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A Series Solution of the Unsteady Von Kármán Swirling Viscous Flows

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Abstract A new analytic technique is applied to solve the unsteady viscous flow due to an infinite rotating disk, governed by a set of two fully coupled nonlinear partial differential equations deduced directly from the exact Navier-Stokes equations. The system of coupled nonlinear partial differential equations is replaced by a sequence of uncoupled systems of linear ordinary differential equations. Different from all other previous analytic results, our series solution is accurate and valid for all time in the whole spatial region. Accurate expressions for skin friction coefficients are given, which are valid for all time. Such kind of series solutions have not been reported, to the best of our knowledge.

Key words Von Kármán swirling viscous flow · unsteady · series solution

Mathematics Subject Classifications (2000) 35Q30 · 76U05

1 Introduction

The steady viscous flows due to an infinite rotating disk were first investigated by Von Kármán [1]. Using a kind of similarity transformations, he reduced the full of Navier-Stokes equations to a pair of nonlinear ordinary differential equations. Then the problem is generalized to include the case where the fluid itself is rotating as a solid body far from the disk with suction or injection at the disk surface. This introduces a parameter, i.e. the ratio of the angular velocity of the fluid at infinity to the angular velocity of the disk. Another generalization is to consider the viscous

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flow between two infinite coaxial rotating disks with suction or injection at both disks, and this introduces another parameter, i.e. the Reynolds number determined by the distance of the two disks. All these problems are studied, theoretically, numerically and experimentally, by many researchers such as Cochran [2], Fettis [3], Rogers and Lance [4], Mellor, Chapple and Stokes [5], Tam [6], Schlichting [7], Bodonyi [8], Zandbergen and Dijkstra [9], Dijkstra [10], Holodniok [11], Dijkstra and Van Heijst [12], Szeri, Schneider, Labbe and Kaufman [13], Bodonyi and Ng [14] and so on. For details, please refer to the review paper of Zandbergen and Dijkstra [15].

In this paper we focus on the unsteady problem of Von Kármán's swirling viscous flows. Unsteady, laminar, axially symmetric viscous flow of incompressible fluid introduced by an infinite disk (z=0) which is started impulsively (at t=0) into steady rotation with constant angular velocity Ω ($\Omega \neq 0$) satisfies the continuity and Navier-Stokes equations

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{1a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{1b}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right), \tag{1c}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right), \tag{1d}$$

subject to the initial and boundary conditions

$$u = v = w = 0, \quad at \ \ t = 0, \tag{1e}$$

$$u = w = 0, \quad v = r\,\Omega, \quad at \ \ z = 0, \tag{1f}$$

$$u = v = 0, \quad as \ \ z \to \infty, \tag{1g}$$

where u, v, w are velocity components in the directions of increasing r, φ and z , t denotes the time, p the pressure, ν the coefficient of kinematic viscosity and ρ the density of the fluid, respectively.

Following Thiriot [16] and Nazar et al. [17], we use the similarity variables

$$\eta = z\sqrt{\frac{\Omega}{\nu\,\xi}}, \quad \xi = 1 - \exp(-\tau), \quad \tau = \Omega\, t. \tag{2}$$

By means of the similarity transformations

$$u = r\,\Omega\ f(\eta,\xi), \qquad v = r\,\Omega\ g(\eta,\xi),$$
$$w = (\Omega\ \nu\ \xi)^{1/2}\ s(\eta,\xi), \qquad p = -\rho\ \nu\ \Omega\ P(\eta,\xi), \tag{3}$$

Eqs. (1a)-(1d) are reduced to

$$\frac{\partial s}{\partial \eta} + 2f = 0. \tag{4a}$$

$$(1-\xi)\left(\xi\frac{\partial f}{\partial \xi} - \frac{\eta}{2}\frac{\partial f}{\partial \eta}\right) - \frac{\partial^2 f}{\partial \eta^2} + \xi\left(s\frac{\partial f}{\partial \eta} + f^2 - g^2\right) = 0, \tag{4b}$$

$$(1-\xi)\left(\xi\frac{\partial g}{\partial \xi} - \frac{\eta}{2}\frac{\partial g}{\partial \eta}\right) - \frac{\partial^2 g}{\partial \eta^2} + \xi s\frac{\partial g}{\partial \eta} + 2\xi fg = 0, \tag{4c}$$

$$(1-\xi)\left(\frac{1}{2}s + \xi\frac{\partial s}{\partial \xi} - \frac{\eta}{2}\frac{\partial s}{\partial \eta}\right) + \xi s\frac{\partial s}{\partial \eta} - \frac{\partial P}{\partial \eta} - \frac{\partial^2 s}{\partial \eta^2} = 0, \tag{4d}$$

subject to the initial and boundary conditions

$$f(0,\xi) = 0, \quad g(0,\xi) = 1, \quad s(0,\xi) = 0, \tag{4e}$$

$$f(\infty,\xi) = 0, \;\; g(\infty,\xi) = 0. \tag{4f}$$

From (4a), it holds

$$f = -\frac{1}{2}\frac{\partial s}{\partial \eta}. \tag{5}$$

Substituting (5) into (4b) and (4c), we obtain

$$\frac{\partial^3 s}{\partial \eta^3} + (1-\xi)\left(\frac{\eta}{2}\frac{\partial^2 s}{\partial \eta^2} - \xi\frac{\partial^2 s}{\partial \xi \partial \eta}\right) + \xi\left[\frac{1}{2}\left(\frac{\partial s}{\partial \eta}\right)^2 - s\frac{\partial^2 s}{\partial \eta^2} - 2g^2\right] = 0, \tag{6a}$$

$$\frac{\partial^2 g}{\partial \eta^2} - (1-\xi)\left(\xi\frac{\partial g}{\partial \xi} - \frac{\eta}{2}\frac{\partial g}{\partial \eta}\right) + \xi\left(g\frac{\partial s}{\partial \eta} - s\frac{\partial g}{\partial \eta}\right) = 0, \tag{6b}$$

subject to the initial and boundary conditions

$$g(0,\xi) = 1, \quad s(0,\xi) = 0, \quad \left.\frac{\partial s(\eta,\xi)}{\partial \eta}\right|_{\eta=0} = 0, \tag{6c}$$

$$g(\infty,\xi) = 0, \qquad \left.\frac{\partial s(\eta,\xi)}{\partial \eta}\right|_{\eta=+\infty} = 0. \tag{6d}$$

Note that Eqs. (6a) and (6b) are *coupled* nonlinear partial differential equations, and thus rather hard to solve by means of analytic methods.

The skin friction coefficients in radial and tangential directions for $\xi \in [0, 1]$ are given by

$$C_f^r(\xi) = \frac{\tau_r}{\rho(r\Omega)^2} = -\frac{f'(0,\xi)}{\sqrt{\xi\,Re_r}} = \frac{s''(0,\xi)}{2\sqrt{\xi\,Re_r}}, \tag{7}$$

$$C_f^\varphi(\xi) = \frac{\tau_\varphi}{\rho(r\Omega)^2} = -\frac{g'(0,\xi)}{\sqrt{\xi\,Re_r}}, \tag{8}$$

where $Re_r = r\sqrt{\Omega/\nu}$ is the local Reynolds number, τ_r and τ_φ are the radial and tangential shear stress, respectively.

When $\xi = 0$, corresponding to $\tau = 0$, we have

$$\frac{\partial^3 s}{\partial \eta^3} + \frac{\eta}{2}\,\frac{\partial^2 s}{\partial \eta^2} = 0, \tag{9a}$$

$$\frac{\partial^2 g}{\partial \eta^2} + \frac{\eta}{2}\,\frac{\partial g}{\partial \eta} = 0, \tag{9b}$$

subject to the boundary conditions:

$$g(0,0) = 1, \quad s(0,0) = 0, \quad \left.\frac{\partial s(\eta,\xi)}{\partial \eta}\right|_{\eta=0,\ \xi=0} = 0, \tag{9c}$$

$$g(\infty,0) = 0, \qquad \left.\frac{\partial s(\eta,\xi)}{\partial \eta}\right|_{\eta=+\infty,\ \xi=0} = 0. \tag{9d}$$

The above equations have the exact solution

$$s(\eta,0) = 0, \quad g(\eta,0) = \text{erfc}\,\left(\frac{\eta}{2}\right), \tag{10}$$

where erfc (η) is the error function defined by

$$\text{erfc}\,(\eta) = \frac{2}{\sqrt{\pi}} \int_{\eta}^{+\infty} \exp(-z^2)dz. \tag{11}$$

This initial solution was found by many researchers. When $\xi = 1$, corresponding to $\tau \to +\infty$, we have from (6a) and (6b) that

$$\frac{\partial^3 s}{\partial \eta^3} + \frac{1}{2}\left(\frac{\partial s}{\partial \eta}\right)^2 - s\frac{\partial^2 s}{\partial \eta^2} - 2g^2 = 0, \tag{12a}$$

$$\frac{\partial^2 g}{\partial \eta^2} + g\frac{\partial s}{\partial \eta} - s\frac{\partial g}{\partial \eta} = 0, \tag{12b}$$

subject to the boundary conditions:

$$g(0,1) = 1, \quad s(0,1) = 0, \quad \left.\frac{\partial s(\eta,\xi)}{\partial \eta}\right|_{\eta=0,\ \xi=1} = 0, \tag{12c}$$

$$g(\infty,1) = 0, \qquad \left.\frac{\partial s(\eta,\xi)}{\partial \eta}\right|_{\eta=+\infty,\ \xi=1} = 0. \tag{12d}$$

These steady-state equations were solved by the above-mentioned researchers. In 2003, Liao [18] gave an analytic steady-state solution of these equations, expressed by a series of exponential functions. It should be emphasized that, at infinity, the properties of the steady-state solution are quite different from the initial solution (10).

Note that Eqs. (6a)–(6d) are deduced directly from the exact Navier-Stokes equations. This might be an important reason why this problem attracted so many researchers. However, because of the complexity mentioned above, they hardly obtained the unsteady solutions valid for *all* time $0 < \tau < \infty$. Thiriot [16] made a start on the impulsively started initial-value rotating disk problem, however, his formulations contain numerical errors. Benton [19] gave a highly accurate solutions by a simple analytical-numerical method and solved the impulsively started initial-value problem. Attia [20] studied the unsteady rotating disk problem combined with heat transfer and gave numerical results. To the best of authors' knowledge, no one reported any a kind of analytic solutions valid for *all* time $0 \leq \tau < \infty$ in the whole region $0 \leq \eta < \infty$ for the considered problem. So, there does not exist such a kind of analytic solutions which smoothly connect the initial solution (10) and the steady-state solution governed by (12a) and (12d).

The homotopy [21] is a basic concept in topology [22]. Based on the homotopy, some numerical techniques such as the continuation method [23] and the homotopy continuation method [24] were developed. There is a suite of FORTRAN subroutines in Netlib for solving nonlinear systems of equations by homotopy methods, called HOMPACK. In 1992, using the concept of homotopy, Liao [25] developed a new analytic method for highly nonlinear problems, namely the homotopy analysis method [26–30], [18]. Different from perturbation techniques [31], the homotopy analysis method does not depend upon any small or large parameters and thus is valid for most of nonlinear problems in science and engineering. Besides, it logically contains other non-perturbation techniques such as Lyapunov's small parameter method [32], the δ-expansion method [33], and Adomian's decomposition method [34], as proved by Liao [18]. The so-called "homotopy perturbation method" [35] proposed in 1999 is only a special case of the homotopy analysis method, as pointed out by Liao [36]. Thus, the homotopy analysis method is more general and is a unification of previous non-perturbation techniques. Besides, different from all previous analytic methods, the homotopy analysis method always gives us a family of series solutions whose convergence region can be adjusted and controlled by an auxiliary parameter. For details, please refer to Liao [18]. The homotopy analysis method has been successfully applied to many nonlinear problems [37–41]. Especially, some new solutions of a few nonlinear problems are found even by means of the homotopy analysis method [42, 43]. All of these verify its power and potential for strongly nonlinear problems. In this paper, we further employ the homotopy analysis method to solve the system of the fully coupled, unsteady nonlinear equations (6a) to (6d) to obtain a series solution valid for *all* time $0 \leq \tau < +\infty$ in the whole spatial region.

2 Homotopy Analysis Solution

2.1 Zero-Order Deformation Equation

Note that ξ and η explicitly appear in Eqs. (6a) and (6b). Thus, $g(\eta, \xi)$ and $s(\eta, \xi)$ must contain the power terms of η and ξ. Besides, as pointed out by many researchers, the velocity profiles decay exponentially in most cases. Therefore, considering the boundary conditions (6c) and (6d), we express $s(\eta,\xi)$ and $g(\eta,\xi)$ by such a set of base functions

$$\{\xi^k\ \eta^m\ \exp(-n\eta)|\ k \geq 0,\ n \geq 0,\ m \geq 0\} \tag{13}$$

in the form

$$s(\eta,\xi) = a^0_{0,0} + \sum_{k=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} a^k_{m,n}\xi^k\eta^m\exp(-n\eta), \tag{14}$$

$$g(\eta,\xi) = \sum_{k=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} b^k_{m,n}\xi^k\eta^m\exp(-n\eta), \tag{15}$$

where $a^k_{m,n}$ and $b^k_{m,n}$ are coefficients. Note that all of our approximations must obey the above expressions. This point is so important in the frame of the homotopy analysis method that it should be regarded as a rule. So, the above expressions provide us with the so-called *Rule of Solution Expressions* (see Liao [18]) for $s(\eta,\xi)$ and $g(\eta,\xi)$, respectively. According to the solution expressions denoted by (14) and (15), and considering the initial/boundary conditions (6c) to (6d), it is straightforward to choose

$$s_0(\eta,\xi) = 0, \quad g_0(\eta,\xi) = \exp(-\eta), \tag{16}$$

as the initial approximations of $s(\eta,\xi)$ and $g(\eta,\xi)$. Besides, to obey the solution expressions (14) and (15), and from the governing equations (6a) and (6b), it is straightforward to choose

$$\mathcal{L}_s[S(\xi,\eta;q)] = \frac{\partial^3 S}{\partial\eta^3} - \frac{\partial S}{\partial\eta}, \tag{17}$$

$$\mathcal{L}_g[G(\xi,\eta;q)] = \frac{\partial^2 G}{\partial\eta^2} - G, \tag{18}$$

as the auxiliary linear operators, which have the following properties

$$\mathcal{L}_s[C_1\exp(-\eta) + C_2\exp(\eta) + C_3] = 0, \tag{19}$$

$$\mathcal{L}_g[C_4\exp(-\eta) + C_5\exp(\eta)] = 0, \tag{20}$$

respectively, where C_1, C_2, C_3, C_4 and C_5 are integral constants. Furthermore, based on Eqs.(6a) and (6b), we define two nonlinear operators

$$\mathcal{N}_s[S(\eta,\xi;q)] = \frac{\partial^3 S}{\partial\eta^3} + (1-\xi)\left(\eta\frac{\partial^2 S}{\partial\eta^2} - \xi\frac{\partial^2 S}{\partial\xi\partial\eta}\right) + \xi\left[\frac{1}{2}\left(\frac{\partial S}{\partial\eta}\right)^2 - S\frac{\partial^2 S}{\partial\eta^2} - 2G^2\right], \tag{21}$$

$$\mathcal{N}_g[G(\eta,\xi;q)] = \frac{\partial^2 G}{\partial\eta^2} - (1-\xi)\left(\xi\frac{\partial G}{\partial\xi} - \frac{\eta}{2}\frac{\partial G}{\partial\eta}\right) + \xi\left(G\frac{\partial S}{\partial\eta} - S\frac{\partial G}{\partial\eta}\right), \tag{22}$$

where $S(\eta, \xi; q)$ and $G(\eta, \xi; q)$ are the mappings of $s(\eta, \xi)$ and $g(\eta, \xi)$, respectively. Let $\hbar$ denote an auxiliary parameter. We construct the so-called zero-order deformation equations

$$(1 - q)\mathcal{L}_s[S(\eta, \xi; q) - s_0(\eta, \xi)] = q\, \hbar\, \mathcal{N}_s[S(\eta, \xi; q), G(\eta, \xi; q)], \tag{23a}$$

$$(1 - q)\mathcal{L}_g[G(\eta, \xi; q) - g_0(\eta, \xi)] = q\, \hbar\, \mathcal{N}_g[S(\eta, \xi; q), G(\eta, \xi; q)], \tag{23b}$$

subject to the boundary conditions

$$G(0, \xi; q) = 1, \quad S(0, \xi; q) = 0, \quad \left.\frac{\partial S(\eta, \xi; q)}{\partial \eta}\right|_{\eta=0} = 0, \tag{23c}$$

$$G(\infty, \xi; q) = 0, \qquad \left.\frac{\partial S(\eta, \xi)}{\partial \eta}\right|_{\eta=+\infty} = 0, \tag{23d}$$

where $q \in [0, 1]$ is an embedding parameter. Obviously, when $q = 0$ and $q = 1$, the above zero-order deformation equations (23a) and (23b) have the solutions

$$S(\eta, \xi; 0) = s_0(\eta, \xi), \quad G(\eta, \xi; 0) = g_0(\eta, \xi), \tag{24}$$

and

$$S(\eta, \xi; 1) = s(\eta, \xi), \quad G(\eta, \xi; 1) = g(\eta, \xi), \tag{25}$$

respectively. Thus, as the embedding parameter q increases from 0 to 1, the mapping $S(\eta, \xi; q)$ varies (or deforms) from the initial guess $s_0(\eta, \xi)$ to the solution $s(\eta, \xi)$, so does $G(\eta, \xi; q)$ from $g_0(\eta, \xi)$ to $g(\eta, \xi)$. And this kind of variation (or deformation) is governed by the so-called zero-order deformation equations (23a) to (23d).

Note that the mappings $S(\eta, \xi; q)$ and $G(\eta, \xi; q)$ contain the embedding parameter q, which has no physical meanings. Expending $S(\eta, \xi; q)$ and $G(\eta, \xi; q)$ in Taylor's series with respect to q, we have

$$S(\eta, \xi; q) = S(\eta, \xi, 0) + \sum_{m=1}^{+\infty} s_m(\eta, \xi) q^m, \tag{26}$$

$$G(\eta, \xi; q) = G(\eta, \xi, 0) + \sum_{m=1}^{+\infty} g_m(\eta, \xi) q^m, \tag{27}$$

where

$$s_m(\eta, \xi) = \frac{1}{m!} \left.\frac{\partial^m S(\eta, \xi; q)}{\partial q^m}\right|_{q=0}, \tag{28}$$

$$g_m(\eta, \xi) = \frac{1}{m!} \left.\frac{\partial^m G(\eta, \xi; q)}{\partial q^m}\right|_{q=0}, \tag{29}$$

respectively. Certainly, it is very important to ensure that the above series (26) and (27) converge at $q = 1$. Fortunately, the zero-order deformation equations (23a) and (23b) contain the auxiliary parameter $\hbar$. Assuming that the value of $\hbar$ is properly chosen so that the series (26) and (27) are convergent at $q = 1$, we have, using (24)–(25), the solution series

$$s(\eta, \xi) = s_0(\eta, \xi) + \sum_{m=1}^{+\infty} s_m(\eta, \xi), \tag{30}$$

$$g(\eta, \xi) = g_0(\eta, \xi) + \sum_{m=1}^{+\infty} g_m(\eta, \xi). \tag{31}$$

As pointed by Liao [18], it is the the auxiliary parameter $\hbar$ which provides us with a convenient way to control and adjust of the convergence of solution series. And this is the reason why the auxiliary parameter $\hbar$ is introduced in the homotopy analysis method.

2.2 High-Order Deformation Equation

Note that the terms $s_m(\eta, \xi)$ and $g_m(\eta, \xi)$ in the above series are unknown for $m \geq 1$. Their governing equations and related boundary conditions can be derived from the zero-order deformation equations (23a) to (23d), as described below.

For the sake of simplicity, write

$$\vec{s}_m = \{s_0, s_1, s_2, ..., s_m\}, \tag{32}$$

$$\vec{g}_m = \{g_0, g_1, g_2, ..., g_m\}. \tag{33}$$

Differentiating the zero-order deformation equations (23a) and (23d) m times with respect to q, then setting $q = 0$, and finally dividing them by $m!$, we obtain the mth-order deformation equations

$$\mathcal{L}_s[s_m(\eta, \xi) - \chi_m s_{m-1}(\eta, \xi)] = \hbar\ R_m^s(\vec{s}_{m-1}, \vec{g}_{m-1}), \tag{34a}$$

$$\mathcal{L}_g[g_m(\eta, \xi) - \chi_m g_{m-1}(\eta, \xi)] = \hbar\ R_m^g(\vec{s}_{m-1}, \vec{g}_{m-1}), \tag{34b}$$

subject to the boundary conditions

$$g_m(0, \xi) = 0, \quad s_m(0, \xi) = 0, \quad \left.\frac{\partial s_m(\eta, \xi)}{\partial \eta}\right|_{\eta=0} = 0, \tag{34c}$$

$$g_m(\infty, \xi) = 0, \quad \left.\frac{\partial s_m(\eta, \xi)}{\partial \eta}\right|_{\eta=\infty} = 0, \tag{34d}$$

where

$$\begin{aligned} R_m^s(\vec{s}_{m-1}, \vec{g}_{m-1}) &= \frac{1}{(m-1)!} \left.\frac{\partial^{m-1} \mathcal{N}_s[S(\eta, \xi; q), G(\eta, \xi; q)]}{\partial q^{m-1}}\right|_{q=0} \\ &= \frac{\partial^3 s_{m-1}}{\partial \eta^3} + (1-\xi)\left(\eta \frac{\partial^2 s_{m-1}}{\partial \eta^2} - \xi \frac{\partial^2 s_{m-1}}{\partial \xi \partial \eta}\right) \\ &\quad + \xi\left[\frac{1}{2}\sum_{n=0}^{m-1} \frac{\partial s_n}{\partial \eta}\frac{\partial s_{m-1-n}}{\partial \eta} - \sum_{n=0}^{m-1} s_n \frac{\partial^2 s_{m-1-n}}{\partial \eta^2} - 2\sum_{n=0}^{m-1} g_n g_{m-1-n}\right] \end{aligned} \tag{35}$$

and

$$R_m^g(\vec{s}_{m-1}, \vec{g}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}_g[S(\eta, \xi; q), G(\eta, \xi; q)]}{\partial q^{m-1}} \Big|_{q=0}$$
$$= \frac{\partial^2 s_{m-1}}{\partial \eta^2} - (1-\xi)\left(\xi \frac{\partial g_{m-1}}{\partial \xi} - \frac{\eta}{2}\frac{\partial g_{m-1}}{\partial \eta}\right)$$
$$+ \xi \left[\sum_{n=0}^{m-1} g_n \frac{\partial s_{m-1-n}}{\partial \eta} - \sum_{n=0}^{m-1} s_n \frac{\partial g_{m-1-n}}{\partial \eta}\right], \quad (36)$$

under the definition

$$\chi_m = \begin{cases} 0, m = 1, \\ 1, m > 1. \end{cases} \quad (37)$$

Note that Eqs. (34a) and (34b) are *linear* and *uncoupled*. Besides, due to the definitions of $\mathcal{L}_s$ and $\mathcal{L}_g$, they are *ordinary differential equations*, and the dimensionless time ξ is regarded only as a parameter. This greatly simplifies solving $g_m(\eta, \xi)$ and $s_m(\eta, \xi)$.

Let $s_m^*(\eta, \xi)$ and $g_m^*(\eta, \xi)$ denote the particular solutions of the high-order deformation equations (34a) to (34d). According to (19) and (20), their general solutions read

$$s_m(\eta, \xi) = s_m^*(\eta, \xi) + C_1 \exp(-\eta) + C_2 \exp(\eta) + C_3, \quad (38)$$

$$g_m(\eta, \xi) = g_m^*(\eta, \xi) + C_4 \exp(-\eta) + C_5 \exp(\eta), \quad (39)$$

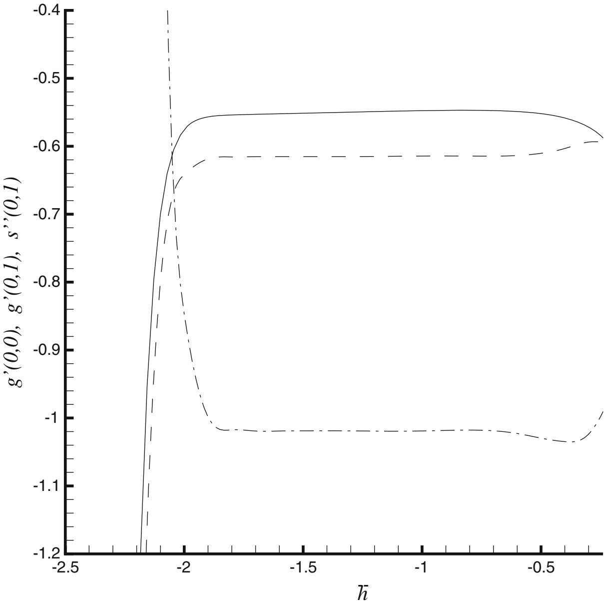
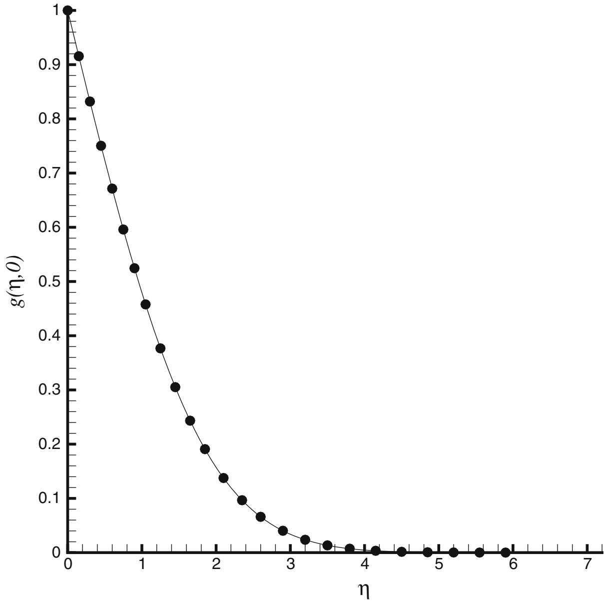


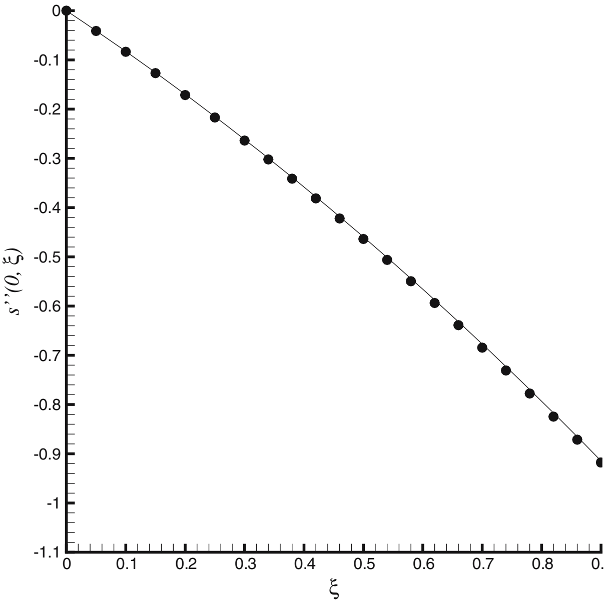
Fig. 1 The $\hbar$-curves of $g'(0, 0)$, $g'(0, 1)$ and $s''(0, 1)$ at the 20th order of approximations given by the homotopy analysis method. Solid line: the $\hbar$-curve of $g'(0, 0)$; Dashed line: the $\hbar$-curve of $g'(0, 1)$; Dash dotted line: the $\hbar$-curve of $s''(0, 1)$

Fig. 2 The comparison of $g(\eta, 0)$ of the exact solution (10) when $\xi = 0$ with the 25th-order approximation when $\hbar = -1$. Solid line: exact solution when $\xi = 0$; symbols: 25th-order approximations given by homotopy analysis method



where the integral constants C_1, C_2, C_3, C_4, and C_5 are determined by the boundary conditions (34c) and (34d). In this way, it is easy to solve the linear equations (34a) and (34d), successively, in the order $m = 1, 2, 3, \cdots$, especially by means of the symbolic computation software such as Mathematica or Maple.

Fig. 3 The approximations of $s''(0, \xi)$ for $0 \leq \xi \leq 1$ when $\hbar$ = -1 given by the homotopy analysis method. Solid line: 10th-order approximation; Symbols: 25th-order approximation



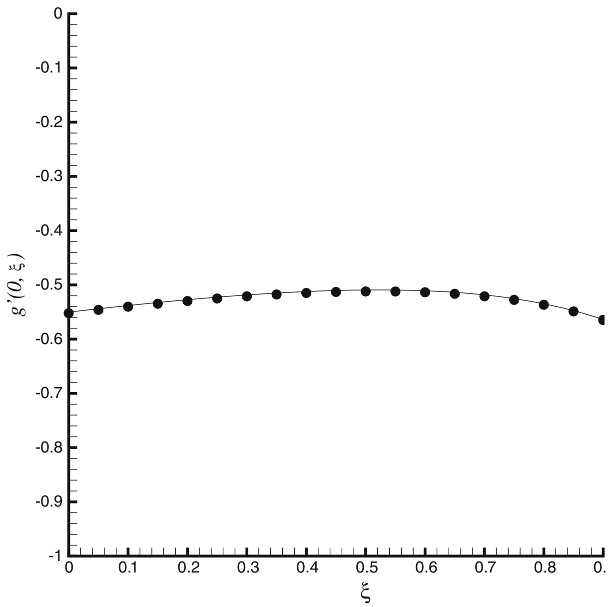


Fig. 4 The approximations of $g'(0, \xi)$ for $0 \leq \xi \leq 1$ when $\hbar = -1$ given by the homotopy analysis method. Solid line: 10th-order approximation; Symbols: 25th-order approximation

3 Analysis of Results

Liao [18] proved that, as long as a solution series given by the homotopy analysis method converges, it must be one of solutions of the considered problem. Note that the solution series (30) and (31) contain one auxiliary parameter $\hbar$, which provides us

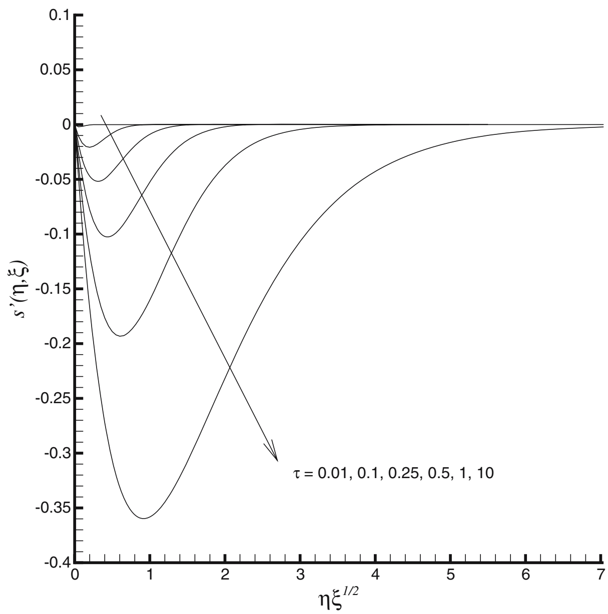
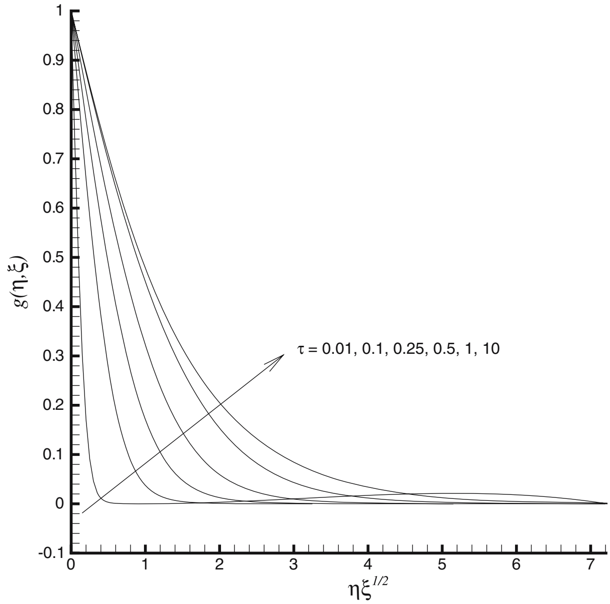


Fig. 5 The 25th-order approximations of $s'(\eta, \xi)$ given by the homotopy analysis method at different dimensionless time $\tau = \Omega t$ when $\hbar = -1$

Fig. 6 The 25th-order approximations of $g(\eta, \xi)$ given by the homotopy analysis method at different dimensionless time $\tau = \Omega t$ when $\hbar = -1$



with a simple way to ensure that the solution series are convergent. As suggested by Liao [18], the value of $\hbar$ is chosen by means of plotting the so-called $\hbar$-curves so as to ensure that the solution series (30) and (31) converge. The $\hbar$-curves of $g'(0, 0) \sim \hbar$, $g'(0, 1) \sim \hbar$ and $s''(0, 1) \sim \hbar$ are as shown in Fig. 1. From Fig. 1, it is clear that, when $-1.7 \leq \hbar \leq -0.7$, we can obtain the convergence results of the series (30) and (31) at $\xi = 0$ and $\xi = 1$, respectively. Similarly, given any a value of $\xi \in [0, 1]$, we can find out a proper value of $\hbar$ to ensure that the solution series (30) and (31) are convergent. It is found that, when $\hbar = -1$, the solution series are convergent for *all* values of $\xi \in [0, 1]$, corresponding to $0 \leq \tau < +\infty$.

Table 1 25*th* order analytic approximations of $s''(0)$ for different ξ

	Numerical results	HAM results	Benton's result [19]
ξ=0.0	0.000000	0.000000	
ξ=0.1	-0.0841074	-0.0841062	
ξ=0.2	-0.172689	-0.172688	
ξ=0.3	-0.266130	-0.266129	
ξ=0.4	-0.364801	-0.364799	
ξ=0.5	-0.468997	-0.468996	
ξ=0.6	-0.578810	-0.578809	
ξ=0.7	-0.693825	-0.693824	
ξ=0.8	-0.812375	-0.812374	
ξ=0.9	-0.929237	-0.929239	
ξ=1.0	-1.02045	-1.020465	-1.0204

Table 2 25*th* order analytic approximations of $g'(0)$ for different ξ

	Numerical results	HAM results	Benton's result [19]
ξ=0.0	-0.56419	-0.56419*	
ξ=0.1	-0.550693	-0.550707	
ξ=0.2	-0.538573	-0.538585	
ξ=0.3	-0.528308	-0.528316	
ξ=0.4	-0.520502	-0.520512	
ξ=0.5	-0.515989	-0.515997	
ξ=0.6	-0.515940	-0.515947	
ξ=0.7	-0.522060	-0.522067	
ξ=0.8	-0.536944	-0.536953	
ξ=0.9	-0.564858	-0.564871	
ξ=1.0	-0.615940	-0.615916	-0.6159

As shown in Fig. 2, at $\xi = 0$, our analytic approximations agree well with the exact solution. To verify the validity and correctness of our analytic approximations in the whole spatial region $0 \le \eta < \infty$ for all time $0 \le \xi \le 1$, we use the Keller-Box method [44] to verify our results. With this numerical technique, the computational domain of η, ranged from 0 to 40, is divided into 1000 intervals, while the time domain, ranged from 0 to 1, is divided into 200 time steps. Fixed step sizes are employed in both directions. The convergence criterion used is based on the Root Mean Square error (RMS) which is 1.0×10^{-6} in the present work. As shown in Fig. 3 and Fig. 4, our 25th-order analytic approximations (30) and (31) agree well with the numerical ones in the whole spatial region $0 \le \eta < \infty$ for all dimensionless time $\xi \in [0, 1]$. Thus, due to Liao's proof [18], they must converge to the solution of the considered problem.

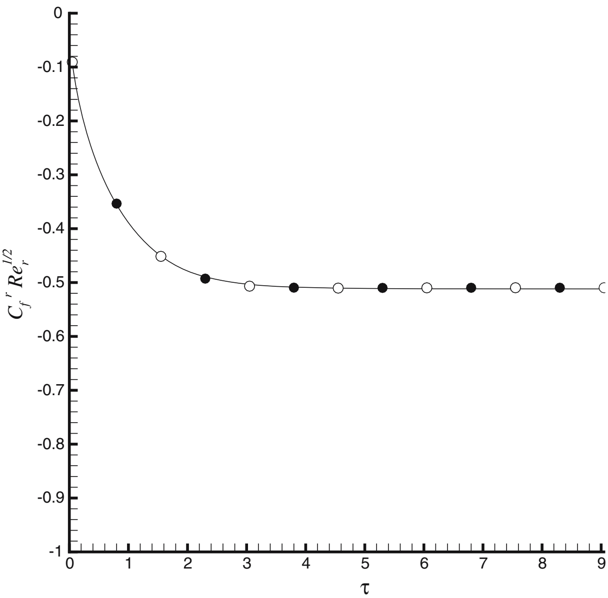


Fig. 7 The comparison of $C_f^r\sqrt{Re_r}$ versus $\tau = \Omega t$. Circle: 10th-order approximations; Filled Circle: 20th-order approximations; Solid line: 25th-order approximations

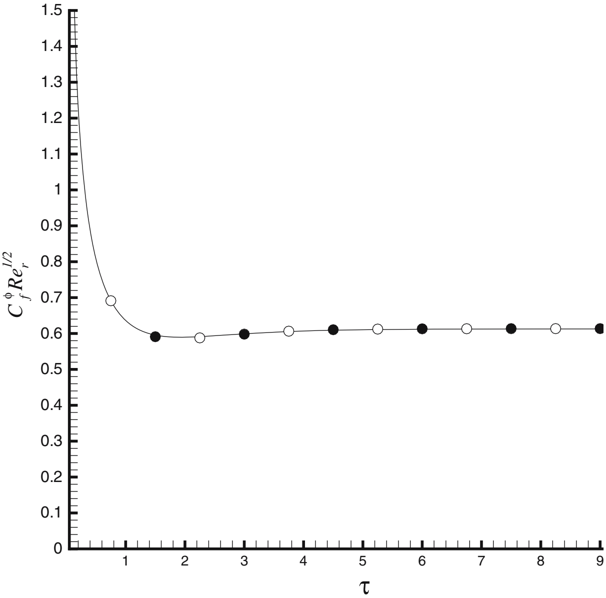


Fig. 8 The comparison of $C_f^{\phi}\sqrt{Re_r}$ versus $\tau = \Omega t$. Circle: 10th-order approximations; Filled Circle: 20th-order approximations; Solid line: 25th-order approximations

Note that $s'(\eta, \xi)$ becomes crookedly as τ increase from 0 to ∞, while $g(\eta, \xi)$ varies smoothly as τ increase from 0 to ∞, as shown in Fig. 5 and Fig. 6, respectively. We find also in Fig. 5 and Fig. 6 that the unsteady procedures are transitory ($\tau = 10$ is almost approaching the steady state). This agrees with Benton's conclusion [19] that after 2 radians of motion the flow is approaching asymptotic steady-state.

The physical interests of this problem are the local skin friction coefficients C_f^r and C_f^{φ}, they are closely related to the $s''(0)$ and $g'(0)$, respectively. Comparing our analytic approximations of $s''(0)$ and $g'(0)$ with numerical ones, agreements are found to be excellent, as shown in Table 1 and Table 2. Note that the homotopy-Padé technique [18] is used here. In the cases of $\xi = 0$ and $\xi = 1$, we obtain [60, 60] homotopy-Padé approximation. While, for other time $0 < \xi < 1$, we have [20, 20] homotopy-Padé approximation. This is mainly because, when $\xi = 0$ and $\xi = 1$, Eqs. (6a) and (6b) become much simpler.

Using $\hbar = -1$, we have the 10th-order approximations

$$\begin{aligned} g'(0, \xi) = & -0.5471711300057143 + 0.13223187929737362\,\xi \\ & -0.05475262380255187\,\xi^2 - 0.05063399307906465\,\xi^3 \\ & -0.0367606438318261\,\xi^4 + 0.03757579321783283\,\xi^5 \\ & -0.5171041283290398\,\xi^6 + 1.8698092416310956\,\xi^7 \\ & -3.414400162429595\,\xi^8 + 2.95442856121643\,\xi^9 \\ & -0.9889087022982035\,\xi^{10}, \end{aligned} \tag{40}$$

Table 3 25*th* order analytic approximations of $s(+\infty)$ for different ξ

	Numerical results	HAM results	Benton's results
ξ=0.0	0.0	0.0	
ξ=0.1	-0.0361299	-0.0361355	
ξ=0.2	-0.0782994	-0.0783056	
ξ=0.3	-0.127749	-0.127758	
ξ=0.4	-0.186021	-0.186034	
ξ=0.5	-0.255031	-0.255049	
ξ=0.6	-0.33714	-0.337165	
ξ=0.7	-0.435198	-0.435202	
ξ=0.8	-0.552471	-0.552517	
ξ=0.9	-0.695393	-0.695411	
ξ=1.0	-0.884456	-0.884322	-0.8845

and

$$\begin{aligned} s''(0,\xi) = &-0.8150263050243334\,\xi - 0.1979911655375893\,\xi^2 \\ &- 0.0534014097220133830\,\xi^3 - 0.010374420142715203\,\xi^4 \\ &+ 0.08225315855656734\,\xi^5 - 0.6329319641976858\,\xi^6 \\ &+ 2.3715178711237166\,\xi^7 - 4.216897308939879\,\xi^8 \\ &+ 3.571393771042506\,\xi^9 - 1.1268468878587505\,\xi^{10}. \end{aligned} \tag{41}$$

Substituting Eqs.(40) and (41) into (7) and (8), respectively, and then comparing them with the corresponding 20*th* and 25*th* order results, it is found that our 10th-order approximations of $C_f^r\sqrt{Re_r}$ and $C_f^\varphi\sqrt{Re_r}$ are accurate enough, and can be used to the engineering practice, as shown in Fig. 7 and Fig. 8.

Another quantity of interest is the axial inflow s at infinity. Our analytic approximations of $s(\infty,\xi)$ also agree well with the Benton's result [19] and the numerical solutions, as shown in Table 3.

4 Conclusions and Discussions

In this paper, we apply the homotopy analysis method [18] to obtain the accurate series solutions for the unsteady viscous flow due to a rotating disk. The *coupled* system of *nonlinear partial difference equations* is replaced by a sequence of *uncoupled* systems of *linear ordinary differential equations*. This greatly simplifies solving the complicated unsteady nonlinear problem. Different from all previous analytic solutions, our series solutions are valid for all time $0 \le \tau < \infty$ in the whole spatial space $0 \le z < \infty$. To the best of our knowledge, such kind of series solutions have not been reported.

It should be emphasized that the homotopy analysis method is a relatively new technique. In this paper, we successfully applied the homotopy analysis method to solve a complicated unsteady nonlinear problem governed by a set of *coupled* partial differential equations. Although the considered problem is a transitory unsteady one, however, the homotopy analysis method itself is not restricted in this type of nonlinear problems: it has general meaning and can be applied widely in a similar way to other more complicated problems. Certainly, it should be further improved by investigating nonlinear phenomena in science and engineering that have not been solved or fully understood.

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