



## The explicit series solution of SIR and SIS epidemic models

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### ABSTRACT

In this paper the SIR and SIS epidemic models in biology are solved by means of an analytic technique for nonlinear problems, namely the homotopy analysis method (HAM). Both of the SIR and SIS models are described by coupled nonlinear differential equations. A one-parameter family of explicit series solutions are obtained for both models. This parameter has no physical meaning but provides us with a simple way to ensure convergent series solutions to the epidemic models. Our analytic results agree well with the numerical ones. This analytic approach is general and can be applied to get convergent series solutions of some other coupled nonlinear differential equations in biology.

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### 1. Introduction

Epidemiology is the branch of biology which deals with the mathematical modeling of spread of diseases. Many problems arising in epidemiology may be described, in a first formulation, by means of differential equations. This means that the models are constructed by averaging some population and keeping only the time variable. To the best of our knowledge the first mathematical model of epidemiology was formulated and solved by Daniel Bernoulli in 1760. Since the time of Kermack and McKendrick [1], the study of mathematical epidemiology has grown rapidly, with a large variety of models having been formulated and applied to infectious diseases [2–4].

Unfortunately, sometimes even the simplest mathematical models of natural phenomena with sets of first-order ordinary differential equations are non-integrable. Nucci and Leach [5] have discussed these features and obtained an integrable SIS model by applying Lie analysis. Many other mathematical tools such as stability theory [6], bifurcation theory [7], Lyapunov method [8], Poincaré–Bendixson type theorems [9], index and topological concepts can be used to study these differential equations arising in biology. There is an extensive literature concerning SIR and SIS models. Considerable attention has been given to these models by several authors see [2–14]. As far as we know, there is no analytic solution of these models, as mentioned by Singh [13].

Consider a population which remains constant and which is divide into three classes: the susceptibles, denoted by  $S$ , who can catch the disease; the infectives, denoted by  $I$ , who are infected and can transmit the disease to the susceptibles, and the removed class, denoted by  $R$ , who had the disease and recovered or died or have developed immunity or have been removed from contact with the other classes. Since from the modeling perspective only the overall state of a person with respect to the disease is relevant, the progress of individuals is schematically described by

$$S \rightarrow I \rightarrow R.$$

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These types of models are known as SIR models. This model is deterministic.

In the SIS model, the infectives,  $I$ , return to the susceptible class,  $S$ , on recovery because the disease confers no immunity against reinfection. Such models are more effective for diseases caused by bacteria or helminth agents and also for most sexually transmitted disease. The progress of individuals in this model is schematically described by

$$S \rightarrow I \rightarrow S.$$

Kermack and McKendrick [1] provided the mathematical models for SIR and SIS.

In this paper our main interest is to obtain explicit series solutions of the SIR and SIS epidemic models provided by Kermack and McKendrick [1] with the help of the homotopy analysis method (HAM) [15,19]. This is a newly developed technique, which works for strongly nonlinear problems [15–28]. In Sections 2 and 3 of the paper we shall study the HAM analysis of SIR and SIS models, respectively. By means of the HAM, the convergent series solutions for SIR and SIS models are obtained. Our HAM results agree well with the results given by Singh [13] qualitatively, and also with the numerical ones, as shown in Sections 2 and 3.

## 2. HAM analysis of SIR model

The classic SIR model [1] is described by

$$s'(t) = -rs(t)i(t), \quad (1)$$

$$i'(t) = rs(t)i(t) - \alpha i(t), \quad (2)$$

subject to the initial conditions,

$$s(0) = S_0, \quad i(0) = I_0, \quad (3)$$

where  $s(t)$  and  $i(t)$  denote the susceptibles and the infectives, respectively,  $r > 0$  is the infectivity coefficient of the typical Lotka–Volterra interaction term, and  $\alpha > 0$  is the recovery coefficient. Here,  $I_0 > 0$  and  $S_0 > 0$  are given constants.

Let us write

$$I_\infty = i(+\infty), \quad S_\infty = s(+\infty).$$

It is known from the previous publications [1–13] that  $i(t)$  decreases monotonously from  $I_0$  to 0 when  $rS_0/\alpha < 1$ . However, in case of  $rS_0/\alpha > 1$ ,  $i(t)$  first increases to the maximum value and then decreases to zero. Thus, it always holds  $I_\infty = 0$ . Besides, it is known [1–13] that

$$S_\infty + I_\infty - \frac{\alpha}{r} \ln S_\infty = S_0 + I_0 - \frac{\alpha}{r} \ln S_0. \quad (4)$$

Thus, from the given values of  $I_0$  and  $S_0$ , one can find  $S_\infty$  from the above equation.

### 2.1. Zeroth-order deformation equations

From (1) and (2), we are led to define the two nonlinear operators,

$$\mathcal{N}_S[S(t; q), I(t; q)] = \frac{\partial S(t; q)}{\partial t} + rS(t; q)I(t; q), \quad (5)$$

$$\mathcal{N}_I[S(t; q), I(t; q)] = \frac{\partial I(t; q)}{\partial t} + \alpha I(t; q) - rS(t; q)I(t; q). \quad (6)$$

Let  $s_0(t)$ ,  $i_0(t)$  denote the initial guesses of  $s(t)$  and  $i(t)$ ,  $\mathcal{L}_S$  and  $\mathcal{L}_I$  the two auxiliary linear operators,  $H_S(t)$  and  $H_I(t)$  the two non-zero auxiliary functions, and  $h$  a non-zero auxiliary parameter, called the convergence-control parameter, respectively. All of them will be determined later. Here, we just emphasize that we have great freedom to choose all of them, besides the initial guess  $s_0(t)$  and  $i_0(t)$  satisfy the initial conditions (3). Let  $q \in [0, 1]$  denote the embedding parameter. Then, we construct the two-parameter family of the differential equations,

$$(1 - q)\mathcal{L}_S[S(t; q) - s_0(t)] = qhH_S(t)\mathcal{N}_S[S(t; q), I(t; q)], \quad (7)$$

$$(1 - q)\mathcal{L}_I[I(t; q) - i_0(t)] = qhH_I(t)\mathcal{N}_I[S(t; q), I(t; q)], \quad (8)$$

subject to the initial conditions,

$$S(0; q) = S_0, \quad I(0; q) = I_0. \quad (9)$$

Obviously, when  $q = 0$ , the above equations have the solution

$$S(0; q) = s_0(t), \quad I(0; q) = i_0(t).$$

Since  $h \neq 0$ ,  $H_S(t) \neq 0$  and  $H_I(t) \neq 0$ , when  $q = 1$ , Eqs. (7) and (8) are equivalent to the original ones (1) and (2), respectively. Thus, we have

$$S(1; q) = s(t), \quad I(1; q) = i(t).$$

Thus, as  $q$  increases from 0 to 1,  $S(t; q)$  varies continuously from the initial guess  $s_0(t)$  to the solution  $s(t)$ , so does  $I(t; q)$  from the initial guess  $i_0(t)$  to the solution  $i(t)$ . Such kinds of continuous variations are called deformation or homotopy in topology. So, Eqs. (7)–(9) are called the zeroth-order deformation equations.

Expand  $S(t; q)$  and  $I(t; q)$  into the Taylor series with respect to  $q$ , i.e.,

$$S(t; q) = s_0(t) + \sum_{m=1}^{+\infty} s_m(t)q^m, \tag{10}$$

$$I(t; q) = i_0(t) + \sum_{m=1}^{+\infty} i_m(t)q^m, \tag{11}$$

where

$$s_m = \frac{1}{m!} \left. \frac{\partial^m S(t; q)}{\partial q^m} \right|_{q=0}, \quad i_m = \frac{1}{m!} \left. \frac{\partial^m I(t; q)}{\partial q^m} \right|_{q=0}. \tag{12}$$

Assuming that  $h, H_S(t)$  and  $H_I(t)$  are properly chosen so that the above two series (10) and (11) converge at  $q = 1$ , we have the solution series,

$$s(t) = s_0(t) + \sum_{m=1}^{+\infty} s_m(t), \tag{13}$$

$$i(t) = i_0(t) + \sum_{m=1}^{+\infty} i_m(t). \tag{14}$$

### 2.2. High-order deformation equations

For brevity, define the vectors,

$$\vec{s}_m = \{s_0(t), s_1(t), s_2(t), \dots, s_m(t)\},$$

$$\vec{i}_m = \{i_0(t), i_1(t), i_2(t), \dots, i_m(t)\}.$$

Differentiating the zeroth-order deformation Eqs. (7) and (8)  $m$  times with respect to the embedding parameter  $q$ , then setting  $q = 0$ , and finally dividing by  $m!$ , we have the so-called  $m$ th-order deformation equations,

$$\mathcal{L}_S [s_m(t) - \chi_m s_{m-1}(t)] = h H_S(t) R_m^S(t), \tag{15}$$

$$\mathcal{L}_I [i_m(t) - \chi_m i_{m-1}(t)] = h H_I(t) R_m^I(t), \tag{16}$$

subject to the initial conditions,

$$s_m(0) = 0, \quad i_m(0) = 0, \tag{17}$$

where

$$R_m^S(t) = s'_{m-1}(t) + r \sum_{k=0}^{m-1} i_k(t) s_{m-1-k}(t), \tag{18}$$

$$R_m^I(t) = i'_{m-1}(t) + \alpha i_{m-1}(t) - r \sum_{k=0}^{m-1} i_k(t) s_{m-1-k}(t), \tag{19}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{20}$$

Note that the  $m$ th-order deformation Eqs. (15) and (16) are decoupled, and governed by the linear auxiliary linear operators  $\mathcal{L}_S$  and  $\mathcal{L}_I$ , which we have great freedom to choose. Besides, according to (18) and (19), the right-hand side terms  $R_m^S(t)$  and  $R_m^I(t)$  are always known to the unknown solutions  $s_m(t)$  and  $i_m(t)$  for  $m = 1, 2, 3, \dots$

### 2.3. Explicit series solution of SIR model

As mentioned before, we have great freedom to choose the initial guesses  $s_0(t)$  and  $i_0(t)$ , the auxiliary linear operators  $\mathcal{L}_S$  and  $\mathcal{L}_I$ , the auxiliary functions  $H_S(t)$  and  $H_I(t)$ , and the convergence-control parameter  $h$ .

First of all, as mentioned in Section 1, we have  $s(t) \rightarrow S_\infty$  and  $i(t) \rightarrow 0$  as  $t \rightarrow +\infty$ . So,  $s(t)$  and  $i(t)$  can be expressed by,

$$i(t) = \sum_{k=1}^{+\infty} a_k e^{-k\beta t}, \quad s(t) = S_\infty + \sum_{k=1}^{+\infty} b_k e^{-k\beta t}, \tag{21}$$

where  $a_k$  and  $b_k$  are coefficients. The above formulas provide us the so-called solution expressions of  $i(t)$  and  $s(t)$ , respectively.

From (1) and (2), we have,

$$i'(0) = I_0(rS_0 - \alpha), \quad s'(0) = -rS_0I_0.$$

To obey the solution expression (21), we choose such the initial guesses  $i_0(t)$  and  $s_0(t)$ :

$$s_0(t) = S_\infty + \gamma_{0,1}e^{-\beta t} + \gamma_{0,2}e^{-2\beta t}, \quad i_0(t) = \delta_{0,1}e^{-\beta t} + \delta_{0,2}e^{-2\beta t}, \quad (22)$$

where

$$\gamma_{0,1} = 2(S_0 - S_\infty) - \frac{rS_0I_0}{\beta}, \quad \gamma_{0,2} = -(S_0 - S_\infty) + \frac{rS_0I_0}{\beta}, \quad (23)$$

$$\delta_{0,1} = 2I_0 + \frac{I_0(rS_0 - \alpha)}{\beta}, \quad \delta_{0,2} = -I_0 - \frac{I_0(rS_0 - \alpha)}{\beta} \quad (24)$$

are determined by means of the initial conditions,

$$i(0) = I_0, \quad s(0) = S_0, \quad i'_0(0) = I_0(rS_0 - \alpha), \quad s'_0(0) = -rS_0I_0. \quad (25)$$

To obtain solutions in the form of (21), we choose the auxiliary linear operators,

$$\mathcal{L}_S u = u'(t) + \beta u(t), \quad \mathcal{L}_I u = u'(t) + \beta u(t), \quad (26)$$

which have the property,

$$\mathcal{L}_S[C_1 e^{-\beta t}] = 0, \quad \mathcal{L}_I[C_2 e^{-\beta t}] = 0, \quad (27)$$

where  $C_1$  and  $C_2$  are the integral constants. Substituting the initial guesses  $i_0(t)$  and  $s_0(t)$  into (18) and (19), we get,

$$R_1^I(t) = \sum_{k=1}^4 a_{1,k} e^{-k\beta t}, \quad R_1^S(t) = \sum_{k=1}^4 b_{1,k} e^{-k\beta t},$$

where

$$a_{1,1} = -\frac{I_0(\alpha - rS_0 - 2\beta)(\alpha - rS_\infty - \beta)}{\beta}, \quad (28)$$

$$b_{1,1} = S_0(rI_0 - 2\beta) + 2S_\infty(rI_0 + \beta) - \frac{rI_0S_\infty(\alpha - rS_0)}{\beta}, \quad (29)$$

⋮

are coefficients. Note that

$$u'(t) + \beta u(t) = Ae^{-\beta t},$$

has the general solution,

$$u(t) = Ate^{-\beta t} + C_1 e^{-\beta t},$$

where  $C_1$  is an integral constant. Obviously, the term  $te^{-\beta t}$  does not satisfy the solution expressions (21). Fortunately, we have freedom to choose the auxiliary functions  $H_S(t)$  and  $H_I(t)$ , and thus we can avoid the appearance of the term  $t \exp(-\beta t)$  simply by means of choosing,

$$H_I(t) = e^{-\beta t}, \quad H_S(t) = e^{-\beta t}. \quad (30)$$

Then the first-order deformation equations become,

$$i'_1(t) + \beta i_1(t) = \hbar \sum_{k=1}^4 a_{1,k} e^{-\beta(k+1)t}, \quad i_1(0) = 0, \quad (31)$$

$$s'_1(t) + \beta s_1(t) = \hbar \sum_{k=1}^4 b_{1,k} e^{-\beta(k+1)t}, \quad s_1(0) = 0, \quad (32)$$

where  $a_{1,k}$  and  $b_{1,k}$  are constants. It is easy to find the solutions of the above two *uncoupled* linear differential equations

$$i_1(t) = -\frac{\hbar}{\beta} \sum_{k=1}^4 \frac{a_{1,k}}{k} e^{-\beta(k+1)t} + \frac{\hbar}{\beta} \left( \sum_{k=1}^4 \frac{a_{1,k}}{k} \right) e^{-\beta t}, \quad (33)$$

$$s_1(t) = -\frac{\hbar}{\beta} \sum_{k=1}^4 \frac{b_{1,k}}{k} e^{-\beta(k+1)t} + \frac{\hbar}{\beta} \left( \sum_{k=1}^4 \frac{b_{1,k}}{k} \right) e^{-\beta t}. \quad (34)$$

In a similar way, it is easy to get  $i_2(t), s_2(t), i_3(t), s_3(t)$  and so on, especially by means of symbolic computation software such as Mathematica, Maple and so on.

It can be found that

$$i_m(t) = \sum_{k=1}^{3m+2} \delta_{m,k} e^{-k\beta t}, \quad s_m(t) = \sum_{k=1}^{3m+2} \gamma_{m,k} e^{-k\beta t}, \tag{35}$$

where  $\delta_{m,k}, \gamma_{m,k}$  are coefficients. Substituting the above expressions into (15)–(17), we have the recurrence formulas,

$$\delta_{m,j} = \chi_m \chi_{3m-j+1} \delta_{m-1,j} - \left(\frac{h}{\beta}\right) \frac{a_{m,j-1}}{(j-1)}, \quad 2 \leq j \leq 3m+2, \tag{36}$$

$$\gamma_{m,j} = \chi_m \chi_{3m-j+1} \gamma_{m-1,j} - \left(\frac{h}{\beta}\right) \frac{b_{m,j-1}}{(j-1)}, \quad 2 \leq j \leq 3m+2, \tag{37}$$

$$\delta_{m,1} = - \sum_{j=2}^{3m+2} \delta_{m,j}, \quad \gamma_{m,1} = - \sum_{j=2}^{3m+2} \gamma_{m,j}, \tag{38}$$

where

$$a_{m,j} = \chi_{3m+1-j} (\alpha - rS_\infty - j\beta) \delta_{m-1,j} - r\chi_j \mu_{m,j}, \quad 1 \leq j \leq 3m+1, \tag{39}$$

$$b_{m,j} = \chi_{3m+1-j} (rS_\infty \delta_{m-1,j} - j\beta \gamma_{m-1,j}) + r\chi_j \mu_{m,j}, \quad 1 \leq j \leq 3m+1, \tag{40}$$

and

$$\mu_{m,j} = \sum_{k=0}^{m-1} \sum_{i=\max\{1, j+3k-3m+1\}}^{\min\{3k+2, j-1\}} \gamma_{k,i} \delta_{m-1-k, j-i}, \quad 2 \leq j \leq 3m+1. \tag{41}$$

Using the above recurrence formulas, we can obtain, one by one, all coefficients from (23) and (24). Thus, we have the explicit series solution

$$i(t) = \sum_{m=1}^{+\infty} \sum_{k=1}^{3m+2} \delta_{m,k} e^{-k\beta t}, \quad s(t) = S_\infty + \sum_{m=1}^{+\infty} \sum_{k=1}^{3m+2} \gamma_{m,k} e^{-k\beta t}. \tag{42}$$

The  $m$ th-order approximation is given by

$$i(t) \approx \sum_{m=1}^M \sum_{k=1}^{3m+2} \delta_{m,k} e^{-k\beta t}, \quad s(t) \approx S_\infty + \sum_{m=1}^M \sum_{k=1}^{3m+2} \gamma_{m,k} e^{-k\beta t}. \tag{43}$$

Note that we have freedom to choose a proper value of the auxiliary parameter  $\beta$ . It is known that  $s(t)$  decreases from  $S_0$  to  $S_\infty$ . When  $rS_0/\alpha < 1$ ,  $i(t)$  also decreases from  $I_0$  to 0. However, in case of  $rS_0/\alpha > 1$ ,  $i(t)$  first increases to the maximum value and then decreases to zero. So, it is important to approximate  $i(t)$  efficiently. Note that  $R_1^l(t)$  denotes the residual error of (8). To ensure that the residual error of (8) decays quickly for large  $t$ , we enforce  $a_{1,1} = 0$ , i.e.,

$$(\alpha - rS_0 - 2\beta)(\alpha - rS_\infty - \beta) = 0, \tag{44}$$

which gives,

$$\beta = \alpha - rS_\infty. \tag{45}$$

The other algebraic solution

$$\beta = \frac{\alpha}{2} \left(1 - \frac{rS_0}{\alpha}\right)$$

is neglected because  $\beta$  must be positive but  $rS_0/\alpha$  can be greater than 1.

#### 2.4. Solution analysis of the SIR model with examples

Singh [13] has provided a qualitative analysis of the SIR model and defined a threshold quantity  $R_0 = \frac{S_0 r}{\alpha}$ . It is assumed that  $s' < 0$  for all  $t$  and  $i' > 0$  when  $R_0 > 1$ . This means  $I$  will initially increase to some maximum if  $R_0 > 1$ , but eventually decreases and approaches zero, since  $S$  is always decreasing. Our series solution for the SIR model provides results that agrees with Singh's [13] qualitative analysis.

The case with  $R_0 > 1$  is of interest as it indicates epidemic. For  $R_0 < 1$ ,  $I$  will simply goes to zero, indicating no epidemic. Our examples given below illustrate that we can always find a proper value of the auxiliary parameter  $h$  to ensure that the explicit series solutions converge to the numerical ones.

#### 2.4.1. Example 1: $\alpha = 2, r = 1/5, I_0 = 25, S_0 = 75$

In this case, we have  $rS_0/\alpha = 7.5 > 1, S_\infty = 0, \beta = 2$ . Note that our explicit series solution (42) contains the convergence-control parameter  $h$  and we have freedom to choose the value of  $h$ . Physically, for given  $\alpha, r, S_0$  and  $I_0$ , if  $i(t)$  is convergent, then the integral,

$$I_w = \int_0^{+\infty} i(t)dt \quad (46)$$

tends to a constant which has physically nothing to do with  $h$ . However, mathematically, the above integral is dependent upon  $h$ , because  $i(t)$  is dependent upon  $h$ . It has been proved by Liao [16] that all convergent series solution given by the HAM converge to the true solution of the original nonlinear equations. So, there is a horizontal line segment of the curve  $I_w$  versus  $h$ , as shown in Fig. 1, and all values of  $h$  in the region  $-0.8 \leq h < -0.2$  under this horizontal line segment give the same convergent value of  $I_w$ . For example, the series solutions are convergent to the numerical results when  $h = -0.5$ , as shown in Figs. 2 and 3. Note that, by means of  $h = -0.8, -0.6, -0.4$  and  $-0.2$ , we obtain the same convergent series solutions of  $i(t)$  and  $s(t)$ . Thus, the convergence-control parameter  $h$  indeed provides us a simple way to ensure the convergence of solution series.

#### 2.4.2. Example 2: $\alpha = 3/2, r = 1/10, I_0 = 10, S_0 = 50$

In this case, we have  $rS_0/\alpha = 3.333 > 1, S_\infty = 0.97, \beta = 1.403$ . According to the curve  $I_w \sim h$  at the 15th-order of approximation, the HAM series is convergent in the region  $-2 \leq h \leq -0.5$ . So, we choose  $h = -1$ , and the corresponding HAM series converge to the numerical ones, as shown in Figs. 4 and 5.

#### 2.4.3. Example 3: $\alpha = 10, r = 1/10, I_0 = 50, S_0 = 80$

In this case, we have  $rS_0/\alpha = 0.8 < 1, S_\infty = 29.19, \beta = 7.081$ . According to the curve  $I_w \sim h$  at the 10th-order of approximation, the HAM series are convergent in the region  $-3 \leq h < 0$ . So we choose  $h = -2$ , and even the second-order HAM series agree well with the numerical ones, as shown in Fig. 6.

#### 2.4.4. Example 4: $\alpha = 2.73, r = 0.0178, I_0 = 7, S_0 = 254$

Let us consider a practical case. Over the period from mid-May to mid-October 1966, the village of Eyam in England suffered an outbreak of bubonic plague. The community was asked to quarantine itself so that the disease could be prevented from spreading to other communities. A thorough record was kept according to which it is believed that out of an initial population of 350 persons only 83 survived the Eyam plague. The disease in Eyam has been used as a case study for the basic SIR modeling. Here we try to fit the basic SIR model, measuring time in months with an initial population of 7 infectives and 254 susceptibles and a final population of 83.

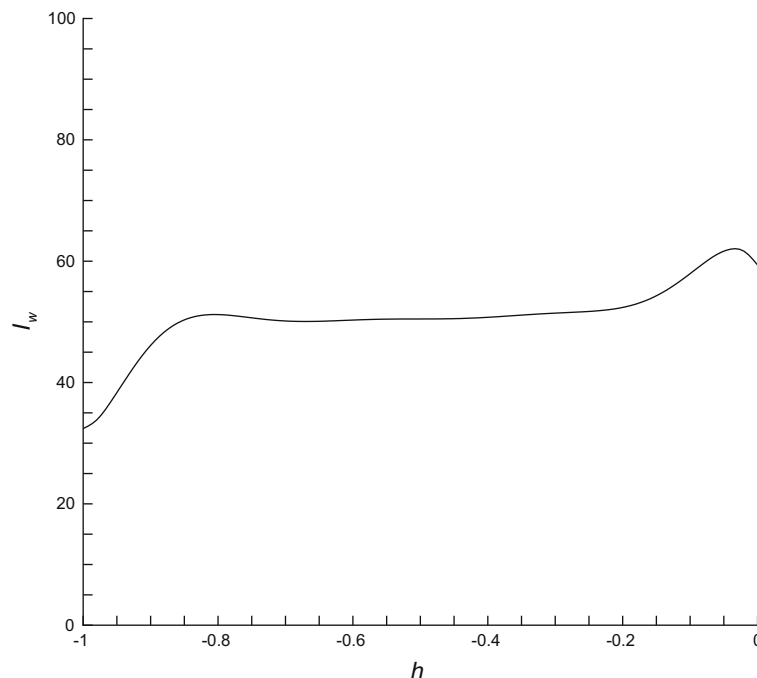
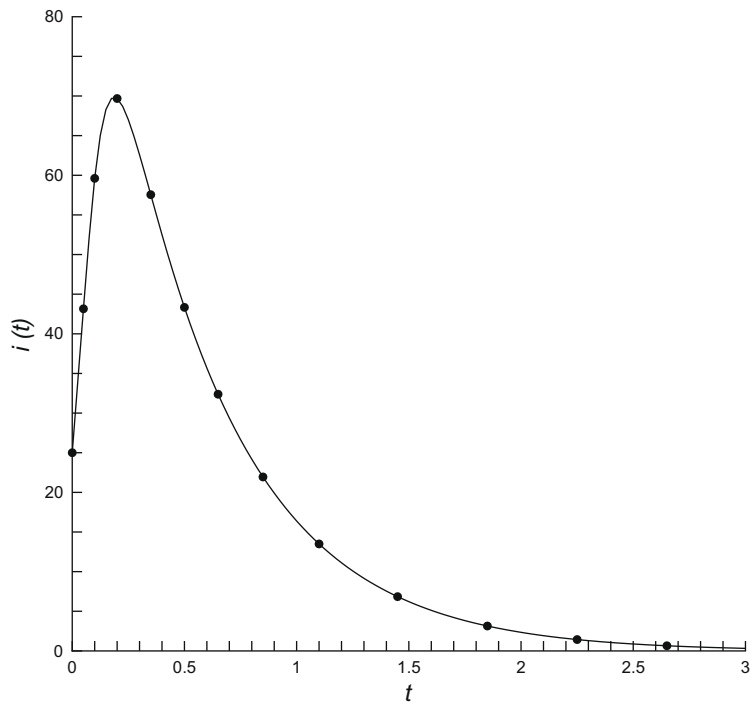
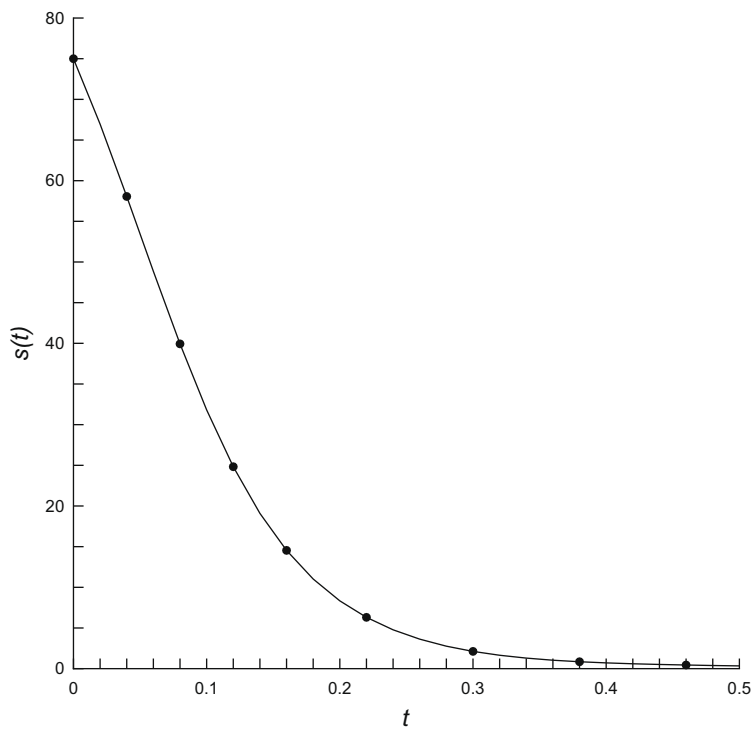


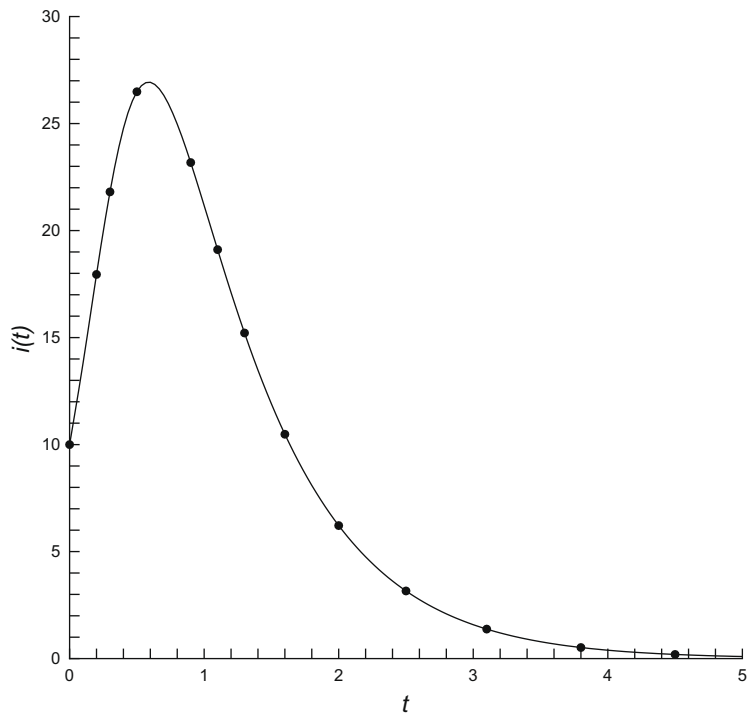
Fig. 1. The curve  $I_w \sim h$  of Example 1 at the 15th-order approximation.



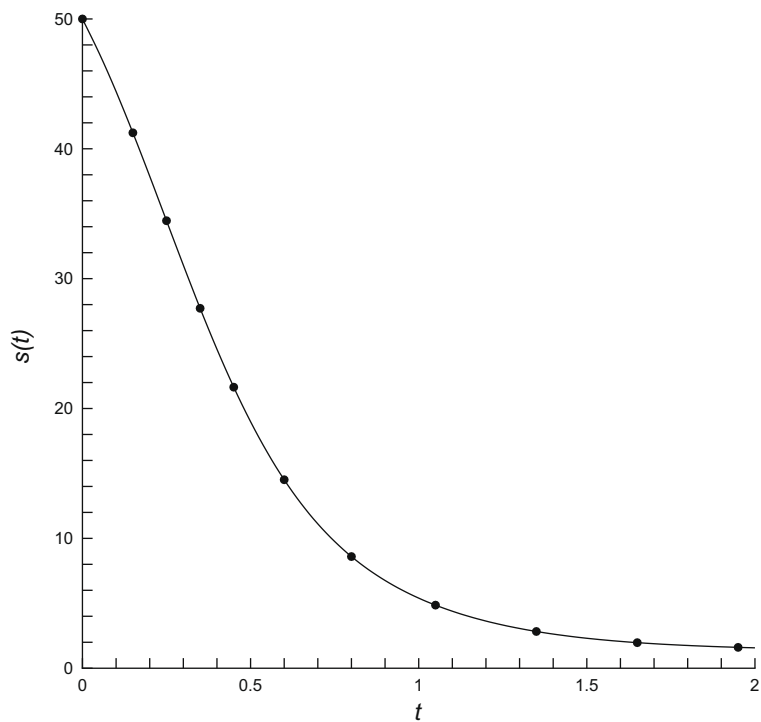
**Fig. 2.** Comparison of the numerical solution of  $i(t)$  with the HAM series solution of Example 1. Symbols: numerical solution; Solid line: 15th-order HAM solution by means of  $h = -1/2$ .



**Fig. 3.** Comparison of the numerical solution of  $s(t)$  with the HAM series solution of Example 1. Symbols: numerical solution; Solid line: 15th-order HAM solution by means of  $h = -1/2$ .

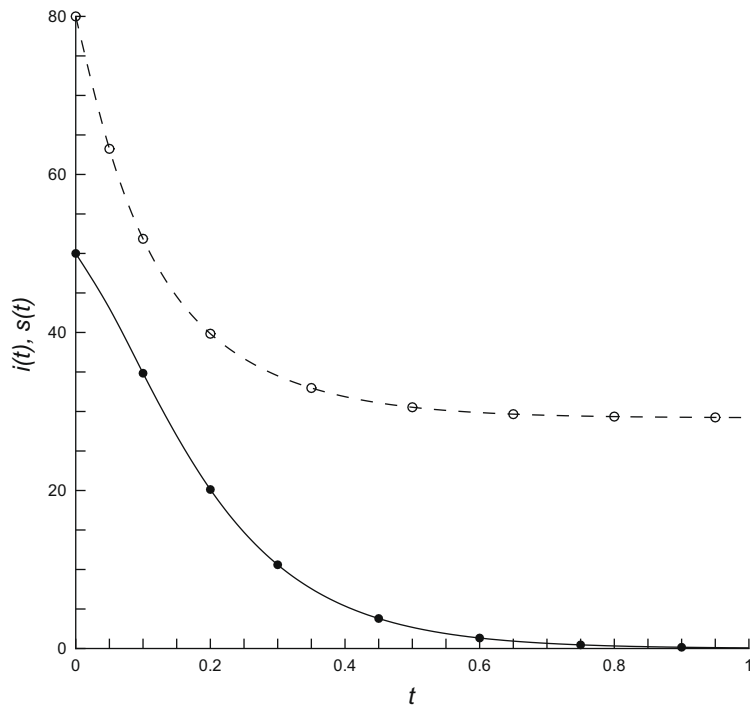


**Fig. 4.** Comparison of the numerical solution of  $i(t)$  with the HAM series solution of Example 2. Symbols: numerical solution; Solid line: 10th-order HAM solution by means of  $h = -1$ .

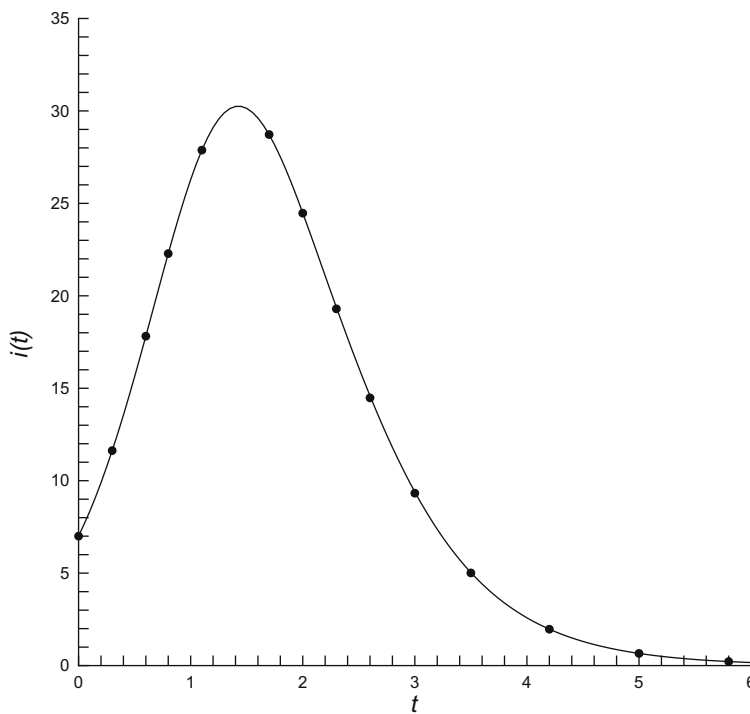


**Fig. 5.** Comparison of the numerical solution of  $s(t)$  with the HAM series solution of Example 2. Symbols: numerical solution; Solid line: 10th-order HAM solution by means of  $h = -1$ .

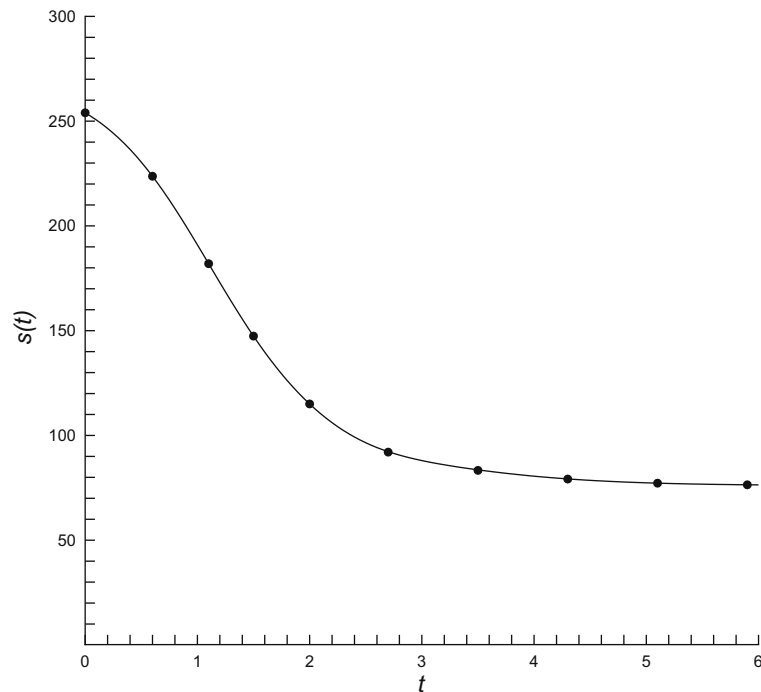




**Fig. 6.** Comparison of the numerical solution of  $i(t)$  and  $s(t)$  with the HAM series solution of Example 3. Filled circles: numerical solution of  $i(t)$ ; Solid line: second-order HAM solution of  $i(t)$  given by  $h = -2$ ; Open circles: numerical solution of  $s(t)$ ; Dashed line: second-order HAM solution of  $s(t)$  given by  $h = -2$ .



**Fig. 7.** Comparison of the numerical solution of  $i(t)$  with the HAM series of Example 4. Symbols: 50th-order result by means of  $h = -2$ ; Solid line: 60th-order result when  $h = -1.5$ .



**Fig. 8.** Comparison of the numerical solution of  $s(t)$  with the HAM series of Example 4. Symbols: 50th-order result by means of  $h = -2$ ; Solid line: 60th-order result by means of  $h = -1.5$ .

In this case, we have  $rS_0/\alpha = 1.656 > 1$ ,  $S_\infty = 76.05$ ,  $\beta = 1.37631$ . According to the curve  $I_w \sim h$  at the 20th-order of approximation, the HAM series is convergent in the region  $-2 \leq h \leq -0.5$ . Our HAM series given in  $h = -2$  and  $h = -1.5$  converge to the same result, as shown in Figs. 7 and 8. This illustrates that our HAM series solution of the basic SIR model is convergent for practical cases.

### 3. HAM analysis of SIS model

The basic SIS model [1] is given by

$$s'(t) = -rs(t)i(t) + \gamma i(t), \quad (47)$$

$$i'(t) = rs(t)i(t) - \gamma i(t), \quad (48)$$

subject to initial conditions,

$$i(0) = I_0, \quad s(0) = S_0, \quad (49)$$

where  $r > 0$ ,  $I_0 > 0$  and  $S_0 > 0$ . Obviously,  $S + I = k$ , where  $k$  is the total population. From (47) and (48), we obtain the threshold value  $rk/\gamma$  for this model. There are two different cases for  $t$  approaching  $\infty$ :

(A) If  $rk/\gamma \leq 1$  for any  $I_0$ , then

$$I(+\infty) = 0, \quad S(+\infty) = k; \quad (50)$$

(B) if  $rk/\gamma > 1$  for any  $I_0$ , then

$$I(+\infty) = k - \gamma/r, \quad S(+\infty) = \gamma/r. \quad (51)$$

Hence, if  $rk/\gamma < 1$ , the infection must vanish, whereas if  $rk/\gamma > 1$ , the infection continues. So the interesting case is  $rk/\gamma > 1$ , the epidemic case. This model is different from the SIR model, because the recovered members return to the class  $S$  at a rate of  $\gamma i$  instead of moving to class  $R$ , where  $\gamma$  is the recovery coefficient.

### 3.1. Deformation equations

The SIS model can be solved in a similar way. First, we construct the zeroth-order deformation equation

$$(1 - q)\mathcal{L}_S[S(t; q) - s_0(t)] = qhH_S(t)\mathcal{N}_S[S(t, q), I(t, q)], \tag{52}$$

$$(1 - q)\mathcal{L}_I[I(t; q) - i_0(t)] = qhH_I(t)\mathcal{N}_I[S(t, q), I(t, q)], \tag{53}$$

subject to the initial conditions,

$$S(0; q) = S_0, \quad I(0; q) = I_0, \tag{54}$$

where  $q \in [0, 1]$  is an embedding parameter, and the nonlinear operators are defined by

$$\mathcal{N}_S[S(t, q), I(t, q)] = S' + rSI - \gamma I, \tag{55}$$

$$\mathcal{N}_I[S(t, q), I(t, q)] = I' - rSI + \gamma I, \tag{56}$$

where the prime denotes the differentiation with respect to  $t$ . And similarly, we have the solution series,

$$s(t) = s_0(t) + \sum_{m=1}^{+\infty} s_m(t), \tag{57}$$

$$i(t) = i_0(t) + \sum_{m=1}^{+\infty} i_m(t), \tag{58}$$

where  $s_m(t)$  and  $i_m(t)$  are governed by the high-order deformation equations

$$\mathcal{L}_S[s_m(t) - \chi_m s_{m-1}(t)] = hH_S(t)R_m^S(t), \tag{59}$$

$$\mathcal{L}_I[i_m(t) - \chi_m i_{m-1}(t)] = hH_I(t)R_m^I(t), \tag{60}$$

subject to the initial conditions,

$$s_m(0) = 0, \quad i_m(0) = 0, \tag{61}$$

with the definitions

$$R_m^S(t) = s'_{m-1} - \gamma i_{m-1} + r \sum_{k=0}^{m-1} i_k s_{m-1-k}, \tag{62}$$

$$R_m^I(t) = i'_{m-1} + \gamma i_{m-1} - r \sum_{k=0}^{m-1} i_k s_{m-1-k}. \tag{63}$$

The above decoupled, linear differential equations are easy to solve, especially by symbolic computation software such as Mathematica, Maple and so on.

### 3.2. Explicit series solution for SIS model

Similarly, we choose the base function

$$\{e^{-m\beta t}, m \geq 1\}, \tag{64}$$

and the solution can be written by

$$i(t) = I_\infty + \sum_{m=1}^{+\infty} a_m e^{-m\beta t}, \tag{65}$$

$$s(t) = S_\infty + \sum_{m=1}^{+\infty} b_m e^{-m\beta t}, \tag{66}$$

where  $S_\infty = S(+\infty)$  and  $I_\infty = I(+\infty)$  are given by (50) or (51),  $a_m$  and  $b_m$  are coefficients to be determined. This provides us the solution expression for  $s(t)$  and  $i(t)$  of the SIS model.

From (47) and (48), we have the additional initial conditions

$$s'(0) = -rS_0I_0 + \gamma I_0, \tag{67}$$

$$i'(0) = rS_0I_0 - \gamma I_0. \tag{68}$$

According to the solution expression (66), it is straightforward to choose the initial solutions,

$$s_0(t) = S_\infty + \gamma_{0,1}e^{-\beta t} + \gamma_{0,2}e^{-2\beta t}, \tag{69}$$

$$i_0(t) = I_\infty + \delta_{0,1}e^{-\beta t} + \delta_{0,2}e^{-2\beta t}, \tag{70}$$

where

$$\begin{aligned}\gamma_{0,1} &= 2(S_0 - S_\infty) + I_0(\gamma - rS_0)/\beta, \\ \gamma_{0,2} &= S_\infty - S_0 + I_0(rS_0 - \gamma)/\beta, \\ \delta_{0,1} &= 2(I_0 - I_\infty) + I_0(rS_0 - \gamma)/\beta, \\ \delta_{0,2} &= I_\infty - I_0 + I_0(\gamma - rS_0)/\beta,\end{aligned}$$

are determined by means of the initial conditions (49) and the additional conditions (67) and (68).

Similarly, to obey the solution expressions (65) and (66), we choose the same auxiliary linear operators  $\mathcal{L}_S$  and  $\mathcal{L}_I$  as defined in (26) for the SIR model. Besides, we also choose the same auxiliary functions  $H_S(t) = H_I(t) = \exp(-\beta t)$  as in the SIR model. Then, similarly, we have the explicit series solutions for the SIS model

$$i(t) = I_\infty + \sum_{m=1}^{+\infty} i_m(t), \quad s(t) = S_\infty + \sum_{m=1}^{+\infty} s_m(t), \quad (71)$$

with,

$$i_m(t) = \sum_{k=1}^{3m+2} \delta_{m,k} e^{-k\beta t}, \quad s_m(t) = \sum_{k=1}^{3m+2} \gamma_{m,k} e^{-k\beta t}, \quad (72)$$

where  $\delta_{m,k}, \gamma_{m,k}$  are coefficients. Substituting the above expressions into (59)–(61), we have the recurrence formulas,

$$\delta_{m,j} = \chi_m \chi_{3m-j+1} \delta_{m-1,j} - \left(\frac{h}{\beta}\right) \frac{c_{m,j-1}}{(j-1)}, \quad 2 \leq j \leq 3m+2, \quad (73)$$

$$\gamma_{m,j} = \chi_m \chi_{3m-j+1} \gamma_{m-1,j} - \left(\frac{h}{\beta}\right) \frac{d_{m,j-1}}{(j-1)}, \quad 2 \leq j \leq 3m+2, \quad (74)$$

$$\delta_{m,1} = -\sum_{j=2}^{3m+2} \delta_{m,j}, \quad \gamma_{m,1} = -\sum_{j=2}^{3m+2} \gamma_{m,j}, \quad (75)$$

where

$$\begin{aligned}c_{m,j} &= \chi_{3m+1-j} \left[ (\gamma - rS_\infty - j\beta) \delta_{m-1,j} - rI_\infty \gamma_{m-1,j} \right] \\ &\quad - r\chi_j \mu_{m,j}, \quad 1 \leq j \leq 3m+1, \quad (76)\end{aligned}$$

$$\begin{aligned}d_{m,j} &= \chi_{3m+1-j} \left[ (rS_\infty - \gamma) \delta_{m-1,j} + (rI_\infty - j\beta) \gamma_{m-1,j} \right] \\ &\quad + r\chi_j \mu_{m,j}, \quad 1 \leq j \leq 3m+1, \quad (77)\end{aligned}$$

and

$$\mu_{m,j} = \sum_{k=0}^{m-1} \sum_{i=\max\{1, j+3k-3m+1\}}^{\min\{3k+2j-1\}} \gamma_{k,i} \delta_{m-1-k,j-i}, \quad 2 \leq j \leq 3m+1. \quad (78)$$

### 3.3. Solution analysis for the SIS model with examples

For the SIS model, we can also get convergent series solutions with good agreement with numerical ones. Here we present a few examples of comparison of our explicit series solutions of the SIS model with numerical results.

#### 3.3.1. Example 5: $\gamma = 3/2, \beta = 2, r = 1/10, I_0 = 10$ and $S_0 = 25$

In this case we have  $rk/\gamma = 2.33 > 1$ . So, using (51), we have  $S_\infty = 15$  and  $I_\infty = 20$ . Similarly, according to the curve  $I_w \sim h$  at the 10th-order approximation, our HAM series are convergent for  $-2.5 \leq h < 0$ . Choosing  $h = -1/2$ , our explicit series solution agree well with numerical ones, as shown in Figs. (9) and (10).

#### 3.3.2. Example 6: $\gamma = 1/10, \beta = 1, r = 1/15, I(0) = 25$ and $S(0) = 10$

This is an example of epidemic because of  $rk/\gamma > 1$ . Our HAM series are convergent for  $-2 \leq h \leq 0$ . Choosing  $h = -1/2$ , our HAM series solution agree well with the numerical ones, as shown in Figs. (11) and (12).

3.3.3. Example 7:  $\gamma = 3/2, \beta = 1, r = 1/20, I(0) = 15$  and  $S(0) = 50$

Similarly, it is found that our HAM series are convergent for  $-3 \leq h < 0$ . Choosing  $h = -2/5$ , our HAM series solution agree well with numerical ones, as shown in Figs. (13) and (14). This is also an example of epidemic for  $rk/\gamma > 1$ .

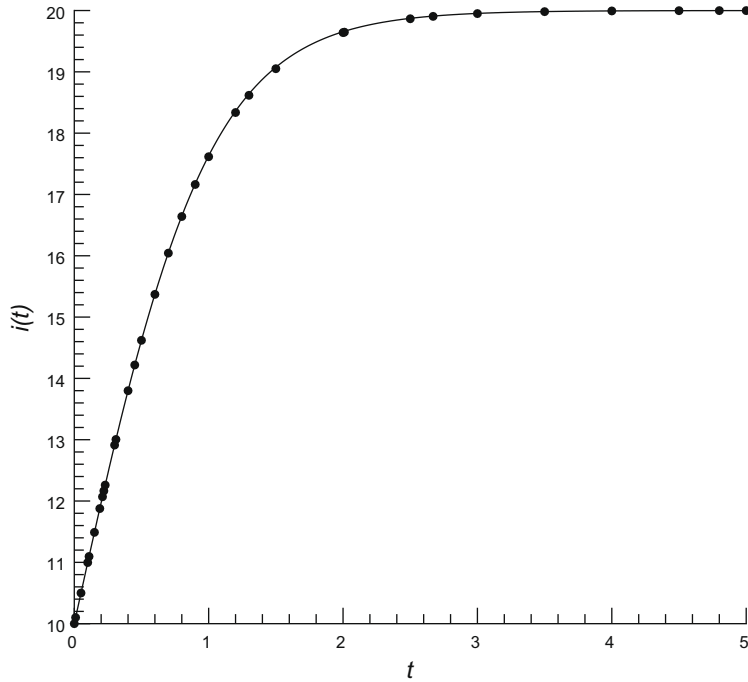


Fig. 9. Comparison of the numerical solution of  $i(t)$  with the HAM series solution of Example 5. Symbols: numerical solution; Solid line: 15th-order HAM solution given by  $h = -1/2$ .

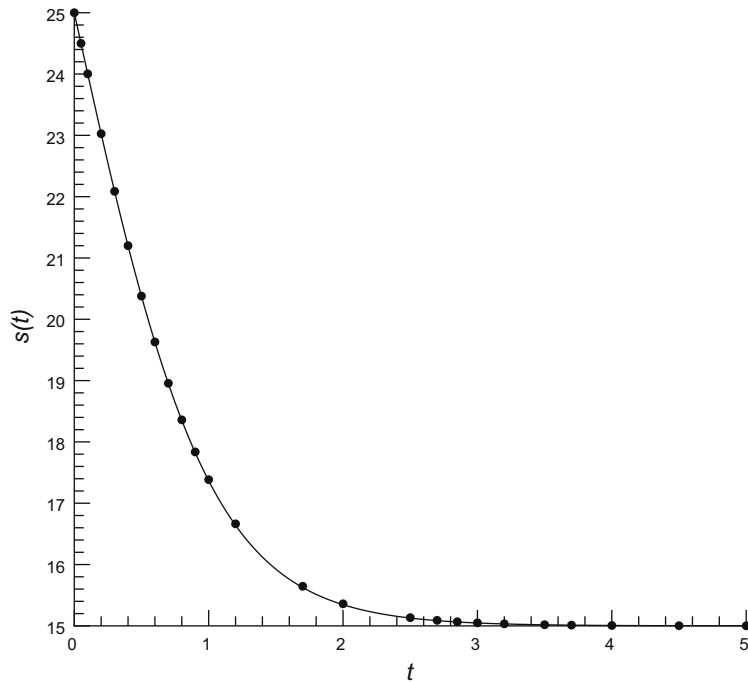


Fig. 10. Comparison of the numerical solution of  $s(t)$  with the HAM series solution of Example 5. Symbols: numerical solution; Line: 15th-order HAM solution given by  $h = -1/2$ .

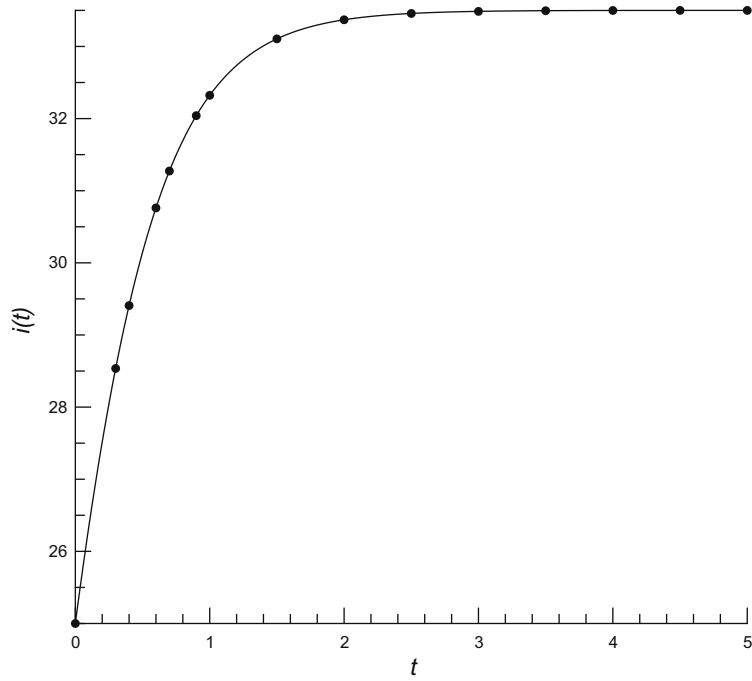


Fig. 11. Comparison of the numerical solution of  $i(t)$  with the HAM series solution of Example 6. Symbols: numerical solution; Line: 40th-order HAM solution given by  $h = -1/2$ .

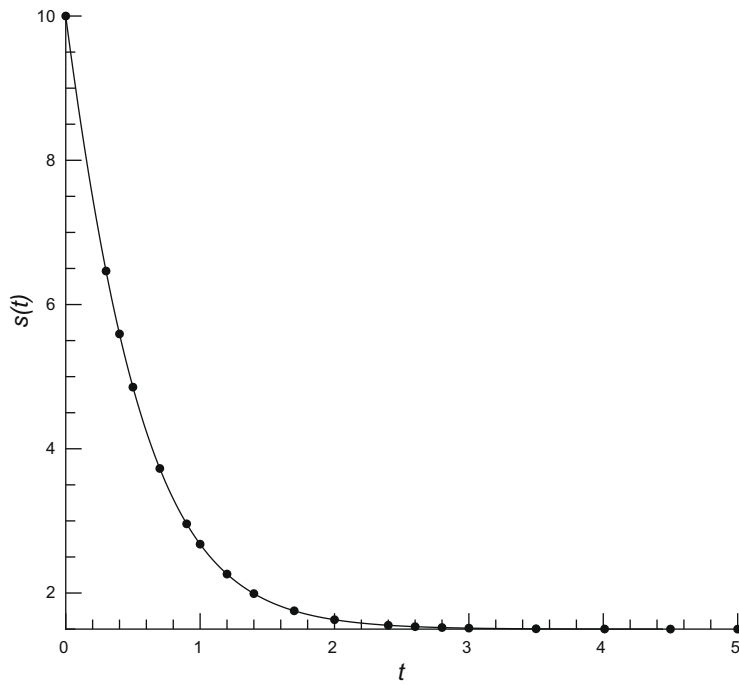
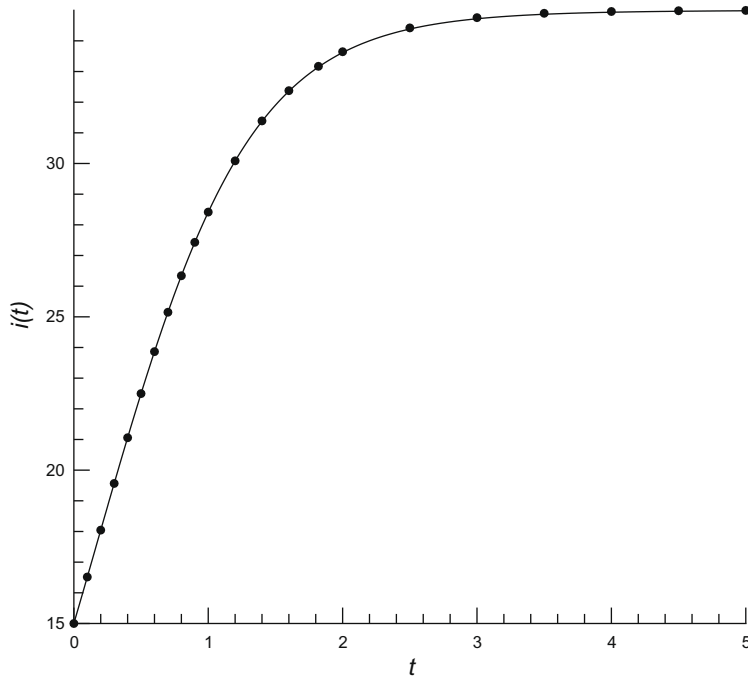
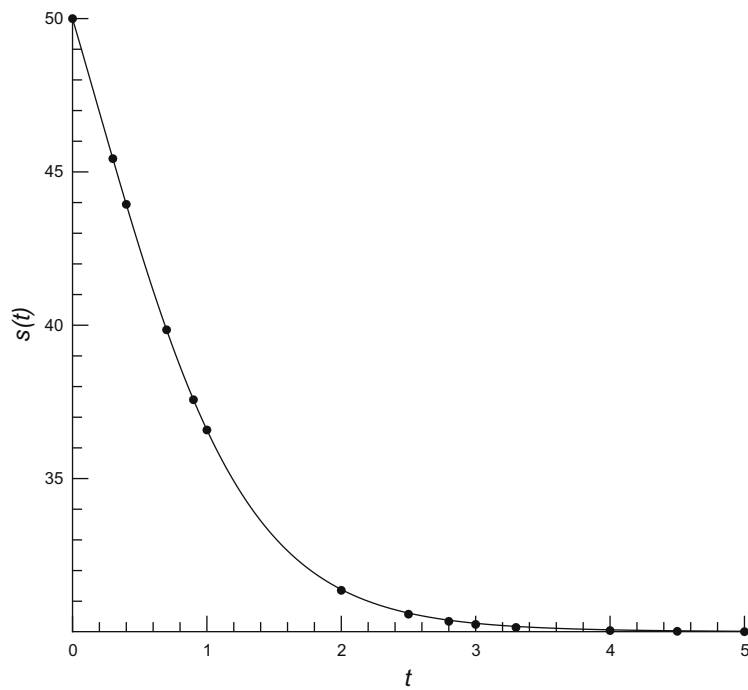


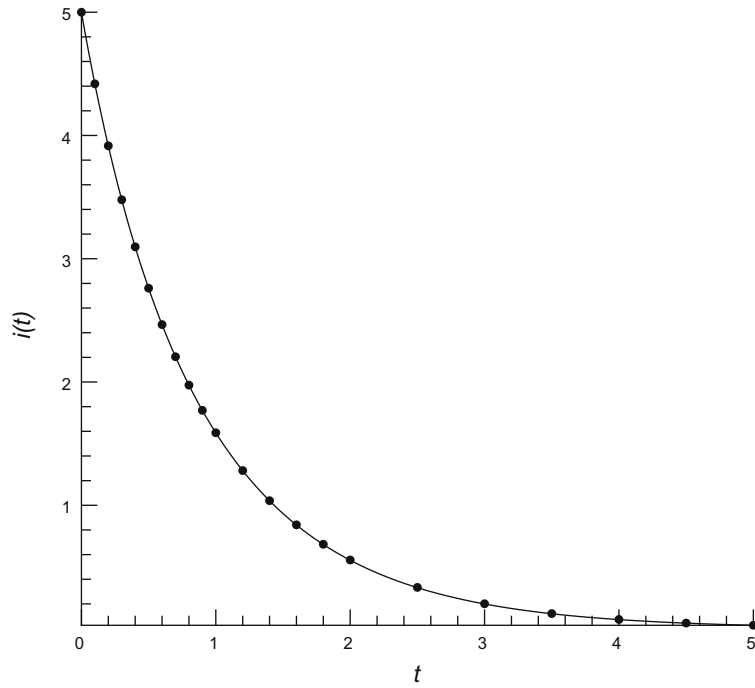
Fig. 12. Comparison of the numerical solution of  $s(t)$  with the HAM series solution of Example 6. Symbols: numerical solution; Line: 40th-order HAM solution given by  $h = -1/2$ .



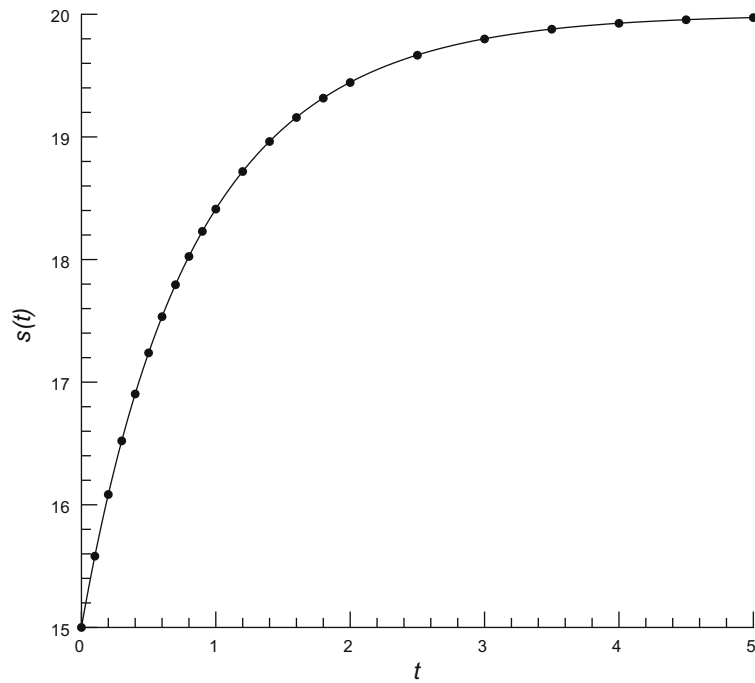
**Fig. 13.** Comparison of the numerical solution of  $i(t)$  with the HAM series solution of Example 7. Symbols: numerical solution; Line: 30th-order HAM solution given by  $h = -2/5$ .



**Fig. 14.** Comparison of the numerical solution of  $s(t)$  with the HAM series solution of Example 7. Symbols: numerical solution; Line: 30th-order HAM solution given by  $h = -2/5$ .



**Fig. 15.** Comparison of the numerical solution of  $i(t)$  with the HAM series solution of Example 8. Symbols: numerical solution; Line: 4th-order HAM solution given by  $h = -3/2$ .



**Fig. 16.** Comparison of the numerical solution of  $s(t)$  with the HAM series solution of Example 8. Symbols: numerical solution; Line: 4th-order HAM solution given by  $h = -3/2$ .



### 3.3.4. Example 8: $\gamma = 2, \beta = 1, r = 1/20, I(0) = 5, S(0) = 15$

In this example we have  $rk/\gamma = 0.5 < 1$  which shows no epidemic. Similarly, it is found that our HAM series are convergent for  $-5/2 \leq h \leq 0$ . Choosing  $h = -3/2$ , even our fourth-order results agree well with the numerical ones, as shown in Figs. (15) and (16).

All of these examples show the validity of our explicit series solutions for the SIR and SIS models.

## 4. Conclusion

In this paper, the analytic method for nonlinear differential equations, namely the homotopy analysis method (HAM), is applied to solve the SIR and SIS models in biology. By means of the HAM, the two coupled original nonlinear differential equations are replaced by an infinite number of linear subproblems with two decoupled linear differential equations. The explicit series solutions are given for both of the SIR and SIS models, and besides an auxiliary parameter, called the convergence-control parameter, is used to ensure the convergence of the explicit series solution. It is illustrated that the convergent analytic series agree quite well with numerical results. Note that such kind of explicit series solution has never been reported for the SIR and SIS models. All of these verify the validity of the homotopy analysis method for the coupled differential equations in biology. It is expected that the explicit series solutions can serve as a complement to the existing literature. The same methodology is general and can be used for some other complicated models.

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