An optimal homotopy-analysis approach for strongly nonlinear differential equations

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In this paper, an optimal homotopy-analysis approach is described by means of the nonlinear Blasius equation as an example. This optimal approach contains at most three convergence-control parameters and is computationally rather efficient. A new kind of averaged residual error is defined, which can be used to find the optimal convergence-control parameters much more efficiently. It is found that all optimal homotopy-analysis approaches greatly accelerate the convergence of series solution. And the optimal approaches with one or two unknown convergence-control parameters are strongly suggested. This optimal approach has general meanings and can be used to get fast convergent series solutions of different types of equations with strong nonlinearity.

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1. Introduction

Nonlinear equations are much more difficult to solve than linear ones, especially by means of analytic methods. Generally speaking, there are two standards for a *satisfactory* analytic method of nonlinear equations:

(a) it can *always* give approximation expressions *efficiently*;
(b) it can guarantee that approximation expressions are *accurate* enough in the *whole* region of *all* physical parameters.

Using above two standards as a criterion, we can discuss the advantages and disadvantages of different analytic techniques for nonlinear problems.

Perturbation techniques [1–6] are widely applied in science and engineering. Most perturbation techniques are based on small (or large) physical parameters in governing equations or boundary conditions, called perturbation quantities. In general, perturbation approximations are expressed in a series of perturbation quantities, and the original nonlinear equations are replaced by an infinite number of linear (sometimes even nonlinear) sub-problems, which are completely determined by the original governing equation and especially by the place where perturbation quantities appear. Perturbation methods are simple, and easy to understand. Especially, based on small physical parameters, perturbation approximations often have clear physical meanings. Unfortunately, *not* every nonlinear problem has such kind of perturbation quantity. Besides, even if there exists a small parameter, the sub-problem might have no solutions, or might be rather complicated so that only a few of the sub-problems can be solved. Thus, it is *not* guaranteed that one can always get perturbation approximations efficiently for any a given nonlinear problem. More importantly, it is well-known that most perturbation approximations are valid only

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for small physical parameters. In general, it is not guaranteed that a perturbation result is valid in the whole region of all physical parameters. Thus, perturbation techniques do not satisfy not only the standard (a) but also the standard (b) mentioned at the beginning of this section.

To overcome the restrictions of perturbation techniques, some traditional nonperturbation methods are developed, such as Lyapunov's artificial small parameter method [7], the δ-expansion method [8,9], Adomian decomposition method [10–15], and so on. In principle, all of these methods are based on a so-called artificial parameter, and approximation solutions are expanded into series of such kind of artificial parameter. This artificial parameter is often used in such a way that one can get approximation solutions efficiently for a given nonlinear equation. Compared with perturbation techniques, this is indeed a great progress. However, in theory, one can put the artificial small parameter in many different ways, but unfortunately there are no theories to guide us how to put it in a better place so as to get a better approximation. For example, Adomian decomposition method simply uses the linear operator d^k/dx in most cases, where k is the highest order of derivative of governing equations, and therefore it is rather easy to get solutions of the corresponding sub-problems by means of integration k times with respect to x. However, such simple linear operator gives approximation solutions in power-series, but unfortunately power-series has often a finite radius of convergence. Thus, Adomian decomposition method cannot ensure the convergence of its approximation series. Generally speaking, all traditional nonperturbation methods, such as Lyapunov's artificial small parameter method [7], the δ-expansion method [8,9] and Adomian decomposition method [10–15], can *not* guarantee the convergence of approximation series. So, these traditional nonperturbation methods satisfy only the standard (a) but *not* the standard (b) mentioned before.

In 1992 Liao [16] took the lead to apply the homotopy [17], a basic concept in topology [18], to get analytic approximations of nonlinear differential equations. Liao [16] described the early form of the homotopy-analysis method (HAM) in 1992. For a given nonlinear differential equation

$$\mathcal{N}[u(x)] = 0, \quad x \in \Omega,$$

where $\mathcal{N}$ is a nonlinear operator and $u(x)$ is a unknown function, Liao constructed a *one-parameter* family of equations in the embedding parameter $q \in [0, 1]$, called the zeroth-order deformation equation

$$(1 - q)\mathcal{L}[U(x;q) - u_0(x)] + q\mathcal{N}[U(x;q)] = 0, \quad x \in \Omega, \ q \in [0,1], \tag{1}$$

where $\mathcal{L}$ is an auxiliary linear operator and $u_0(x)$ is an initial guess. The homotopy provides us larger freedom to choose both of the auxiliary linear operator $\mathcal{L}$ and the initial guess than the traditional nonperturbation methods mentioned before, as pointed out later by Liao [19–21]. At $q = 0$ and $q = 1$, we have $U(x;0) = u_0(x)$ and $U(x;1) = u(x)$, respectively. So, if the Taylor series

$$U(x;q) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x)q^n \tag{2}$$

converges at $q = 1$, we have the so-called homotopy-series solution

$$u(x) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x), \tag{3}$$

which must satisfy the original equation $\mathcal{N}[u(x)] = 0$, as proved by Liao [19,20] in general. Here, $u_n(x)$ is governed by a linear differential equation related to the auxiliary linear operator $\mathcal{L}$ and therefore is easy to solve, as long as we choose the auxiliary linear operator properly. In some cases, one can get convergent series of nonlinear differential equations by choosing proper linear operator and initial guess. However, Liao [22,20] found that this early homotopy-analysis method can not always guarantee the convergence of approximation series. To overcome this restriction, Liao [22] in 1997 introduced such a nonzero auxiliary parameter c_0 to construct a *two-parameter* family of equations, i.e. the zeroth-order deformation equation.[1]

$$(1 - q)\mathcal{L}[U(x;q) - u_0(x)] = c_0 q\mathcal{N}[U(x;q)], \quad x \in \Omega, \ q \in [0,1]. \tag{4}$$

In this way, the homotopy-series solution (3) is not only dependent upon x but also the auxiliary parameter c_0. It was found [22,19,20] that the auxiliary parameter c_0 can adjust and control the convergence region and rate of homotopy-series solutions. In essence, the use of the auxiliary parameter c_0 introduces us one more "artificial" degree of freedom, which has no physical meaning but greatly improved the early homotopy-analysis method: it is the auxiliary parameter c_0 which provides us a convenient way to guarantee the convergence of homotopy-series solution [22,20]. Currently, Liang and Jeffrey [23] used a simple example to illustrate the importance of the auxiliary parameter c_0. Besides, Liao [24] revealed the relationship between the homotopy-analysis method (in some special cases) and the famous Euler transform, which explains clearly why the homotopy-analysis method can ensure the convergence of homotopy-series solution. Due to this reason, c_0 was renamed currently as the *convergence-control parameter* [25].

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[1] Liao [22] originally used the symbol $\hbar$ to denote the auxiliary parameter. But, $\hbar$ is well-known as Planck's constant in quantum mechanics. To avoid misunderstanding, we suggest to use the symbol c_0 to denote the "basic" convergence-control parameter.

The use of the convergence-control parameter c_0 is indeed a great progress. It indicates that more "artificial" degrees of free-dom imply larger possibility to get better approximations by means of the homotopy-analysis method. Thus, Liao [19] in 1999 further introduced more "artificial" degrees of freedom by using the zeroth-order deformation equation in a more general form:

$$[1 - B(q)]\mathscr{L}[U(x;q) - u_0(x)] = c_0 A(q)\mathscr{N}[U(x;q)], \quad x \in \Omega, \quad q \in [0,1], \tag{5}$$

where $A(q)$ and $B(q)$ are the so-called *deformation functions*[2] satisfying

$$A(0) = B(0) = 0, \quad A(1) = B(1) = 1, \tag{6}$$

whose Taylor series

$$A(q) = \sum_{m=1}^{+\infty} \mu_m q^m, \quad B(q) = \sum_{m=1}^{+\infty} \sigma_m q^m, \tag{7}$$

exist and are convergent for $|q| \leqslant 1$. The zeroth-order deformation Eq. (5) can be further generalized, as shown by Liao [20,26,25]. Obviously, there are an infinite number of the deformation functions as defined above. Thus, the approximation series given by the HAM can contain so many "artificial" degrees of freedom that we have more ways to guarantee the convergence of homotopy-series solution and to get better approximations. Note that $u_n(x)$ is always governed by the same auxiliary linear operator $\mathscr{L}$, and we have great freedom to choose $\mathscr{L}$ in such a way that $u_n(x)$ is easy to obtain and besides $u_n(x)$ is expressed by a set of proper base functions. More importantly, for given auxiliary linear operator $\mathscr{L}$ and initial guess, we can always get con-vergent homotopy-series solution by choosing proper convergence-control parameter c_0 and proper deformation functions $A(q)$ and $B(q)$. In fact, the guarantee of the convergence of homotopy-series solutions provides us much larger freedom to choose the auxiliary linear operator $\mathscr{L}$ and initial guess: with such kind of guarantee in the frame of the HAM, a nonlinear ODE with *variable* coefficients can be transferred into a sequence of linear ODEs with *constant* coefficients [27], a nonlinear *PDE* can be transferred into an infinite number of linear *ODEs* [28,29], several *coupled* nonlinear ODEs can be transferred into an infinite number of linear *decoupled* ODEs [30], and even a *2nd*-order nonlinear PDE can be replaced by an infinite number of *4th*-order linear PDEs [21]. Indeed, it is such kind of guarantee for convergence of series solutions, together with the freedom in choice of the auxiliary linear operators, that greatly simplifies finding convergent series of nonlinear equations in the frame of the HAM, as illustrated in above-mentioned articles [21,27–30]. On the other hand, without such kind of guarantee of convergence, we have in practice no *true* freedom to choose the auxiliary linear operator $\mathscr{L}$, because the freedom to get a divergent series solution has no meanings at all and is thus useless! For example, Liang and Jeffrey [23] pointed out that the series solution given by means of the so-called "homotopy-perturbation method" [31] may be divergent at *all* points except the initial guess, and thus has completely no scientific meanings. So, unlike perturbation techniques and the traditional nonperturbation methods men-tioned above, the homotopy-analysis method satisfies both the standard (a) and (b). Besides, it is proved by Liao [20] that the HAM logically contains the traditional nonperturbation methods such as Lyapunov's artificial small parameter method [7], the δ-expansion method [8,9] and Adomian decomposition method [10–15]. Note that the homotopy-perturbation method [31] in 1999 is also a special case of the HAM, as proved by Sajid and Hayat [32] and pointed out by other researchers [23,33–36]. Thus, the HAM is a rather general method for nonlinear problems, especially for those with strong nonlinearity. The HAM has been widely applied to solve different types of nonlinear problems in science, finance and engineering [37,32,23,33,38–43,34,44–48,35,36,49,50]. Especially, a few *new* solutions of some nonlinear problems have been found by means of the HAM [51–53], which were neglected by other analytic methods and even by numerical techniques. All of these show the potential of the HAM for strongly nonlinear problems.

How to find a proper convergence-control parameter c_0 to get a convergent series solution, or even better, to get a faster con-vergent one? A straight-forward way to check the convergence of a homotopy-series solution is to substitute it into original gov-erning equations and boundary/initial conditions, and then to check the corresponding square residual errors integrated in the whole region: the more quickly the residual error decays to zero, the faster the homotopy-series converges. However, when the approximations contain unknown convergence-control parameters and/or other physical parameters, it is time-consuming to calculate the square residual error at high-order of approximations. To avoid the time-consuming computation, Liao [22,20,19] suggested to investigate the convergence of some special quantities, which often have important physical meanings. For exam-ple, one can consider the convergence of $u'(0)$ and $u''(0)$ of a nonlinear differential equation $\mathscr{N}[u(x)] = 0$, if they are unknown. It is found by Liao [22,20,19] that there often exists such a region $\mathbf{R}_c$ that any a value of $c_0 \in \mathbf{R}_c$ gives a convergent series solution of such kind of quantities. Besides, such a region can be found, although approximately, by plotting the curves of these unknown quantities (mostly with important physical meanings) versus c_0. For example, for a nonlinear differential equation $\mathscr{N}[u(x)] = 0$, one may plot curves $u'(0) \sim c_0$, $u''(0) \sim c_0$ and so on. These curves are called " c_0-curves" or "curves for convergence-control parameter", [3] which have been successfully applied in many nonlinear problems [20].

However, it is a pity that curves for convergence-control parameter (i.e. c_0-curves) can not tell us which value of $c_0 \in \mathbf{R}_c$ gives the fastest convergent series. In 2007, Yabushita et al. [37] applied the HAM to solve two coupled nonlinear ODEs, and suggested the so-called "optimization method" to find out two optimal convergence-control parameters by means of the

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[2] $A(q)$ and $B(q)$ were called approaching function in previous articles about the homotopy-analysis method. Here, we suggest to use this new name which better reveals its relationship with the deformation equations.

[3] The c_0-curve here was originally called the *h*-curve, and $\mathbf{R}_c$ was originally denoted by $\mathbf{R}_h$.

minimum of the square residual error integrated in the whole region having physical meanings. In 2008, Marinca et al. [54] combined c_0 and $A(q)$ in the zeroth-order deformation Eq. (5) as one function $H(q)$ with $H(0) = 0$ but $H(1) \neq 1$, and considered such a family of equations

$$[1 - q]\mathscr{L}[U(x;q) - u_0(x)] = H(q)\mathscr{N}[U(x;q)], \quad q \in [0,1], \tag{8}$$

where the Taylor series

$$H(q) = \sum_{n=1}^{+\infty} c_n q^n$$

converges at $q = 1$. The above equation is a special case of (5), if one sets $B(q) = q$ and

$$H(q) = c_0 A(q) = c_0 \sum_{n=1}^{+\infty} \mu_n q^n, \ \text{i.e.} \ \ c_n = c_0 \mu_n.$$

So, the so-called "homotopy-asymptotic method" [54–56] is also in the frame of the homotopy-analysis method. However, Marinca et al. [54] suggested a very interesting approach, which has the advantage that $H(1) = 1$ is unnecessary so that one has more freedom to choose the parameters c_n. Marinca et al. [54] developed the so-called "optimal homotopy-asymptotic method" by minimizing the square residual error: at the Mth-order of approximation, a set of nonlinear algebraic equations about $c_1, c_2, \ldots, c_M$ is solved so as to find their optimal values. In theory, the more the convergence-control parameters are used, the better approximation one should obtain by this optimal HAM approach. However, it is a pity that, with so many unknown parameters, it is time-consuming to calculate the corresponding square residual errors, especially at high-order of approximations for a complicated nonlinear problem. It is reported [57] that the optimal approach given by Marinca et al. [54–56] often does not work in practice, especially at high-order approximations for complicated nonlinear problems.

In this paper, we propose a new kind of optimal homotopy-analysis approach. Our optimal homotopy-analysis approach is also based on the generalized zeroth-order deformation equation (5). However, we use here special deformation functions which are determined completely by only one characteristic parameter $|c_1| < 1$ and $|c_2| < 1$, respectively. In this way, there exist at most only three convergence-control parameters c_0, c_1 and c_2 at *any* order of approximations. Besides, a new definition of residual error is introduced so as to efficiently find out the unknown optimal convergence-control parameters c_0, c_1 and c_2. The basic ideas of the optimal homotopy-analysis approach is described in Section 2. Detailed comparisons of different optimal approaches are shown in Section 3. A brief description of the optimal homotopy-analysis approach for general nonlinear problems is given in 4, and some comments and suggestions are discussed in Section 5.

2. Basic ideas

For the sake of simplicity, let us consider the so-called Blasius boundary-layer flows in fluid mechanics, governed by the nonlinear differential equation

$$f'''(\eta) + \frac{1}{2}f(\eta)f''(\eta) = 0, \quad f(0) = f'(0) = 0, \quad f'(+\infty) = 1, \tag{9}$$

where η is a similarity variable, $f(\eta)$ is related to the stream-function, and the prime denotes the derivative with respect to η, respectively. Let $\lambda > 0$ denote a kind of scale-parameter and introduce the transformation

$$f(\eta) = \lambda^{-1} F(\xi), \quad \xi = \lambda \eta. \tag{10}$$

Then, Eq. (9) becomes

$$F'''(\xi) + \left(\frac{1}{2\lambda^2}\right)F(\xi)F''(\xi) = 0, \quad F(0) = F'(0) = 0, \quad F'(+\infty) = 1, \tag{11}$$

where the prime denotes the derivative with respect to ξ. Following Liao [19], we use here $\lambda = 4$.

Due to the boundary condition $F'(+\infty) = 1$, one has $F \sim \xi$ as $\xi \to +\infty$. Besides, according to the physical meaning of boundary-layer flows, the velocity tends to the main stream flow exponentially. Thus, $F(\xi)$ should be expressed in the form

$$F(\xi) = A_{0,0} + \xi + \sum_{m=1}^{+\infty} \sum_{n=0}^{+\infty} A_{m,n} \xi^n \exp(-m\xi), \tag{12}$$

where $A_{m,n}$ is a constant. The above expression provides us the so-called solution expression of $F(\xi)$, which plays a key role in the homotopy-analysis method, as shown later.

According to the solution expression (12) and the boundary conditions, we choose such an initial guess

$$F_0(\xi) = \xi - 1 + e^{-\xi}, \tag{13}$$

which satisfies all boundary-conditions. Besides, according to the solution expression (12), we choose such an auxiliary linear operator

$$\mathscr{L}F = F''' + F'', \tag{14}$$

which possesses the property

$$\mathscr{L}[C_0 + C_1\xi + C_2 e^{-\xi}] = 0, \tag{15}$$

where the prime denotes the derivative with respect to ξ, and C_0, C_1 and C_2 are integration coefficients. Furthermore, based on the governing equation (11), we define such a nonlinear operator

$$\mathscr{N}[F] = F'''(\xi) + \left(\frac{1}{2\lambda^2}\right) F(\xi) F''(\xi). \tag{16}$$

There are an infinite number of deformation functions satisfying the properties (6) and (7). For the sake of computation efficiency, we use here the following one-parameter deformation functions:

$$A_1(q; c_1) = \sum_{m=1}^{+\infty} \mu_m(c_1) q^m, \quad B_1(q; c_2) = \sum_{m=1}^{+\infty} \sigma_m(c_2) q^m, \tag{17}$$

where $|c_1| < 1$ and $|c_2| < 1$ are constants, called *the convergence-control parameter*, and

$$\mu_1(c_1) = 1 - c_1, \quad \mu_m(c_1) = (1 - c_1)\, c_1^{m-1}, \quad m > 1, \tag{18}$$

$$\sigma_1(c_1) = 1 - c_2, \quad \sigma_m(c_2) = (1 - c_2)\, c_2^{m-1}, \quad m > 1. \tag{19}$$

The different values of c_1 give different paths of $A_1(q; c_1)$, as shown in Fig. 1.

Let $q \in [0, 1]$ denote the embedding parameter, $c_0 \neq 0$ an auxiliary parameter, called *the convergence-control parameter*, and $\phi(\xi; q)$ a kind of continuous mapping of $F(\xi)$, respectively. We construct the so-called zeroth-order deformation equation

$$[1 - B_1(q; c_2)]\mathscr{L}[\phi(\xi; q) - F_0(\xi)] = c_0 A_1(q; c_1) \mathscr{N}[\phi(\xi; q)], \quad 0 \leqslant \xi < +\infty, \ q \in [0, 1], \tag{20}$$

subject to the boundary conditions

$$\phi = 0, \quad \frac{\partial \phi}{\partial \xi} = 0, \quad \text{at } \xi = 0 \tag{21}$$

and

$$\frac{\partial \phi}{\partial \xi} = 1, \quad \text{as } \xi \to +\infty. \tag{22}$$

Note that $A_1(q; c_1)$ and $B_1(q; c_2)$ contain the convergence-control parameters c_1 and c_2, respectively. So, we have at most three unknown convergence-control parameters c_0, c_1 and c_2, which can be used to ensure the convergence of solutions series, as shown later.
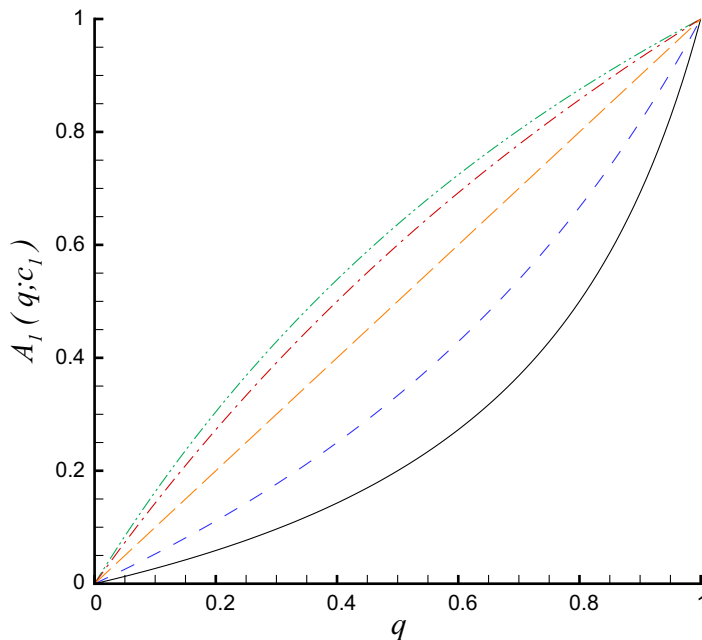

Fig. 1. Deformation function $A_1(q; c_1)$ defined by (17) and (18). Solid line: $c_1 = 3/4$; Dashed line: $c_1 = 1/2$, Long-dashed line: $c_1 = 0$; Dash-dotted line: $c_1 = -1/2$; Dash-dot-dotted line: $c_1 = -3/4$.

When $q = 0$, according to the definition (14) of $\mathscr{L}$ and the definition (13) of $F_0(\xi)$, it is obvious that

$$\phi(\xi; 0) = F_0(\xi). \tag{23}$$

When $q = 1$, according to the definition (6), the zeroth-order deformation equations (20)–(22) are equivalent to the original Eq. (11), provided

$$\phi(\xi; 1) = F(\xi). \tag{24}$$

Thus, as q increases from 0 to 1, the solution $\phi(\xi; q)$ varies (or deforms) continuously from the initial guess $F_0(\xi)$ to the solution $F(\xi)$ of Eq. (11). Obviously, $\phi(\xi; q)$ is determined by the auxiliary linear operator $\mathscr{L}$, the initial guess $F_0(\xi)$, and the convergence-control parameters c_0, c_1 and c_2. Note that we have great freedom to choose all of them. Assuming that all of them are so properly chosen that the Taylor series

$$\phi(\xi; q) = F_0(\xi) + \sum_{k=1}^{+\infty} F_k(\xi) q^k \tag{25}$$

exists and besides converges at $q = 1$, we have using (24) the homotopy-series solution

$$F(\xi) = F_0(\xi) + \sum_{k=1}^{+\infty} F_k(\xi), \tag{26}$$

where

$$F_m(\xi) = \frac{1}{m!} \frac{\partial^m \phi(\xi; q)}{\partial q^m}\bigg|_{q=0}.$$

Let G denote a function of $q \in [0, 1]$ and define the so-called mth-order homotopy-derivative [25]:

$$D_m[G] = \frac{1}{m!} \frac{\partial^m G}{\partial q^m}\bigg|_{q=0}. \tag{27}$$

Taking the above operator on both sides of the zeroth-order deformation equation (20) and the boundary conditions (21) and (22), we have the mth-order deformation equation

$$\mathscr{L}\left[F_m(\xi) - \sum_{k=1}^{m-1} \sigma_{m-k}(c_2) F_k(\xi)\right] = c_0 \sum_{k=0}^{m-1} \mu_{m-k}(c_1) \delta_k(\xi), \tag{28}$$

subject to the boundary conditions

$$F_m(0) = F'_m(0) = 0, \quad F'_m(+\infty) = 0, \tag{29}$$

where

$$\delta_k(\xi) = D_k \mathscr{N}[\phi(\xi; q)] = \frac{1}{k!} \frac{\partial^k \mathscr{N}[\phi(\xi; q)]}{\partial q^k}\bigg|_{q=0} = F'''_k(\xi) + \left(\frac{1}{2\lambda^2}\right) \sum_{j=0}^{k} F''_j(\xi) F_{k-j}(\xi), \tag{30}$$

and the coefficients $\mu_k(c_1)$ and $\sigma_k(c_2)$ are defined by (18) and (19), respectively. For details about the operator (27), please refer to Liao [25].

Let $F^*_m(\xi)$ denote a special solution of (28) and $\mathscr{L}^{-1}$ the inverse operator of $\mathscr{L}$, respectively. We have

$$F^*_m(\xi) = \sum_{k=1}^{m-1} \sigma_{m-k}(c_2) F_k(\xi) + c_0 \sum_{k=0}^{m-1} \mu_{m-k}(c_1) S_k(\xi), \tag{31}$$

where

$$S_k(\xi) = \mathscr{L}^{-1}[\delta_k(\xi)]. \tag{32}$$

The common solution reads

$$F_m(\xi) = F^*_m(\xi) + C_0 + C_1 \xi + C_2 e^{-\xi},$$

where the integral coefficients

$$C_1 = 0, \quad C_2 = \frac{dF^*_m}{d\xi}\bigg|_{\xi=0}, \quad C_0 = -F^*_m(0) - C_2,$$

are determined by the boundary conditions (29).

It should be emphasized that $F_m(\xi)$ contains at most three unknown convergence-control parameters c_0, c_1 and c_2, which determine the convergence region and rate of the homotopy-series solution (26). Obviously, if the convergence-control

parameters c_0, c_1 and c_2 are properly chosen, the homotopy-series solution (26) may converge fast. So, we should find out the good enough values of c_0, c_1 and c_2 so that the homotopy-series solution (26) converges fast enough.

In theory, at the mth-order of approximation, one can define the exact square residual error

$$\Delta_m = \int_0^{+\infty} \left(\mathcal{N} \left[\sum_{i=0}^{m} F_i(\xi) \right] \right)^2 d\xi. \tag{33}$$

Note that Δ_m contains at most three unknown convergence-control parameters c_0, c_1 and c_2, even at very high order of approximation. Obviously, the more quickly Δ_m decreases to zero, the faster the corresponding homotopy-series solution converges. So, at the given order of approximation m, the corresponding optimal values of the convergence-control parameters c_0, c_1 and c_2 are given by the minimum of Δ_m, corresponding to a set of three nonlinear algebraic equations

$$\frac{\partial \Delta_m}{\partial c_0} = 0, \quad \frac{\partial \Delta_m}{\partial c_1} = 0, \quad \frac{\partial \Delta_m}{\partial c_2} = 0. \tag{34}$$

However, it is a pity that the exact square residual error Δ_m defined by (33) needs too much CPU time to calculate even if the order of approximation is not very high, and thus is often useless in practice. To overcome this disadvantage, we will introduce a more efficient definition of the residual error to replace (33), which will be described in the following section in details.

3. Comparisons of different approaches

Note that we have at most three unknown convergence-control parameters c_0, c_1 and c_2. In case of $c_1 = c_2 = 0$, one has the plain deformation functions $A_1(q; c_1) = B_1(q; c_2) = q$, which was used by Liao [22] for Blasius problem and also by many other users [37,32,23,33,38–43,34,44–48,35,36,49] of the HAM. Here, we will give optimal homotopy-analysis approaches with different numbers of unknown convergence-control parameters, and compare them in details.

3.1. Optimal c_0 in case of $c_1 = c_2 = 0$

In this case, only one convergence-parameter c_0 is unknown. For given order of approximation M, the optimal value of c_0 is given by the minimum of Δ_m, corresponding to a nonlinear algebraic equation

$$\frac{d\Delta_M}{dc_0} = 0.$$

The curves of Δ_M versus c_0 at different order of approximation $M = 6$, 8 and 10 are shown in Fig. 2, which indicate that the optimal value of c_0 is about $-3/2$.

However, more and more CPU time is needed to calculate the exact residual error Δ_m, especially for large M, the order of approximation. For example, even in case of $c_1 = c_2 = 0$, it needs 68.13 s, 272.7 s and 1089.5 s to calculate the corresponding exact residual error (33) for $M = 6$, 8 and 10, respectively. It is found that, when there are more than one unknown parameters, the CPU time increases exponentially so that the exact residual error (33) is often useless in practice. Thus, to greatly decrease the CPU time, we use here the so-called averaged residual error defined by

$$E_m = \frac{1}{K} \sum_{j=0}^{K} \left[\mathcal{N} \left(\sum_{k=0}^{m} F_k(j\Delta x) \right) \right]^2, \tag{35}$$

where $\Delta x = 10/K$ and $K = 20$ for Blasius flow problem. The curves of the averaged residual error E_m versus c_0 indicate that the optimal value of c_0 is also about $-3/2$, as shown in Fig. 3. It is found that, as the order of approximation increases, the optimal value of c_0 given by the minimum of the averaged residual error (35) is more and more close to $-3/2$, as shown in Table 1. Thus, the averaged residual error E_m defined by (35) can give good enough approximation of the optimal convergence-control parameter. However, the CPU time to get the averaged residual error E_m is much less than that to calculate the exact residual error Δ_m in case of $m \geqslant 5$: it takes only 0.30 s, 1.11 s and 1.58 s to get the optimal c_0 of E_6, E_8 and E_{10}, respectively, i.e. only 0.44%, 0.41% and 0.15% CPU time when the exact definition (33) is used. Therefore, in the following part of this article, we will use the averaged residual error (35) to find the optimal values of the unknown convergence-control parameters. For the sake of impartial comparisons, E_{10} given by the 10th-order approximation is used in the whole paper to search for the unknown optimal convergence-control parameters.

In case of $c_1 = c_2 = 0$ there is only one unknown convergence-control parameter c_0, thus the optimal value of c_0 is determined by the minimum of E_{10}, corresponding to the nonlinear algebraic equation $E\prime_{10}(c_0) = 0$. Using the symbolic computation software *Mathematica*, we directly employ the command **Minimize** to get the optimal convergence-control parameter c_0. According to Table 1, E_{10} has its minimum value at $c_0 = -1.400$. By means of $c_0 = -7/5$, the value of $f''(0)$ converges much faster to 0.3320573 than the corresponding homotopy-series solution given by Liao [19] in case of $c_0 = -1$ and $c_1 = c_2 = 0$, as shown in Table 2, and the corresponding square averaged residual error $\sqrt{E_m}$ decreases much more quickly, as shown in Table 3. So, even the one-parameter optimal homotopy-analysis approach can give much better approximations.
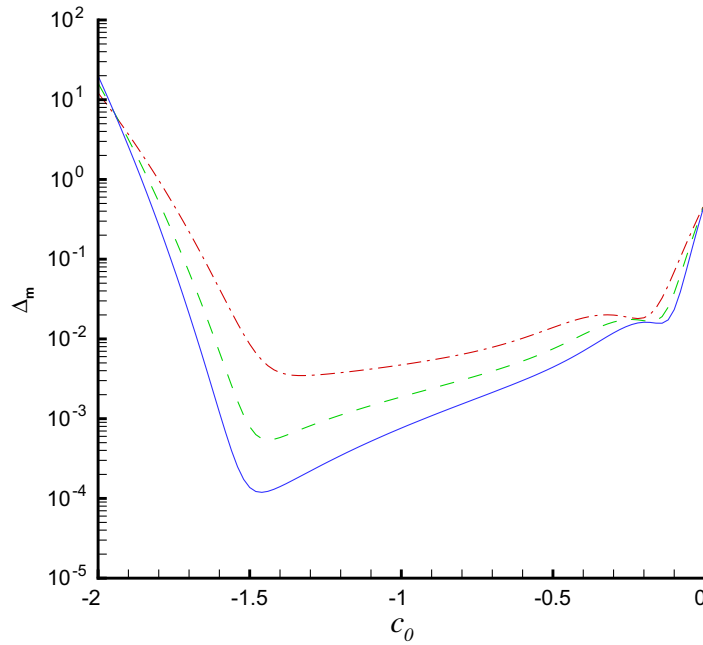
Fig. 2. Exact residual error Δ_m versus c_0 in case of $c_1 = c_2 = 0$. Solid line: 10th-order approximation; Dashed line: 8th-order approximation; Dash-dotted line: 6th-order approximation.

Fig. 3. Averaged residual error E_m versus c_0 in case of $c_1 = c_2 = 0$. Solid line: 10th-order approximation; Dashed line: 8th-order approximation; Dash-dotted line: 6th-order approximation.
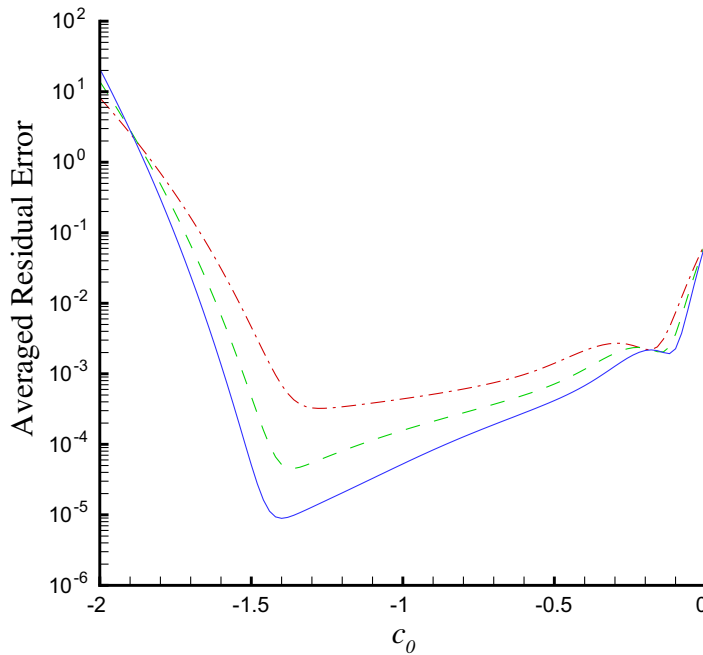
3.2. Optimal $c_1 = c_2$ in case of $c_0 = -1$

In Section 3.1, we show that the optimal convergence-control parameter $c_0 = -7/5$ in case of $c_1 = c_2 = 0$ gives a homotopy-series solution which converges much faster than that given by Liao [19] in case of $c_0 = -1$ and $c_1 = c_2 = 0$. Here, we investigate another one-parameter optimal approach in case of $c_0 = -1$ with the unknown $c_1 = c_2$.

Table 1
Optimal value of c_0 in case of $c_1 = c_2 = 0$.

m, order of approximation	Optimal value of c_0	Minimum value of E_m
6	−0.187	2.18×10^{-3}
8	−1.366	4.53×10^{-5}
10	−1.400	8.91×10^{-6}
12	−1.431	3.12×10^{-6}
14	−1.476	4.76×10^{-7}
16	−1.482	6.13×10^{-8}
18	−1.491	8.37×10^{-9}
20	−1.498	1.87×10^{-9}

Table 2
Comparison of $f''(0)$ given by different approaches.

Order of appr.	$c_1 = 0,\ c_2 = 0,$ $c_0 = -1$	$c_1 = 0,\ c_2 = 0,$ $c_0 = -7/5$	$c_1 = -2/5,\ c_2 = -2/5,$ $c_0 = -1$	$c_1 = 0.187,\ c_2 = 0.187,$ $c_0 = -1.723$	$c_1 = 0.165,\ c_2 = 0.107,$ $c_0 = -1.791$
5	0.2563899	0.2731356	0.2731356	0.2728099	0.3167887
10	0.3277556	0.3310852	0.3310852	0.3310700	0.3290419
15	0.3312560	0.3320500	0.3320500	0.3320498	0.3317650
20	0.3318513	0.3320453	0.3320453	0.3320456	0.3321357
25	0.3320037	0.3320520	0.3320520	0.3320520	0.3320844
30	0.3320403	0.3320564	0.3320564	0.3320564	0.3320580
35	0.3320519	0.3320572	0.3320572	0.3320572	0.3320559
40	0.3320554	0.3320573	0.3320573	0.3320573	0.3320570

Table 3
Comparison of the averaged residual error $\sqrt{E_m}$ given by different approaches.

Order of appr.	$c_1 = 0,\ c_2 = 0,$ $c_0 = -1$	$c_1 = 0,\ c_2 = 0,$ $c_0 = -7/5$	$c_1 = -2/5,\ c_2 = -2/5,$ $c_0 = -1$	$c_1 = 0.187,\ c_2 = 0.187,$ $c_0 = -1.723$	$c_1 = 0.165,\ c_2 = 0.107,$ $c_0 = -1.791$
5	6.03×10^{-3}	1.10×10^{-2}	1.10×10^{-2}	1.10×10^{-2}	1.22×10^{-2}
10	1.62×10^{-3}	6.68×10^{-4}	6.68×10^{-4}	6.68×10^{-4}	3.56×10^{-4}
15	5.13×10^{-4}	1.32×10^{-4}	1.32×10^{-4}	1.32×10^{-4}	1.27×10^{-4}
20	1.87×10^{-4}	1.43×10^{-5}	1.43×10^{-5}	1.42×10^{-5}	1.45×10^{-5}
25	5.05×10^{-5}	3.44×10^{-6}	3.44×10^{-6}	3.43×10^{-6}	1.02×10^{-6}
30	1.06×10^{-5}	6.32×10^{-7}	6.32×10^{-7}	6.27×10^{-7}	2.66×10^{-7}
35	3.43×10^{-6}	6.15×10^{-8}	6.15×10^{-8}	6.07×10^{-8}	5.42×10^{-8}
40	1.44×10^{-6}	2.90×10^{-9}	2.90×10^{-9}	2.86×10^{-9}	1.41×10^{-8}

In this case, E_{10} has the minimum 8.91×10^{-6} at the optimal value $c_1 = c_2 = -0.400$. It is found that the homotopy-approximations given by $c_0 = -1$ and $c_1 = c_2 = -2/5$ converges much faster than those given by Liao [19] in case of $c_0 = -1$ and $c_1 = c_2 = 0$, as shown in Tables 2 and 3. Besides, it is very interesting that the homotopy-approximations given by $c_0 = -1$ and $c_1 = c_2 = -2/5$ are exactly the same as those given by $c_0 = -7/5$ and $c_1 = c_2 = 0$ at *every* order of approximation! It is further found that the homotopy-approximations given by the following three cases:

(A) $c_0 = -3/2,\ c_1 = c_2 = 0$,
(B) $c_0 = -1,\ c_1 = c_2 = -1/2$,
(C) $c_0 = -4/3,\ c_1 = c_2 = -1/8$,

are also exactly the same at every order of approximation, and all of them converges much faster than the approximations given by Liao [19] in case of $c_0 = -1$ and $c_1 = c_2 = 0$. This illustrates that the second one-parameter optimal homotopy-analysis approach is as good as the first one mentioned in Section 3.1.

3.3. Optimal c_0 and $c_1 = c_2$

Let us consider the optimal approach with the two unknown convergence-control parameters c_0 and c_1 in case of $c_2 = c_1$. The corresponding averaged residual error E_{10} is now a function of both c_0 and c_1, which has the minimum 8.91×10^{-6} at the optimal values $c_0 = -1.723$ and $c_1 = c_2 = 0.187$. The corresponding homotopy-approximations converges much faster than those given by Liao [19] in case of $c_0 = -1$ and $c_1 = c_2 = 0$, as shown in Tables 2 and 3. Note that the square averaged residual

error $\sqrt{E_m}$ given by the two-parameter optimal homotopy-analysis approach decreases a little faster than that of the homotopy-approximations given by one-parameter optimal approaches in case of $c_0 = -7/5$, $c_1 = c_2 = 0$ or $c_0 = 0$, $c_1 = c_2 = -2/5$, although all of them give nearly the same $f''(0)$ at the same order of approximation.

3.4. Optimal c_0, c_1 and c_2 in case of $c_2 \neq c_1$

This gives a three-parameter optimal homotopy-analysis approach. The corresponding E_{10} is now a function of c_0, c_1 and c_2, which has the minimum 2.53×10^{-6} at the optimal values $c_0 = -1.791$, $c_1 = 0.165$ and $c_2 = 0.107$. The corresponding homotopy-series solution converges much faster than that given by Liao [19] in case of $c_0 = -1$ and $c_1 = c_2 = 0$, as shown in Tables 2 and 3. However, the homotopy-approximations given by the three-parameter optimal homotopy-analysis approach are a little worse than those given by one-parameter optimal homotopy-analysis approach in case of $c_0 = -7/5$, $c_1 = c_2 = 0$ and $c_0 = 0$, $c_1 = c_2 = -2/5$, and also by the two-parameter optimal homotopy-analysis approach in case of $c_0 = -1.723$ and $c_1 = c_2 = 0.187$, as shown in Tables 2 and 3. This is mainly because the averaged residual error E_m defined by (35) is an approximation of the exact residual error Δ_m defined by (33) so that the corresponding optimal values of the convergence-control parameters are also approximate, too. The corresponding 20th-order homotopy-analysis approximation of $f'(\eta)$ agrees quite well with the numerical ones, as shown in Fig. 4. In fact, all optimal homotopy-analysis approaches mentioned above give rather accurate result of $f'(\eta)$ in the whole region $0 \leqslant \eta < +\infty$.

Based on the above calculations for Blasius flow problem, we have the following conclusions:

(1) All optimal homotopy-analysis approaches can give much better approximations which converges much faster than those without optimal convergence-control parameters.
(2) Two-parameter optimal homotopy-analysis approaches can often give better homotopy-analysis approximations than one-parameter optimal approaches, although the modification might be not very obvious, as illustrated by the example in this article.
(3) The homotopy-analysis approximations given by three-parameter optimal approaches might be not obviously better than those given by one and two-parameter optimal homotopy-analysis approaches.

Thus, it is strongly suggested to use at first one or two-parameter optimal homotopy-analysis approaches, together with a properly defined averaged residual error like (35).

3.5. Homotopy-Padé acceleration

The so-called homotopy-Padé technique [20] was proposed to accelerate the convergence of the homotopy-series solution. Its idea is simple: one first applied the traditional Padé technique to (25) so as to get $[m, m]$ Padé approximant with
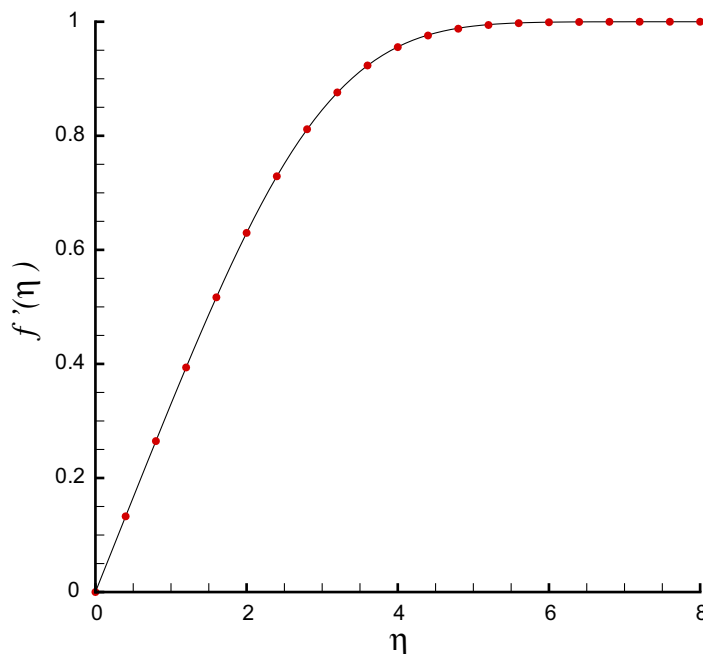

Fig. 4. Comparison of analytic approximation of $f'(\eta)$ with numerical ones. Symbols: numerical result; Solid line: 20th-order homotopy approximation given by the three-parameter optimal approach in case of $c_0 = -1.791, c_1 = 0.165$ and $c_2 = 0.107$.

Table 4
Comparison of $[m, m]$ homotopy-Padé approximations of $f''(0)$ given by different approaches.

m	$c_1 = 0,\ c_2 = 0,$ $c_0 = -1$	$c_1 = 0,\ c_2 = 0,$ $c_0 = -7/5$	$c_1 = -2/5,\ c_2 = -2/5,$ $c_0 = -1$	$c_1 = 0.187,\ c_2 = 0.187,$ $c_0 = -1.723$	$c_1 = 0.165,\ c_2 = 0.107,$ $c_0 = -1.791$
2	0.4752626	0.4752626	0.4752626	0.4752626	0.3068023
4	0.3446753	0.3446753	0.3446753	0.3446753	0.3262952
6	0.3326957	0.3326957	0.3326957	0.3326957	0.3315596
8	0.3320548	0.3320548	0.3320548	0.3320548	0.3320060
10	0.3320495	0.3320495	0.3320495	0.3320495	0.3320405
12	0.3320565	0.3320565	0.3320565	0.3320565	0.3320578
14	0.3320573	0.3320573	0.3320573	0.3320573	0.3320573
16	0.3320573	0.3320573	0.3320573	0.3320573	0.3320573
18	0.3320573	0.3320573	0.3320573	0.3320573	0.3320573
20	0.3320573	0.3320573	0.3320573	0.3320573	0.3320573

respect to the embedding parameter q, and then set $q = 1$. It is found that the so-called homotopy-Padé technique can greatly accelerate the convergence of homotopy-series solution in general, and besides the $[m, m]$ homotopy-Padé approximations are often *independent* of the convergence-control parameter c_0 in case of $c_1 = c_2 = 0$, as pointed out by Liao [20].

It is straightforward to apply the homotopy-Padé technique to accelerate the homotopy-series solution obtained by the optimal homotopy-analysis approaches mentioned above. In case of three unknown convergence-control parameters c_0, c_1 and c_2, the $[1, 1]$ homotopy-Padé approximation of $f''(0)$ reads

$$f''(0) \approx \frac{1728(c_2 - c_1) + 53c_0(c_1 - 1)}{864(c_2 - c_1) + 877c_0(c_1 - 1)}. \tag{36}$$

It is found that the homotopy-Padé technique indeed greatly accelerates the convergence of $f''(0)$ given by all optimal homotopy-analysis approaches mentioned above, as shown in Table 4. Especially, it is very interesting that all homotopy-Padé approximations of $f''(0)$ in case of $c_1 = c_2$ are the same, as shown in Table 4. Why? To explain it, we regard c_0 and c_1 as unknown parameters in case of $c_2 = c_1$, and calculate the 20th-order homotopy-analysis approximations by symbolic computation. In a surprise, it is found that *all* $[m, m]$ homotopy-Padé approximations of $f''(0)$ are *independent* of the three convergence-control parameters c_0, c_1 and c_2 in case of $c_1 = c_2$. This is also clear from the $[1, 1]$ homotopy-Padé approximation (36) of $f''(0)$. It seems that the homotopy-Padé technique might give better approximations even than the above-mentioned optimal homotopy-analysis approaches. However, it is a pity that we can not give a mathematical proof in general cases for this guess.

Note that, as shown in Table 4, the $[m, m]$ homotopy-Padé approximations of $f''(0)$ given by the three-parameter optimal homotopy-analysis approach in case of $c_0 = -1.791$, $c_1 = 0.165$ and $c_2 = 0.107$ are a little better than those given by other one and two-parameter optimal approaches in case of $c_1 = c_2$. And the homotopy-series solutions given by all optimal homotopy-analysis approaches mentioned above converge to the same result $f''(0) = 0.3320573$, which is more accurate than the result $f''(0) = 0.33206$ reported by Liao [22] in case of $c_0 = -1$ and $c_1 = c_2 = 0$.

4. The optimal approach in general cases

Note that the definition of the averaged residual error (35) and the above-mentioned optimal homotopy-analysis approaches have general meanings and thus can be easily applied to solve different types of equations with strong nonlinearity. Here, we give a brief description in general cases.

For a general nonlinear differential/integral equation

$$\mathcal{N}[u(\mathbf{r}, t)] = 0, \tag{37}$$

where $u(\mathbf{r}, t)$ is a unknown funciton, $\mathbf{r}$ and t denote respectively spatial and temporal independent variables, we can always choose a proper initial guess $u_0(\mathbf{r}, t)$ and a proper auxiliary linear operator $\mathcal{L}$ to construct the so-called zeroth-order deformation equation

$$[1 - B_{n'}(q; \mathbf{b})]\mathcal{L}[\Phi(\mathbf{r}, t; q) - u_0(\mathbf{r}, t)] = c_0 A_{m'}(q; \mathbf{a})\mathcal{N}[\Phi(\mathbf{r}, t; q)], \quad q \in [0, 1], \tag{38}$$

where $A_{m'}(q; \mathbf{a})$ is a deformation function with m' ($m' \geqslant 0$) unknown convergence-control parameters denotes by $c_1, c_2, \ldots, c_{m'}$, and $B_{n'}(q; \mathbf{b})$ is a deformation function with n' ($n' \geqslant 0$) unknown convergence-control parameters denoted by $c_{m'+1}, c_{m'+2}, \ldots, c_{m'+n'}$, respectively. Here,

$$\mathbf{a} = \{c_1, c_2, \cdots, c_{m'}\}, \quad \mathbf{b} = \{c_{m'+1}, c_{m'+2}, \ldots, c_{m'+n'}\}.$$

Write $\kappa = m' + n'$. Then, we have $\kappa + 1$ unknown convergence-control parameters $c_0, c_1, \ldots, c_\kappa$.

Assuming that the initial guess $u_0(\mathbf{r}, t)$, the auxiliary linear operator $\mathcal{L}$, and the $\kappa + 1$ convergence-control parameters $c_0, c_1, \ldots, c_\kappa$ are so properly chosen that the Taylor series

$$\Phi(\mathbf{r}, t; q) = u_0(\mathbf{r}, t) + \sum_{n=1}^{+\infty} u_n(\mathbf{r}, t) q^n \tag{39}$$

converges at $q = 1$, we have the homotopy-series solution

$$u(\mathbf{r}, t) = u_0(\mathbf{r}, t) + \sum_{n=1}^{+\infty} u_n(\mathbf{r}, t). \tag{40}$$

Substituting the series (39) into the zeroth-order deformation equation (38) and then equating the coefficients of the like-power of the embedding parameter q, we have the high-order deformation equation.[4]

$$\mathscr{L}\left[u_m(\mathbf{r}, t) - \sum_{k=1}^{m-1} \sigma_{m-k}(\mathbf{b}) u_k(\mathbf{r}, t)\right] = c_0 \sum_{k=0}^{m-1} \mu_{m-k}(\mathbf{a}) \delta_k(\mathbf{r}, t), \tag{41}$$

where

$$\delta_k(\mathbf{r}, t) = \left(\frac{1}{k!} \frac{\partial^k}{\partial q^k} \mathscr{N}\left[\sum_{n=0}^{+\infty} u_n(\mathbf{r}, t) q^n\right]\right)\Bigg|_{q=0} \tag{42}$$

and $\mu_k(\mathbf{a})$, $\sigma_k(\mathbf{b})$ are coefficients of the Taylor series

$$A_{m'}(q; \mathbf{a}) = \sum_{k=1}^{+\infty} \mu_k(\mathbf{a}) q^k, \quad B_{n'}(q; \mathbf{b}) = \sum_{k=1}^{+\infty} \sigma_k(\mathbf{b}) q^k. \tag{43}$$

The special solution $u_m^*(\mathbf{r}, t)$ of (41) is given by

$$u_m^*(\mathbf{r}, t) = \sum_{k=1}^{m-1} \sigma_{m-k}(\mathbf{b}) u_k(\mathbf{r}, t) + c_0 \sum_{k=0}^{m-1} \mu_{m-k}(\mathbf{a}) S_k(\mathbf{r}, t), \tag{44}$$

where

$$S_k(\mathbf{r}, t) = \mathscr{L}^{-1}[\delta_k(\mathbf{r}, t)] \tag{45}$$

and $\mathscr{L}^{-1}$ is the inverse operator of $\mathscr{L}$.

To avoid time-consuming computation for the exact square residual error, at the mth-order of approximation, we define a kind of averaged residual error E_m in a similar way to (35). Note that E_m contains $\kappa + 1$ unknown convergence-control parameters $c_0, c_1, \ldots, c_\kappa$, whose optimal values are given by the minimum of E_m, corresponding to a set of $\kappa + 1$ nonlinear algebraic equations

$$\frac{\partial E_m}{\partial c_j} = 0, \quad 0 \leqslant j \leqslant \kappa.$$

So, the above approach is called the $(\kappa + 1)$-parameter optimal homotopy-analysis approach.

In general, the above-mentioned optimal homotopy-analysis approaches can greatly modify the convergence of homotopy-series solution. And the optimal homotopy-analysis approaches with one or two unknown convergence-control parameters are strongly suggested: an optimal approach with too many unknown convergence-control parameters are not efficient computationally. Besides, the homotopy-Padé technique can be used to get even better approximations in most cases.

Note that the nonlinear operator $\mathscr{N}$ in (37) is rather general so that the above-mentioned optimal homotopy-analysis approach can be employed to different types of equations with strong nonlinearity, such as ordinary/partial differential equations, integral equations, differential-integral equations, time-delayed equations, and so on.

5. Discussions and conclusions

In this article, the famous Blasius equation in fluid mechanics is used to describe an optimal homotopy-analysis approach for highly nonlinear problems. With the deformation functions defined by (17), our optimal homotopy-analysis approach contains at most only three unknown convergence-control parameters c_0, c_1 and c_2. To increase the computational efficiency, an averaged residual error like (35) is defined, which can give good approximations of the optimal convergence-control parameters of the exact residual error (33), as shown in Fig. 3 and Table 1. It is found that *all* optimal homotopy-analysis approaches may give much better approximations, which converge much faster than those given by Liao [19] in case of $c_0 = -1$ and $c_1 = c_2 = 0$. So, it is strongly suggested to use at least one optimal convergence-control parameter to accelerate the convergence of homotopy-series solution. In general, two optimal convergence-control parameters c_0 and c_1 (with $c_2 = c_1$) may give good enough approximations. However, because the averaged residual error (35) is a kind of approximation of the exact residual error (33), the three-parameter optimal homotopy-analysis approach might not give better approximations than one or

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[4] Using the so-called mth-order homotopy-derivative defined by (27), one can obtain exactly the same high-order deformation equation.

two-parameter optimal approaches, as shown in this paper for Blasius flow problem. Considering the fact that much CPU time is needed when the approximations have more than three unknown parameters, it is strongly suggested to use at first one or two optimal convergene-control parameters in the homotopy-analysis approach. Besides, it is found that the homotopy-Padé technique can greatly accelerate the convergence of approximations given by all optimal homotopy-analysis approach mentioned in this paper. Therefore, the homotopy-Padé technique is strongly suggested to use, if possible.

The optimal homotopy-analysis approach mentioned above is based on the deformation function with a few unknown convergence-control parameters. For example,

$$A_1(q;a) = (1-a)\sum_{n=1}^{+\infty} a^{n-1}q^n, \quad |a| < 1. \tag{46}$$

Note that $A_1(q;0) = q$ is only a special case of it, although widely used today. There exist an infinite number of deformation functions satisfying the properties (6) and (7). For example, we can define the following one-parameter deformation function:

$$\widehat{A}_1(q;b) = \frac{1}{\zeta(b)}\sum_{n=1}^{+\infty} \frac{q^n}{n^b}, \quad b > 1, \tag{47}$$

where $\zeta(b)$ is Riemann zeta function, and $b > 1$ is a convergence-control parameter. We call $A_1(q;a)$ and $\widehat{A}_1(q;b)$ the first and second-type of one-parameter deformation functions, respectively.

Any two different deformation functions may create a new one. For example

$$A_2(q;a,b) = A_1(q;a)\widehat{A}_1(q;b) \tag{48}$$

gives a two-parameter deformation function with two convergence-control parameters a and b. Currently, Zhao and Wong [50] suggested a kind of deformation function which can define the $(m+1)$-parameter deformation function

$$A_{m+1}(q;a,\mathbf{b}) = \frac{a\,A_m(q;\mathbf{b})}{1+(a-1)A_m(q;\mathbf{b})}, \tag{49}$$

where $a \neq 0$ is a convergence-control parameter and $A_m(q;\mathbf{b})$ is a m-parameter deformation function with m convergence-control parameters $\mathbf{b} = \{b_1, b_2, \ldots, b_m\}$. In theory, given any a convergent series

$$S = \sum_{n=1}^{+\infty} s_n,$$

we can always define a corresponding deformation function

$$A(q;\mathbf{s}) = \frac{1}{S}\sum_{n=1}^{+\infty} s_n q^n,$$

where

$$\mathbf{s} = \{s_1, s_2, \ldots\}.$$

Note that it is straightforward to apply the basic ideas of the optimal homotopy-analysis approach mentioned in this paper for all different types of deformation functions. Thus, it is very interesting to study whether or not there exists the "best" deformation function among all of these possible ones, which gives the fastest convergent homotopy-series solution.

Note that the auxiliary linear operator $\mathscr{L}$ defined by (14) is rather different from the linear term of the original governing equation (11). Using this auxiliary linear operator, it is convenient to solve the high-order deformation equations, and more importantly, it is much easier to ensure that the homotopy-series solution converges even at infinity $\eta \to +\infty$. This is mainly because the homotopy-analysis method provides us great freedom to choose the auxiliary linear operator $\mathscr{L}$. Besides, the optimal convergence-control parameters guarantee the fast convergence of the homotopy-series solution. Thus, the example in this paper illustrates that the homotopy-analysis method indeed satisfies the two standard (a) and (b) mentioned at the beginning. This is the advantage of the homotopy analysis method (HAM), compared to perturbation techniques and other nonperturbation methods. It is the reason why the HAM is valid for different types of highly nonlinear problems.

Acknowledgements

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