-order	17th-order	19th-order
0	100	100	100	100	100
3	82.65	82.61	82.60	82.61	82.62
6	79.24	79.29	79.34	79.37	79.39
9	77.37	77.44	77.47	77.49	77.49
12	76.12	76.16	76.17	76.15	76.11

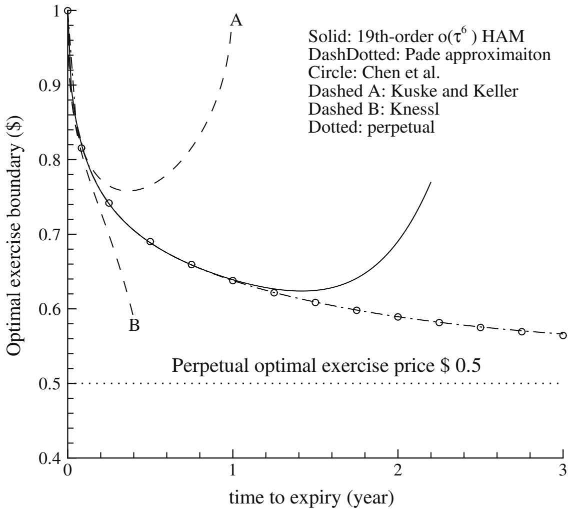

Fig. 2. The optimal exercise boundary for Example 2: $X = \$1$, $r = 0.08$, $\sigma = 0.4$ and $T = 3$ (year). Solid line: the 19th-order $o(\tau^6)$ HAM approximation (52); Dash-Dotted line: the Padé approximation (56); Circles: Chen et al.'s [11] front tracking and extrapolation solution; Dashed line: Kuske and Keller's [18] approximation (54) and Knessl's [22] approximation (55); Dotted line: perpetual optimal exercise price \$0.5.

Table 2
American put values using the 19th-order $o(\tau^6)$ HAM approximation and three numerical methods: the binomial methods with 150 and 10,000 time steps, and Huang et al.'s [40] recursive four-point extrapolation. The risk-free interest rate is 0.0488, and the stock price is \$40. The strike price, volatility and time to maturity in months are denoted by X, σ and T, respectively.

	$\sigma = 0.2$			$\sigma = 0.3$			$\sigma = 0.4$		
	$T = 1$	$T = 4$	$T = 7$	$T = 1$	$T = 4$	$T = 7$	$T = 1$	$T = 4$	$T = 7$
Panel A: $X = 35$									
Binomial, $N = 150$	0.0061	0.1995	0.4340	0.0775	0.6993	1.2239	0.2454	1.3505	2.1602
Binomial, $N = 10,000$	0.0062	0.2004	0.4328	0.0774	0.6975	1.2198	0.2466	1.3460	2.1549
Huang et al.	0.0062	0.2004	0.4337	0.0775	0.6973	1.2233	0.2467	1.3468	2.1603
HAM	0.0062	0.2008	0.4335	0.0776	0.6989	1.2219	0.2470	1.3487	2.1585
Panel B: $X = 40$									
Binomial, $N = 150$	0.8512	1.5783	1.9886	1.3083	2.4799	3.1665	1.7658	3.3836	4.3480
Binomial, $N = 10,000$	0.8522	1.5798	1.9904	1.3099	2.4825	3.1696	1.7681	3.3874	4.3526
Huang et al.	0.8543	1.5873	1.9987	1.3116	2.4919	3.1842	1.7694	3.3970	4.3699
HAM	0.8542	1.5822	1.9922	1.3127	2.4872	3.1739	1.7715	3.3938	4.3592
Panel C: $X = 45$									
Binomial, $N = 150$	5.0000	5.0886	5.2677	5.0600	5.7065	6.2448	5.2875	6.5103	7.3897
Binomial, $N = 10,000$	5.0000	5.0883	5.2670	5.0597	5.7056	6.2436	5.2868	6.5099	7.3830
Huang et al.	5.0020	5.0594	5.2631	5.0604	5.6970	6.2303	5.2853	6.5128	7.3865
HAM	5.0038	5.0912	5.2677	5.0707	5.7141	6.2493	5.2975	6.5209	7.3921

Table 3
Root-mean-square error (RMSE) for a set of 27 American put option prices. The benchmark value is the binomial method with 10,000 time steps. Values for the finite-difference, the Geske and Johnson [41] are those reported in Tables 1 and 2 of Huang et al. [40] and also Table 3 of Bunch and Johnson [38].

Method	RMSE (cent)
Binomial-150	0.264
Huang et al.	0.860
Finite-difference	4.104
Geske and Johnson	0.534
HAM	0.528

where V_i and $\widetilde{V}_i$ are the option values by means of the corresponding method and the benchmark binomial method with 10,000 steps, respectively. Note that the effective digit number of the option prices itself is E-02 cent. The RMSE of the 19th-order $o(\tau^6)$ HAM approximation is 0.534 cent. From Table 3 it is seen that the HAM approximation is as accurate as

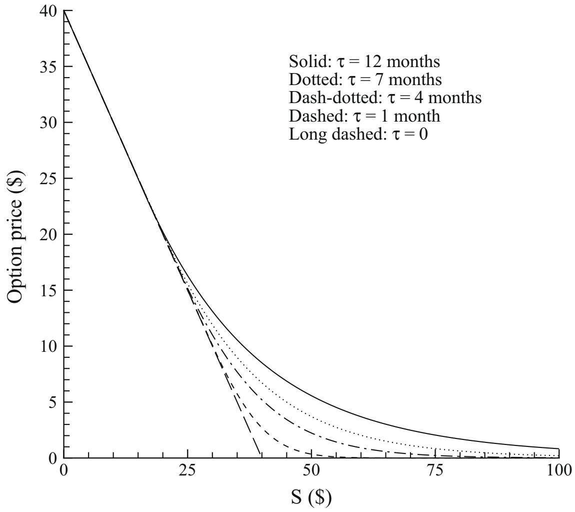

Fig. 3. Option prices at different times to expiry by substitution of the 19th-order $o(\tau^6)$ HAM approximation (52) into the price formula (7). The parameters are $X = \$40$, $r = 0.0488$, $\sigma = 0.3$ and $T = 12$ months.

the numerical results by Geske and Johnson [41], and better than Huang et al.'s [40] recursive four-point extrapolation solution. Fig. 3 illustrate the value of an American put option at different times to expiry under the parameters: $X = \$40$, $r = 0.0488$, $\sigma = 0.3$ and $T = 12$ months.

5. Conclusions

This paper presents an analytic algorithm (51) and its power series approximation (52) to the optimal exercise boundary of an American put option. The homotopy analysis method (HAM) is used. We do not use Laudau transform: $x = \ln[B(\tau)/S]$ (or equivalent transform) in the solution procedure. So, for some moving boundary problems where Laudau transform (or equivalent transforms) is not valid, this approach can still be used. Therefore, this approach is general and can be applied to solve many other moving boundary problems in finance and engineering.

We compare the 19th-order $o(\tau^6)$ HAM approximation (52) with perturbation/asymptotic solutions and numerical solutions. We show that the perturbation/asymptotic solutions are valid for very short time prior to expiry, usually a couple of weeks. In comparison, the HAM approximation can largely extend the valid region of the solutions. Thanks to its form of series expression, the present approximation (52) can also be combined with Padé approximation method. Results show that Padé approximation method can indeed extend the valid region. Comparison with numerical methods show that the HAM approximations as accurate as many numerical methods. The RMSE for a set of 27 American put options is less than 1 cent.

Extending our approach could be a fruitful area for future research. First, the problem of an American put option with continuous dividend could be directly solved. Second, other types of American style derivatives and American options described by more complicated models could be investigated by the present approach in a similar way.

Acknowledgement

The first author thanks to China Scholarship Council and the University of Wollongong for the finance support.

References

[1] Cox J, Ross S, Rubinstein M. Option pricing: a simplified approach. J Financ Econ 1979;7:229–63.
[2] Broadie M, Detemple J. American option valuation: new bounds, approximations, and a comparison of existing methods. Rev Financ Stud 1996;9(4):1211–50.
[3] Grant D, Vora G, Weeks D. Simulation and the early-exercise option problem. J Financ Eng 1996;5:211–27.
[4] Longstaff F, Schwartz ES. A radial basis function method for solving options pricing model. Rev Financ Stud 2001;14:113–47.
[5] Jaillet P, Lamberton D, Lapeyre B. Variational inequalities and the pricing of American options. Acta Applicandae Math 1990;21:263–89.
[6] Dempster M. Fast numerical valuation of American, exotic and complex options. Colchester, England: Department of Mathematics research report, University of Essex; 1994.
[7] Brennan M, Schwartz E. The valuation of American put options. J Financ 1977;32:449–62.
[8] Wu L, Kwok YK. A front-fixing finite difference method for the valuation of American options. J Financ Eng 1997;6:83–97.
[9] Allegretto W, Lin Y, Yang H. Simulation and the early-exercise option problem. Discr Contin Dyn Syst Ser B, Appl Algor 2001;8:127–36.
[10] Hon YC, Mao XZ. A radial basis function method for solving options pricing model. J Financ Eng 1997;8:31–49.
[11] Chen XF, Chadam J, Stamicar R. The optimal exercise boundary for American put options: analytic and numerical approximations. Working paper. University of Pittsburgh; 2000. Available from: http://www.math.pitt.edu/-xfc/Option/CCSFinal.ps.

[12] Broadie M, Detemple J. Recent advances in numerical methods for pricing derivative securities. In: Rogers LCG, Talay D, editors. Numerical methods in finance. England: Cambridge University Press; 1997.
[13] Kim IJ. The analytic valuation of American options. Rev Financ Stud 1990;3:547–72.
[14] Carr P, Jarrow R, Myneni R. Alternative characterizations of American put options. Math Financ 1992;2:87–106.
[15] Van Moerbeke P. An optimal stopping problem with linear reward. Acta Math 1974;132:111–51.
[16] Blanchet A. On the regularity of the free boundary in the parabolic obstacle problem. Application to American options. Nonlinear Anal 2006;65:1362–78.
[17] Barles G, Burdeau J, Romano M, Samsoen N. Critical stock price near expiration. Math Financ 1995;5(2):77–95.
[18] Kuske RA, Keller JB. Optional exercise boundary for an American put option. Appl Math Financ 1998;5:107–16.
[19] Allobaidi G, Mallier R. On the optimal exercise boundary for an American put option. J Appl Math 2001;1(1):39–45.
[20] Evans JD, Kuske R, Keller JB. American options on asserts with dividends near expiry. Math Financ 2002;12(3):219–37.
[21] Zhang JE, Li TC. Pricing and hedging American options analytically: a perturbation method. Working paper. The University of Hong Kong; 2006.
[22] Knessl C. A note on a moving boundary problem arising in the American put option. Stud Appl Math 2001;107:157–83.
[23] Chen XF, Chadam J. A mathematical analysis for the optimal exericise boundary American put option. Working paper. University of Pittsburgh; 2005. Available from: http://www.pitt.edu/-chadam/papers/2CC9-30-05.pdf.
[24] Liao SJ. The proposed homotopy analysis technique for the solution of nonlinear problems. PhD thesis, Shanghai Jiao Tong University; 1992.
[25] Liao SJ. Beyond perturbation: introduction to the homotopy analysis method. Boca Raton: Chapman & Hall/CRC Press; 2003.
[26] Liao SJ, Tan Y. A general approach to obtain series solutions of nonlinear differential equations. Stud Appl Math 2007;119:297–355.
[27] Liao SJ. Notes on the homotopy analysis method Some definitions and theorems. Commun Nonlinear Sci Numer Simul 2009;14(4):983–97.
[28] Liao SJ, Cheung KF. Homotopy analysis of nonlinear progressive waves in deep water. J Eng Math 2003;45(2):105–16.
[29] Liao SJ. A new branch of solutions of boundary-layer flows over a permeable stretching plate. Int J Non-Linear Mech 2007;42:819–30.
[30] Hayat T, Abbas Z, Sajid M, Asghar S. The influence of thermal radiation on MHD flow of a second grade fluid. Int J Heat Mass Transfer 2007;50:931–41.
[31] Abbasbandy S. Solitary wave equations to the Kuramoto–Sivashinsky equation by means of the homotopy analysis method. Nonlinear Dyn 2007;52:35–40.
[32] Mustafa Inc. On exact solution of Laplace equation with Dirichlet and Neumann boundary conditions by the homotopy analysis method. Phys Lett A 2007;365:412–15.
[33] Cheng J, Liao SJ, Mohapatra RN, Vajravelu K. Series solutions of nano boundary layer flows by means of the homotopy analysis method. J Math Anal Appl 2008;343(1):233–45.
[34] Cheng J, Cang J, Liao SJ. On the interaction of deep water waves and exponential shear currents. ZAMP. Online.
[35] Zhu SP. An exact and explicit solution for the valuation of American put options. Quant Financ 2006;6:229–42.
[36] Landau HG. Heat conduction in melting solid. Quart Appl Math 1950;8:81–94.
[37] Carr P, Faguet D. Fast accurate valuation of American options. Working paper. Cornell University; 1994.
[38] Bunch DS, Johnson H. The American put option and its critical stock price. J Financ 2000;5:2333–56.
[39] Van Dyke M. Extension of Goldstein's series for the Oseen drag of a sphere. J Fluid Mech 1970;44:365–72.
[40] Huang JZ, Marti GS, Yu GG. Pricing and Hedging American options: a recursive integration method. Rev Financ Stud 1996;9:277–300.
[41] Geske R, Johnson HE. The American put option valued analytically. J Financ 1984;5:1511–23.