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A new branch of solutions of boundary-layer flows over an impermeable stretched plate

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Abstract

The boundary-layer flows over a stretched impermeable wall are solved by means of an analytic technique, namely the homotopy analysis method. Two branches of solutions are found. The first agrees well with numerical results given by Banks in 1983. The second branch of solutions can be divided into two parts. One is mathematically equivalent to a branch of solutions of Cheng and Minkowycz's equation reported by Ingham and Brown in 1986, the other is new and has never been reported. Different from the first branch of solutions, the second branch of solutions shows reversed velocity flows. It is found that the difference of skin frictions of the two branches of solutions is rather small, even when the corresponding profiles of velocity are clearly distinct. Thus, from a practical point of view, we need not worry about large variations of the skin friction on the wall when the profile of the velocity changes from one to the other.

Keywords: Self-similarity; Boundary-layer; Stretched wall; Impermeable surface; Multiple solution; Skin friction; Homotopy analysis method

1. Introduction

Investigation of boundary-layer flows of incompressible fluid over a stretched surface has many important applications in engineering, such as aerodynamic extrusion of plastic sheets, boundary layer flows along liquid film in condensation processes, cooling of a metallic plate in a cooling bath, and applications in the glass and polymer industries. The investigations were made by many researchers, such as Sakiadis [1], Crane [2], Banks [3], Banks and Zaturska [4], Grubka and Bobba [5], Ali [6] for the impermeable plate, and Erickson et al. [7], Gupta and Gupta [8], Chen and Char [9], Chaudhary et al. [10], Elbashbeshy [11], Magyari and Keller [12] for the permeable plate. Using numerical methods, Banks [3] gave a kind of solutions without velocity reversed flows, but did not find dual solutions for impermeable stretched surfaces.

Consider the boundary layer viscous flow over a stretched impermeable plate [2,3], governed by

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2}$$

subject to the boundary conditions

$$u=a(x+b)^{\lambda},\quad v=0\quad \text{at } y=0, \tag{3}$$

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Nomenclature

a, b	physical parameter related with stretched surface
f, F	reduced stream functions
$\mathscr{L}$	auxiliary linear operator
$\mathscr{N}$	non-linear operator
q	embedding parameter
u, v	velocity components along x and y directions, respectively
x, y	Cartesian coordinates along the plate and normal to it, respectively
w	reduced stream functions
Greek symbols	
β	parameter defined by (8)
γ	parameter defined by (19)
δ	parameter defined by Eq. (13)
λ	physical parameter related with stretched surface
ξ, η, ζ	similarity variables
ν	kinematic viscosity
τ_w	skin friction on the stretched surface
ϕ	non-dimensional stream function
Subscript	
∞	ambient temperature condition
Superscript	
$'$	differentiation with respect to similarity variable

$$u \to 0 \quad \text{as } y \to +\infty, \tag{4}$$

where (x, y) denotes the Cartesian coordinates along the plate and normal to it, u, v are velocity components of fluid in the x and y directions, ν is the kinematic viscosity, a, b and λ are parameters related to the surface stretching speed, respectively. Let ψ denote the stream-function. Under the transformation

$$\psi = a\sqrt{\frac{\nu}{a(1+\lambda)}}(x+b)^{\frac{\lambda+1}{2}}F(\xi),$$
$$\xi = \sqrt{\frac{a(1+\lambda)}{\nu}}(x+b)^{\frac{\lambda-1}{2}}y, \tag{5}$$

where $a \neq 0$ and $a(1+\lambda) > 0$, the above equations become

$$F'''(\xi) + \frac{1}{2}F(\xi)F''(\xi) - \beta F'^2(\xi) = 0, \tag{6}$$

subject to the boundary conditions

$$F(0) = 0, \quad F'(0) = 1, \quad F'(+\infty) = 0, \tag{7}$$

where

$$\beta = \frac{\lambda}{1+\lambda}. \tag{8}$$

Since $a(1+\lambda) > 0$, we have $a > 0$ when $\lambda > -1$, corresponding to $-1 < \beta \leqslant 1$; and $a < 0$ when $\lambda < -1$, corresponding to $\beta > 1$. When, $a > 0$ and $-1 < \beta \leqslant 1$, the stretching velocity of the surface is positive. However, when $a < 0$ and $\beta > 1$, the stretching velocity is negative, corresponding to the so-called backward boundary layer that has physical meanings, as mentioned by Goldstein [13].

From (5), we deduce the velocity of the fluid

$$u(x, y) = a(x+b)^{\lambda}F'(\xi), \tag{9}$$

$$v(x, y) = -\frac{1}{2}\sqrt{a(1+\lambda)\nu}(x+b)^{\frac{\lambda-1}{2}}[F(\xi) + (2\beta-1)\xi F'(\xi)]. \tag{10}$$

The entrainment velocity of the fluid at $\xi \to +\infty$ is

$$v(x, +\infty) = -\frac{1}{2}\sqrt{a(1+\lambda)\nu}(x+b)^{\frac{\lambda-1}{2}}F(+\infty) \tag{11}$$

and the skin friction on the stretched surface becomes

$$\tau_w = \rho\nu\frac{\partial u}{\partial y}\bigg|_{y=0} = a\rho\sqrt{a(1+\lambda)\nu}(x+b)^{\frac{3\lambda-1}{2}}F''(0). \tag{12}$$

Using a numerical method, Banks [3] gave a branch of solutions for $-1 < \beta < +\infty$. All of Banks' solutions have the property $F'(\xi) > 0$ for $0 \leqslant \xi < +\infty$. As pointed by Chaudhary et al. [10], Magyari and Keller [14], and reproduced by Liao and Pop [15], the considered problem is mathematically equivalent to the steady free convection flow over a vertical semi-infinite plate that is embedded in a fluid saturated porous medium, described by the famous Cheng and Minkowycz's equation [16,17], if and *only* if $\lambda > -1/2$, corresponding to $-1 < \beta \leqslant 1$. It is easy to verify that the first branch of solutions reported by Ingham and Brown [17] is indeed mathematically equivalent to those given by Banks [3] in a *finite* region $-1 < \beta \leqslant 1$. Note that the two problems are not *completely* equivalent even mathematically. Ingham and Brown [17] proved that there exist no solutions corresponding to forward boundary layer flows for the Cheng and Minkowycz's equation when $\lambda \leqslant -1$, corresponding to $\beta > 1$. However, Banks [3] found numerical results even for $\beta = 200$. It is interesting that Ingham and Brown [17] reported a second branch of solution for $1 \leqslant \lambda < +\infty$, corresponding to $1/2 \leqslant \beta < 1$. According to the above-mentioned equivalence between the two problems, Eqs. (6) and (7) should have multiple solutions at least in the region $-1/2 < \beta \leqslant 1$. But, Banks [3] did not mention such kind of solutions. Employing the homotopy analysis method [18–20], a new analytic technique for nonlinear problems, Liao and Pop [15] successfully found the first branch of solutions which agree well with Banks' numerical results, but equally failed to find the second ones. And to the best of our knowledge, dual solutions of the boundary-layer flows over a stretched impermeable wall for $\beta > 1$ have not been reported.

In this paper, a new branch of solutions is found for $1/2 < \beta < +\infty$ by means of the homotopy analysis method [18–20], which has been successfully applied to solve many nonlinear problems [21–26,28,27]. In a finite region $1/2 \leqslant \beta \leqslant 1$, this kind of branch of solution is mathematically equivalent to the second branch of solutions of the Cheng and Minkowycz's equation given by Ingham and Brown [17]. However, to the best of our knowledge, such kind of solution for $1 < \beta < +\infty$ has been never reported. This new branch of solutions might deepen our understanding for such kind of flows and widen its applications in engineering.

2. Mathematical formulations

2.1. Asymptotic property

According to the boundary condition $F'(+\infty) = 0$, it is straightforward to define

$$\delta = F(+\infty). \tag{13}$$

It is well-know that most boundary-layer flows have exponential property at infinity. Thus, it is natural to express $F(\xi)$ as

$$F(\xi) = \delta + \sum_{j=1}^{+\infty} A_j \exp(-j\mu\xi),$$

where $\mu > 0$ is the spatial-scale parameter and A_j is a coefficient. When $\xi \to +\infty$, the dominant term of the above expression is

$$F(\xi) \approx \delta + A_1 \exp(-\mu\xi).$$

Substituting it into Eq. (6), we have

$$-\frac{A_1\mu^2}{2}(2\mu - \delta)\exp(-\mu\xi) - \frac{A_1^2\mu^2}{2}\exp(-2\mu\xi) + \cdots = 0, \quad \xi \to +\infty.$$

To enforce that the dominant term of the above equation is zero, we set

$$\mu = \delta/2, \tag{14}$$

which enforces

$$\delta > 0. \tag{15}$$

Thus, under the transformation

$$F(\zeta) = \delta[1 - w(\zeta)], \quad \zeta = \left(\frac{\delta}{2}\right)\xi, \tag{16}$$

Eq. (6) becomes

$$w'''(\zeta) + [1 - w(\zeta)]w''(\zeta) + 2\beta w'^2(\zeta) = 0, \quad \zeta \in [0, +\infty), \tag{17}$$

subject to the boundary conditions

$$w(0) = 1, \quad w'(0) + 2\gamma = 0, \quad w'(+\infty) = 0, \quad w(+\infty) = 0, \tag{18}$$

where

$$\gamma = \frac{1}{\delta^2} \tag{19}$$

depends on β. Note that γ does not appear in Eq. (17). If β is given, then the unknown γ must be determined. Similarly, if γ is given, we should regard β as an unknown parameter.

In the frame of the homotopy analysis method [18–20], Eqs. (17) and (18) can be solved by three different approaches, as described below.

2.2. First approach for given β

2.2.1. Zeroth-order deformation equation

Obviously, $w(\zeta)$ can be expressed by the base functions

$$\{\exp(-n\zeta) \mid n \geqslant 1\}$$

in the form

$$w(\zeta) = \sum_{j=1}^{+\infty} A_j \exp(-j\zeta), \tag{20}$$

where A_j is a coefficient. This provides us with the so-called Rule of Solution expression for $w(\zeta)$, as named by Liao [18,20], which plays an important role in the frame of the homotopy analysis method.

For given β, γ is unknown. Let γ_0 denote the initial approximation of γ, where γ_0 is unknown. Under the Rule of Solution Expression (20) and using the boundary conditions (18), it is straightforward to choose

$$w_0(\zeta) = 2(1 - \gamma_0)\exp(-\zeta) + (2\gamma_0 - 1)\exp(-2\zeta) \tag{21}$$

as the initial approximation of $w(\zeta)$. Under the Rule of Solution Expression (20) and with the aid of the governing equation (17), we are led to choose the auxiliary linear operator

$$\mathscr{L}[\phi(\zeta;q)] = \frac{\partial^3\phi(\zeta;q)}{\partial\zeta^3} + 2\frac{\partial^2\phi(\zeta;q)}{\partial\zeta^2} - \frac{\partial\phi(\zeta;q)}{\partial\zeta} - 2\phi(\zeta;q). \tag{22}$$

It possesses the property

$$\mathscr{L}[C_1\exp(-\zeta) + C_2\exp(-2\zeta) + C_3\exp(\zeta)] = 0 \tag{23}$$

for any constants C_1, C_2, and C_3. Furthermore, Eq. (17) suggests to define the nonlinear operator

$$\mathscr{N}[\phi(\zeta;q)] = \frac{\partial^3\phi(\zeta;q)}{\partial\zeta^3} + [1-\phi(\zeta;q)]\frac{\partial^2\phi(\zeta;q)}{\partial\zeta^2} + 2\beta\left[\frac{\partial\phi(\zeta;q)}{\partial\zeta}\right]^2. \tag{24}$$

Let $q \in [0,1]$ denote an embedding parameter, $\hbar \neq 0$ an auxiliary parameter, and $H(\zeta) \neq 0$ an auxiliary function. Using above definitions, we construct the zeroth-order deformation equation

$$(1-q)\mathscr{L}[\phi(\zeta;q) - w_0(\zeta)] = \hbar H(\zeta)\mathscr{N}[\phi(\zeta;q)], \tag{25}$$

subject to the boundary conditions

$$\phi(0;q) = 1, 2\Gamma(q) + \left.\frac{\partial\phi(\zeta;q)}{\partial\zeta}\right|_{\zeta=0} = 0, \ \lim_{\zeta\to+\infty}\frac{\partial\phi(\zeta;q)}{\partial\zeta} = 0. \tag{26}$$

When $q = 0$ and $q = 1$, we have from the zeroth-order deformation equations (25) and (26) that

$$\phi(\zeta;0) = w_0(\zeta), \Gamma(0) = \gamma_0, \tag{27}$$

and

$$\phi(\zeta;1) = w(\zeta), \Gamma(1) = \gamma, \tag{28}$$

respectively. Defining

$$w_n(\zeta) = \frac{1}{n!}\left.\frac{\partial^n\phi(\zeta;q)}{\partial q^n}\right|_{q=0}, \quad \gamma_n = \frac{1}{n!}\left.\frac{\partial^n\Gamma(q)}{\partial q^n}\right|_{q=0}$$

and expanding $\phi(\zeta;q)$ and $\Gamma(q)$ in Taylor series with respect to the embedding parameter q, we have

$$\phi(\zeta;q) = \phi(\zeta;0) + \sum_{n=1}^{+\infty} w_n(\zeta)q^n, \tag{29}$$

$$\Gamma(q) = \Gamma(0) + \sum_{n=1}^{+\infty}\gamma_n q^n. \tag{30}$$

Assuming that $\hbar$ and $H(\zeta)$ are properly chosen so that the above series converge at $q = 1$, we have from (27) and (28) that

$$w(\zeta) = w_0(\zeta) + \sum_{n=1}^{+\infty} w_n(\zeta), \tag{31}$$

$$\gamma = \gamma_0 + \sum_{n=1}^{+\infty}\gamma_n. \tag{32}$$

2.2.2. High-order deformation equation

For the sake of simplicity, define the vector

$$\vec{w}_m(\zeta) = \{w_0(\zeta), w_1(\zeta), w_2(\zeta), \ldots, w_m(\zeta)\}.$$

Differentiating the zeroth-order deformation equations (25) and (26) n times with respect to the imbedding parameter q, then setting $q = 0$, and finally dividing by $n!$, we have the nth-order deformation equation

$$\mathscr{L}[w_n(\zeta) - \chi_n w_{n-1}(\zeta)] = \hbar H(\zeta)R_n[\vec{w}_{n-1}(\zeta), \gamma_{n-1}], \tag{33}$$

subject to the boundary conditions

$$w_n(0) = 0, \ w_n'(0) + 2\gamma_n = 0, \ w_n'(+\infty) = 0, \tag{34}$$

under the definitions

$$R_n[\vec{w}_{n-1}(\zeta), \gamma_{n-1}] = w_{n-1}'''(\zeta) + w_{n-1}''(\zeta) + \sum_{i=0}^{n-1}\left[2\beta w_i'(\zeta)w_{n-1-i}'(\zeta) - w_i''(\zeta)w_{n-1-i}(\zeta)\right], \tag{35}$$

and

$$\chi_k = \begin{cases} 0, & k \leqslant 1, \\ 1, & k > 1. \end{cases} \tag{36}$$

Due to the boundary conditions (34), $w_k(\zeta)$ contains the unknown γ_k for $k = 0, 1, 2, 3, \ldots$. Thus, the term $R_n[\vec{w}_{n-1}(\zeta), \gamma_{n-1}]$ contains the unknown γ_{n-1}. Note that both $w_n(\zeta)$ and γ_{n-1} are unknown, but we have only Eqs. (33) and (34) for $w_n(\zeta)$. Thus, the problem is not closed and an additional algebraic equation is needed to determine γ_{n-1}.

Substituting (21) into (35), we have

$$R_1[w_0(\zeta), \gamma_0] = \sum_{j=2}^{4} B_{1,j}(\gamma_0)\exp(-j\zeta),$$

where $B_{1,j}(\gamma_0)$ is a coefficient dependent upon γ_0. According to the Rule of Solution Expression (20), the auxiliary function $H(\zeta)$ must have the form

$$H(\zeta) = \exp(\kappa\zeta),$$

where κ is an integer. So, we have

$$H(\zeta)R_1[\vec{w}_0(\zeta), \gamma_0] = \sum_{i=2}^{4} B_{1,i}(\gamma_0)\exp[-(\mathrm{i}-\kappa)\zeta]. \tag{37}$$

When $\kappa \geqslant 1$, the term $\exp(-\zeta)$ appears on the right-hand side of Eq. (33). Thus, according to (23), the corresponding solution $w_1(\zeta)$ contains

$$\zeta\exp(-\zeta),$$

which, however, disobeys the Rule of Solution Expression (20). When $\kappa \leqslant -2$, the solution of Eq. (33) does not contain the term $\exp(-3\zeta)$, and this disobeys the Rule of Coefficient Ergodicity, i.e. all coefficients in the solution expression (20) can be modified to ensure the completeness of the set of the base functions, as mentioned by Liao ([18], p. 21). When $\kappa = -1$, there does not exist an algebraic equation to determine γ_0 and therefore the problem is not closed. This however disobeys the Rule of Solution Existence described by Liao ([18, p. 21]). When $\kappa = 0$, i.e.

$$H(\zeta) = 1, \tag{38}$$

we can set

$$B_{1,2}(\gamma_0) = 0, \tag{39}$$

which provides us with an algebraic equation to determine the unknown γ_0. And in this way, the Rule of Solution Expression, the Rule of Coefficient Ergodicity, and the Rule of Solution Existence, are satisfied. Using (21) and (39) and imposing henceforth $\kappa = 0$ in (37), we have the algebraic equation

$$(2\beta - 1)\gamma_0^2 - 4\beta\gamma_0 + 2\beta = 0, \tag{40}$$

which has two different positive solutions

$$\gamma_0 = \frac{\sqrt{2\beta}}{\sqrt{2\beta}+1}, \quad \beta \geqslant 0, \tag{41}$$

and

$$\gamma_0 = \frac{\sqrt{2\beta}}{\sqrt{2\beta}-1}, \quad \beta > 1/2, \tag{42}$$

where the range of β is determined by the definition (19). Each of the above expressions corresponds to a branch of solutions.

It is found that, when $H(\zeta) = 1$, we always have

$$R_n[\vec{w}_{n-1}(\zeta), \gamma_{n-1}] = \sum_{i=2}^{2n+2} B_{n,i}(\gamma_{n-1}) \exp[-i\zeta]. \tag{43}$$

Thus, in general, we can always obtain γ_{n-1} by solving the algebraic equation

$$B_{n,2}(\gamma_{n-1}) = 0, \tag{44}$$

It is found that the above algebraic equation is always linear when ($n \geqslant 2$), and the solution $w_n(\zeta)$ obeys the Rule of Solution Expression (20). In this way, it is convenient to solve the linear high-order deformation equations (33) and (34) by means of a symbolic software such as Mathematica.

2.2.3. Multiple solutions

For a given β, the two different values of γ_0, given by (41) and (42), correspond to the two different values of γ and the two different solutions of $w(\zeta)$, respectively. For example, let us first consider the case of $\beta = 1$. For each γ_0, we can obtain

$$w_1(\zeta), \gamma_1, w_2(\zeta), \gamma_2, \ldots,$$

successively. Obviously, it is important to ensure that the series (31) and (32) are convergent. Fortunately, there exists another auxiliary parameter $\hbar$, which can control the convergence of these series.

Note that γ is a function of $\hbar$. The so-called $\hbar$-curves (see Liao [18]) of γ at the 10th-order of approximation are as shown in Fig. 1. Obviously, the series of γ converges when $\hbar = -1/2$ for the first branch of solutions, or when $\hbar = -3/4$ for the second. This is indeed true,

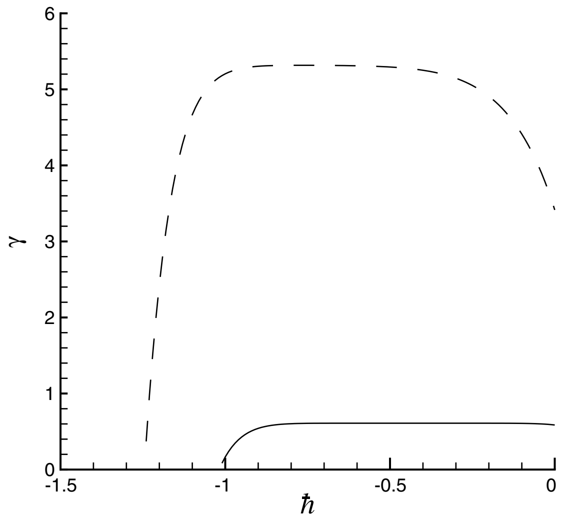

Fig. 1. $\hbar$-Curves of γ at the 10th-order of approximations when $\beta = 1$: (solid line) for the first branch of solution when $\gamma_0 = 2 - \sqrt{2}$ and (dashed line) for the second branch of solution when $\gamma_0 = 2 + \sqrt{2}$.

as shown in Tables 1 and 2. Besides, much more accurate results can be obtained by means of the so-called homotopy-Padé technique [18,23], as shown in Tables 3 and 4. The corresponding series of $w(\zeta)$ also converge to the numerical results given by Runge–Kutta's method using the analytic results of $F''(0)$, as shown in Fig. 2.

Thus, when $\beta = 1$, our first approach gives two different solutions, corresponding to

$$F(+\infty) = 1.2807737812, \quad F''(0) = -0.9063755237$$

and

$$F(+\infty) = 0.4336537219, \quad F''(0) = -0.9133389388,$$

respectively. Our first branch of solution agrees well with Banks' numerical results [3]. As shown in Fig. 2, the second branch of solution shows in some region reversed velocity flows. The second one was not reported for the stretched impermeable wall. Note that, although there exist obvious differences between $F(+\infty)$ of the two solutions, the difference between $F''(0)$ is small. This might be the reason why the second branch of solution was not found by the shooting method.

As mentioned before, when $\beta = 1$, the above-mentioned second branch should be mathematically equivalent to the corresponding second branch of solutions of the Cheng and Minkowycz's equation at $\lambda \to +\infty$. We compare the two solutions and find that this is indeed true. This verifies the validity of our approach. And our approach is valid for other values of β, as shown later. So, different from Banks [3] and Liao and Pop [15], we successfully find the two branches of solutions for the boundary-layer flows over a stretched impermeable plate.

In general, for given β ($\beta > 0$), we can find the first branch of solution ($0 \leqslant \beta < +\infty$) by means of (41) and the second one ($1/2 < \beta < +\infty$) by (42). In each case, the convergent result can be obtained by choosing a proper $\hbar$ according to the corresponding $\hbar$-curves of γ. Besides, the homotopy-Padé technique can be applied to accelerate the convergence of the solution series, when necessary. The values of $F(+\infty)$ and $F''(0)$ of the two branches of solutions are listed in Tables 5 and 6. It is found that, different from the first branch of solutions, the second branch of solutions shows reversed velocity flows in some regions that become larger and larger as β tends to 1/2, as shown in Fig. 3. The second branch of solutions for $1/2 < \beta \leqslant 1$ is mathematically equivalent to the Cheng and Minkowycz's equation. However, it should be emphasized that the second branch of solution for $1 < \beta < +\infty$ has never been reported, to the best of our knowledge.

In summary, by means of the above approach, we not only find the first branch of the solutions obtained by Banks [3], but also the second branch for $1/2 < \beta < +\infty$. Although the second branch of solutions for $1/2 < \beta \leqslant 1$ is mathematically equivalent to the second branch of solutions found by Ingham and Brown

Table 1
Approximations of γ, $F(+\infty)$, and $F''(0)$ of the first branch of solutions when $\beta = 1$, $\gamma_0 = 2 - \sqrt{2}$ and $\hbar = -1/2$

Order of approximation	γ	$F(+\infty)$	$F''(0)$
5	0.60957	1.28083	−0.90621
10	0.60961	1.28077	−0.90638
20	0.60961	1.28077	−0.90638
30	0.60961	1.28077	−0.90638
40	0.60961	1.28077	−0.90638

Table 2
Approximations of γ, $F(+\infty)$, and $F''(0)$ of the second branch of solutions when $\beta = 1$, $\gamma_0 = 2 + \sqrt{2}$ and $\hbar = -3/4$

Order of approximation	γ	$F(+\infty)$	$F''(0)$
10	5.31532	0.43375	−0.91243
20	5.31758	0.43365	−0.91334
30	5.31758	0.43365	−0.91334
40	5.31758	0.43365	−0.91334

Table 3
The $[m,m]$ homotopy-Padé approximations of γ, $F(+\infty)$, and $F''(0)$ of the first branch of solution when $\beta = 1$ and $\gamma_0 = 2 - \sqrt{2}$

$[m,m]$	γ	$F(+\infty)$	$F''(0)$
[5,5]	0.6096142954	1.2807737816	−0.9063755218
[10,10]	0.6096142958	1.2807737812	−0.9063755237
[15,15]	0.6096142958	1.2807737812	−0.9063755237
[20,20]	0.6096142958	1.2807737812	−0.9063755237

Table 4
The $[m,m]$ homotopy-Padé approximations of γ, $F(+\infty)$, and $F''(0)$ of the second branch of solution when $\beta = 1$ and $\gamma_0 = 2 + \sqrt{2}$

$[m,m]$	γ	$F(+\infty)$	$F''(0)$
[5,5]	5.3174870195	0.4336574191	−0.9133447576
[10,10]	5.3175776654	0.4336537229	−0.9133389444
[15,15]	5.3175776896	0.4336537219	−0.9133389388
[20,20]	5.3175776896	0.4336537219	−0.9133389388

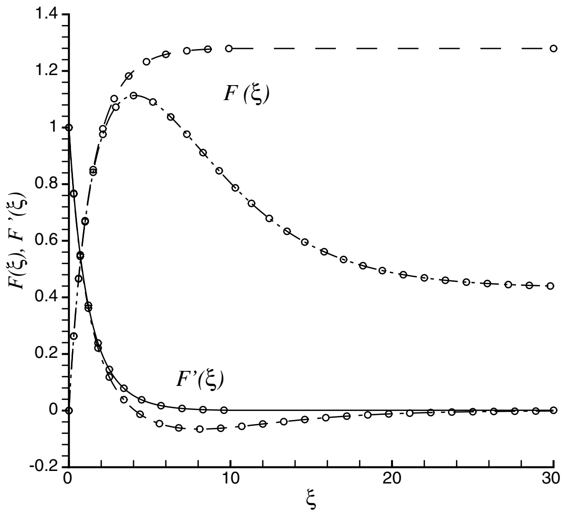

Fig. 2. $F(\xi)$ and $F'(\xi)$ of the two branches of solutions when $\beta = 1$: (dashed line) $F(\xi)$ of the first branch of solution; solid line: $F'(\xi)$ of the first branch of solution; (dash-dot-dotted line) $F(\xi)$ of the second branch of solution; (dash-dotted line) $F'(\xi)$ of the second branch of solution; (symbols) numerical results given by Runge–Kutta's method.

Table 5
$F(+\infty)$ and $F''(0)$ of the first branch of solutions when $\beta > 0$

β	h	$F(+\infty)$	$F''(0)$
0.1	−1	1.5671987677	−0.5044714296
0.2	−1	1.5233211707	−0.5604081070
0.3	−1	1.4836193076	−0.6124206246
0.4	−1	1.4474244267	−0.6611548740
0.5	−1	1.4142135624	−0.7071067812
0.6	−1	1.3835703539	−0.7506652338
0.7	−1	1.3551581436	−0.7921407683
0.8	−1	1.3287010210	−0.8317853005
0.9	−1	1.3039701608	−0.8698060404
1	−1	1.2807737812	−0.9063755237
2	−1/2	1.1065468058	−1.2160186992
5	−1/3	0.8466059564	−1.8632196025
10	−1/5	0.6583880016	−2.6081483726
20	−1/7	0.4958786514	−3.6698608598

Table 6
$F(+\infty)$ and $F''(0)$ of the second branch of solutions when $1/2 < \beta < +\infty$

β	h	$F(+\infty)$	$F''(0)$
0.505	−1/40	0.1203123736	−0.71680431
0.51	−1/20	0.1350947640	−0.7194636996
0.53	−1/10	0.1708129121	−0.7290554847
0.55	−1/4	0.1956425368	−0.7381289899
0.6	−1/2	0.2423611636	−0.7598846445
0.8	−1/2	0.3597660152	−0.8401331462
1	−1/2	0.4336537219	−0.9133389388
1.5	−1/2	0.5402172761	−1.0760357869
2	−1/2	0.5934991205	−1.2185531166
3	−1/2	0.6351815454	−1.4644223721
4	−1/3	0.6416251516	−1.6756327445
5	−1/3	0.6347755047	−1.8634603629
10	−1/4	0.5646581867	−2.6081685246
20	−1/8	0.4579147612	−3.6698616153

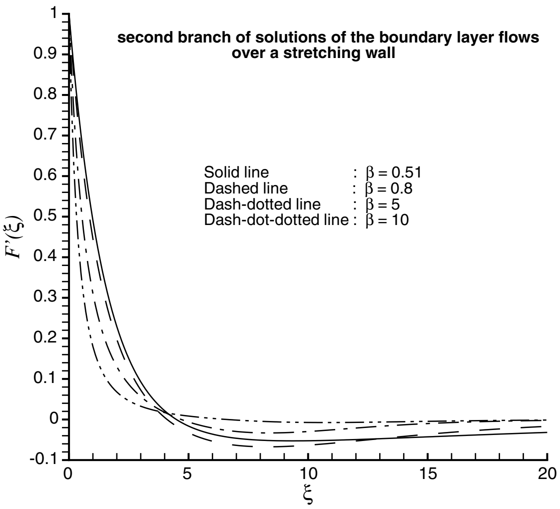

Fig. 3. The second branch of the analytic solutions of the boundary layer flows over a stretched wall.

[17] for the Cheng and Minkowycz's equation, our second branch of solutions for $1 < \beta < +\infty$ has never been reported.

2.3. Second approach for given β

By means of the above-mentioned approach we can find the first branch of solutions in the region $\beta > 0$. However, as reported by Banks[3], the first branch of solutions exist for $-1 < \beta < +\infty$. So, we should give an analytic approach valid for $-1 < \beta < 0$.

For this purpose, we construct the zeroth-order deformation equation

$$(1-q)\hat{\mathscr{L}}[\phi(\zeta;q) - w_0(\zeta)] = q\hbar\widehat{H}(\zeta)\mathscr{N}[\phi(\zeta;q)], \tag{45}$$

subject to the boundary conditions

$$\phi(0;q) = 1, \lim_{\zeta\to+\infty} \frac{\partial\phi(\zeta;q)}{\partial\zeta} = 0 \tag{46}$$

and

$$\left.\frac{\partial\phi(\zeta;q)}{\partial\zeta}\right|_{\zeta=0} + 2(1-q)\gamma_0 + 2q\Gamma(q) = 0, \tag{47}$$

where

$$\hat{\mathscr{L}}[\phi(\zeta;q)] = \frac{\partial^3\phi(\zeta;q)}{\partial\zeta^3} - \frac{\partial\phi(\zeta;q)}{\partial\zeta} \tag{48}$$

with the property

$$\hat{\mathscr{L}}[C_1\exp(-\zeta) + C_2 + C_3\exp(\zeta)]. \tag{49}$$

The initial approximation $w_0(\zeta)$ and the nonlinear operator $\mathscr{N}$ are the same as (21) and (24), respectively. Similarly, the series (31) and (32) hold, and $w_n(\zeta)$ is now governed by the high-order deformation equations

$$\hat{\mathscr{L}}[w_n(\zeta) - \chi_n w_{n-1}(\zeta)] = \hbar\widehat{H}(\zeta)R_n[\vec{w}_{n-1}(\zeta), \gamma_{n-1}], \tag{50}$$

subject to the boundary conditions

$$w_n(0) = 0, \quad w_n'(+\infty) = 0, \quad w_n(+\infty) = 0 \tag{51}$$

and

$$w_n'(0) - 2\gamma_0(1-\chi_n) + 2\gamma_{n-1} = 0, \tag{52}$$

where $R_n[\vec{w}_{n-1}(\zeta), \gamma_{n-1}]$ is defined by (35).

Similarly, to obey the Rule of Solution Expression (20), we should choose

$$\widehat{H}(\zeta) = 1. \tag{53}$$

Let $w_n^*(\zeta)$ denote the special solution of high-order deformation equation (50), which obeys the Rule of Solution Expression (20). Then, using the property (49), we have the general solution

$$w_n(\zeta) = w_n^*(\zeta) + C_1\exp(-\zeta) + C_2 + C_3\exp(\zeta).$$

From (51), it is obvious that $C_2 = C_3 = 0$. Then, C_1 is determined by the boundary condition $w_n(0) = 0$. For $n = 1$, $w_1(\zeta)$ contains the unknown initial approximation γ_0, which is determined by the boundary condition (52), i.e.

$$w_1'(0) = 0. \tag{54}$$

It is found that Eq. (54) has two different solutions

$$\gamma_0 = \frac{51 + 8\beta - \sqrt{5(429 - 16\beta - 64\beta^2)}}{4(3+4\beta)} \tag{55}$$

and

$$\gamma_0 = \frac{51 + 8\beta + \sqrt{5(429 - 16\beta - 64\beta^2)}}{4(3+4\beta)}. \tag{56}$$

For $n > 1$, we have from (52) that

$$\gamma_{n-1} = -\frac{w_n'(0)}{2}, \quad n \geqslant 2. \tag{57}$$

In this way, γ_{n-1} is obtained and all boundary conditions are satisfied. Note that, different from the first approach, the solution $w_n(\zeta)$ given by the second approach does not contain the unknown γ_n.

Note that γ_0 given by (55) is valid for

$$-\frac{\sqrt{430}+1}{8} < \beta < \frac{\sqrt{430}-1}{8}. \tag{58}$$

Besides, it tends to 2/9 as $\beta \to -3/4$. So, (55) is valid for $-1 \leqslant \beta \leqslant 0$. Similarly, for given β, we can choose a proper $\hbar$ by plotting the corresponding $\hbar$-curves of γ, and besides we can employ the homotopy-Padé technique to accelerate the convergence of the solution series, when necessary. In this way, using the initial approximation γ_0 given by (55), we obtain the first branch of the solution in region $-1 < \beta \leqslant 0$, as shown in Table 7. We can not find convergent analytic results for $\beta \leqslant -1$. All these analytic solutions agree well with Banks' numerical results [3] without reversed velocity flows.

Therefore, by means of the above two approaches for given β, we can find the whole first branch of solutions given by Banks' numerical method [3] and besides the whole second branch of solutions that shows reversed velocity flows in some regions. The values of $F(+\infty)$ of the two branches of solutions are as shown in Fig. 4. Note that the second branch of solutions for $1 < \beta < +\infty$ has never been reported, although the solutions for $1/2 < \beta \leqslant 1$ are mathematically equivalent to Ingham and Brown's numerical solutions [17] for the Cheng and Minkowycz's equation.

2.4. Approach for given entrainment velocity

From (11), $F(+\infty)$ is related to the entrainment velocity of the fluid. Suppose that we would determine the movement of the stretched wall for a given entrainment velocity. This is an inverse problem and a corresponding approach is given below.

Table 7
$F(+\infty)$ and $F''(0)$ of the first branch of solutions when $-1 < \beta \leqslant 0$

β	h	$F(+\infty)$	$F''(0)$
−0.95	−1	3.7343486357	2.777454210
−0.90	−1	3.1198300898	1.5176890220
−0.8	−1	2.5922842960	0.7135604366
−0.7	−1	2.3170430417	0.3660958444
−0.6	−1	2.1346791652	0.1528510095
−0.5	−1	2	0
−0.4	−1	1.8941059198	−0.1194986175
−0.3	−1	1.8073695138	−0.2181647105
−0.2	−1	1.7342480756	−0.3026969209
−0.1	−1	1.6712706083	−0.3770531959
0	−1	1.6161254468	−0.4437483134

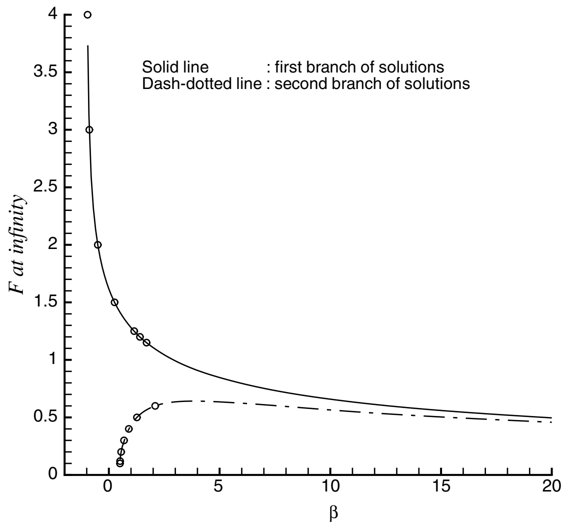

Fig. 4. Comparison of $F(+\infty)$ of the two branches of solutions of the boundary layer flows over a stretched wall: (solid line) first branch solutions; (dash-dotted line) second branch solutions; (symbols) results obtained by the approach for given entrainment velocity of the fluid.

For a given $\gamma = 1/F^2(+\infty)$, we need to search for the corresponding value of β. Now, γ is known but β is unknown. So, we define a nonlinear operator

$$\widehat{\mathscr{N}}[\phi(\zeta;q),\Lambda(q)] = \frac{\partial^3\phi(\zeta;q)}{\partial\zeta^3} + [1-\phi(\zeta;q)]\frac{\partial^2\phi(\zeta;q)}{\partial\zeta^2} + 2\Lambda(q)\left[\frac{\partial\phi(\zeta;q)}{\partial\zeta}\right]^2, \tag{59}$$

where $\Lambda(q)$ corresponds to β. Using the initial guess

$$w_0(\zeta) = 2(1-\gamma)\exp(-\zeta) + (2\gamma-1)\exp(-2\zeta) \tag{60}$$

and the same auxiliary linear operator as in (22), we construct the zeroth-order deformation equation

$$(1-q)\mathscr{L}[\phi(\zeta;q) - w_0(\zeta)] = q\hbar H(\zeta)\widehat{\mathscr{N}}[\phi(\zeta;q),\Lambda(q)], \tag{61}$$

subject to the boundary conditions

$$\phi(0;q) = 1, \quad 2\gamma + \left.\frac{\partial\phi(\zeta;q)}{\partial\zeta}\right|_{\zeta=0} = 0, \quad \lim_{\zeta\to+\infty}\frac{\partial\phi(\zeta;q)}{\partial\zeta} = 0, \tag{62}$$

where $\gamma = 1/F^2(+\infty)$ is known. Let β_0 denote the initial approximation of β. Write

$$\Lambda(q) = \beta_0 + \sum_{n=1}^{+\infty}\beta_n q^n \tag{63}$$

and

$$\vec{\beta}_n = \{\beta_0, \beta_1, \beta_2, \ldots, \beta_n\},$$

where

$$\beta_n = \frac{1}{n!}\frac{\partial^n \Lambda(q)}{\partial q^n}\bigg|_{q=0}.$$

Similarly, from the zeroth-order deformation equations (61) and (62), we have the corresponding high-order deformation equation

$$\mathscr{L}[w_n(\zeta) - \chi_n w_{n-1}(\zeta)] = \hbar H(\zeta)\widehat{R}_n(\vec{w}_{n-1}, \vec{\beta}_{n-1}), \tag{64}$$

subject to the boundary conditions

$$w_n(0) = 0, \quad w_n'(0) = 0, \quad w_n'(+\infty) = 0, \tag{65}$$

where

$$\widehat{R}_n(\vec{w}_{n-1}, \vec{\beta}_{n-1}) = w'''_{n-1}(\zeta) + w''_{n-1}(\zeta) - \sum_{k=0}^{n-1} w_{n-1-k}(\zeta) w''_k(\zeta) + 2\sum_{k=0}^{n-1}\beta_{n-1-k}\sum_{i=0}^{k} w'_i(\zeta)w'_{k-i}(\zeta). \tag{66}$$

Similarly, we choose the auxiliary function

$$H(\zeta) = 1.$$

Let $\widehat{B}_{n,2}(\vec{\beta}_{n-1})$ denote the coefficient of $\exp(-2\zeta)$ of $\widehat{R}_n(\vec{w}_{n-1}, \vec{\beta})$. Similarly, using the same auxiliary function as in (38) and enforcing

$$\widehat{B}_{n,2}(\vec{\beta}_{n-1}) = 0, \tag{67}$$

we can obtain the unknown β_{n-1}. Thereafter, it is easy to get $w_n(\zeta)$ of the high-order deformation equations (64) and (65).

When $n = 1$, we have from Eq. (67) the initial approximation

$$\beta_0 = \frac{\gamma^2}{2(1-\gamma)^2} = \frac{1}{2(1-\delta^2)^2}. \tag{68}$$

The above expression is invalid near $F(+\infty) = \delta = 1$. Similarly, by means of plotting the $\hbar$-curves of β and $F''(0)$, a proper value of $\hbar$ can be found to ensure that the solution series converge, and besides the homotopy-Padé technique may be applied to accelerate the convergence. For example, when $F(+\infty) = 2$, our analytic approximation converges to the exact solution

$$F(\xi) = 2\tanh(\xi/2), \quad F''(0) = 0, \beta = 1/2,$$

as shown in Table 8. The values of $F''(0)$ and β for some given $F(+\infty)$ are listed in Table 9. All these results agree well with those given by the above-mentioned two approaches for the given β, as shown in Fig. 4. This indicates the validity and the flexibility of the homotopy analysis method. Besides, it also verifies the correctness of our second branch of solutions.

Table 8
The $[m, m]$ homotopy-Padé approximations of β and $F''(0)$ when $F(+\infty) = 2$ is given

$[m,m]$	β	$F''(0)$
[5,5]	−0.4998755351	4.6×10^{-4}
[10,10]	−0.4999999350	4.4×10^{-8}
[15,15]	−0.4999999998	1.3×10^{-11}
[20,20]	−0.5000000000	4.0×10^{-15}

Table 9
Approximations of β and $F''(0)$ for given $F(+\infty)$

δ	h	β	$F''(0)$
4	−3/4	−0.96177340	3.45489516
3	−3/4	−0.88397276	1.31559436
2.5	−3/4	−0.77171006	0.59275642
2	−3/4	−0.50000000	0.00000000
1.5	−3/4	0.25757399	−0.59078760
1.25	−1/10	1.14276743	−0.95635271
1.2	−1/50	1.40233554	−1.04147974
1.15	−1/50	1.70164153	−1.13211564
0.6	−3/5	2.09104978	−1.24285249
0.5	−3/4	1.26616358	−1.00298966
0.4	−2/5	0.89950383	−0.87728143
0.3	−3/10	0.68378517	−0.79457316
0.2	−1/10	0.55399162	−0.73990719
0.12	−3/50	0.50491551	−0.71675655
0.1	−1/25	0.50121417	−0.71436667
0.08	−3/100	0.50011008	−0.7129335

3. Analysis of the results and discussion

Using the above-mentioned three analytic approaches, we successfully found two branches of solutions for the boundary-layer flows over a stretched impermeable wall. The first branch of solutions ($-1 < \beta < +\infty$) agrees well with Banks' numerical results [3] given by a shooting method, and do not show the reversed velocity flows. The second branch ($1/2 < \beta < +\infty$) show the reversed velocity flows in some regions that becomes larger and larger as β tends to 1/2. The second branch of solutions can be divided into two parts. One ($1/2 < \beta \leqslant 1$) is mathematically equivalent to Ingham and Brown's second branch of numerical solutions of the Cheng and Minkowycz's equation [17], the other ($1 < \beta < +\infty$) is new and has never been reported, to the best of our knowledge. The difference between the values of $F(+\infty)$ for the two branches of solutions is obvious near $\beta = 1/2$, but becomes smaller and smaller as β increases, as shown in Fig. 4. So, according to (9)–(11), the velocity of the fluid and especially the entrainment velocity of the fluid of the two branches of solutions are different, especially near $\beta = 1/2$. However, the difference between the values of $F''(0)$ of the two branches of solutions is so small that it is even hard to distinguish them in Fig. 5. For example, the relative differences of $F''(0)$ when $\beta = 1{,}5$ and 10 are 0.77%, 0.013%, and 0.00077%, respectively. According to (12), the local skin friction coefficient on the stretched wall

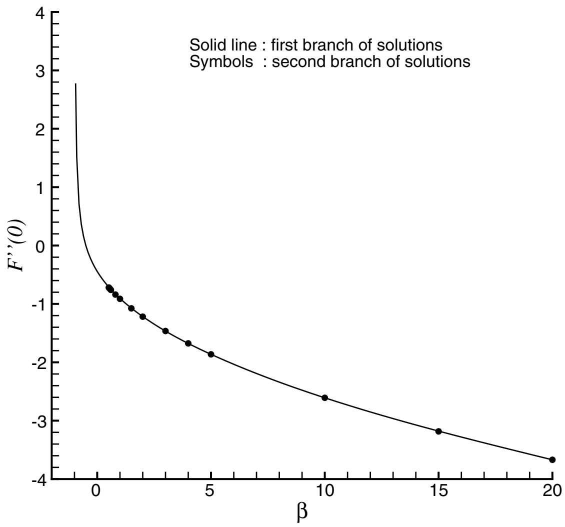

Fig. 5. Comparison of $F''(0)$ of the two branches of solutions of the boundary layer flows over a stretched wall. The difference of solutions of the two branches is not visible, because it is very small, as mentioned in main text.

is directly proportional to $F''(0)$. Thus, although the velocity profiles of the two branches of solutions of the boundary layer flows over a stretched wall might be obviously different, the skin frictions on the wall are nearly the same. So, from a practical point of view, we need not worry about the great increase of the skin friction on the wall when, owing to some reasons, the profile of the velocity and the entrainment velocity change from one of the two branches of solutions to the other. This result is interesting.

It is well-known that, for some unsteady nonlinear problems, the tiny differences of initial conditions might lead to obviously different solutions. In this paper we show that the dual solutions of the boundary-layer flows over a stretched wall are sensitive to the boundary value $F''(0)$. Thus, mathematically speaking, for some nonlinear boundary value problems, the small difference of boundary conditions might also lead to obviously different solutions. This might be the reason why it is not easy to find the second branch of solutions by numerical methods.

Recently, Pop and Na [29] reported multiple solutions for MHD flows over a stretched permeable surface. Zaturska and Banks [30] found multiple solutions for Blasius boundary-layer flows. Magyari et al. [31] found multiple solutions of boundary-layer flows over a moving plane surface in a special case. In this paper we assume that the solutions tend to zero exponentially at infinity. However, Magyari et al. [32] found that, when $\beta = -1/2$, Eqs. (6) and (7) have an infinite number of solutions with algebraic asymptotic property at infinity. It seems that the boundary-layer flows might have multiple solutions in general, and thus further investigations are necessary. And the homotopy analysis method provides us with a useful analytic tool for this purpose.

One can find applications of the new branch of analytic solutions in industries, such as aerodynamic extrusion of plastic sheets, boundary layer flows along liquid film in condensation processes, cooling of a metallic plate in a cooling bath, applications in the glass and polymer industries, and so on. In all these application fields, one should pay more attentions on the existence of multiple solutions. Besides, there exist many problems which are physically different from but mathematically identical with the considered problem [12]. Therefore, all of these problems might have dual solutions. Besides, the impermeable wall is only a special case of the permeable wall. To the best of our knowledge, the dual solutions of the boundary-layer flows over a stretched permeable wall were not reported [12]. Using the above approaches, the author has successfully found a new branch of solutions of boundary-layer flows over a stretched permeable wall in a similar way. Besides, it is interesting to investigate the related unsteady boundary-layer flows when there exist multiple steady-state solutions in case of $\beta > 1/2$.

The discover of the new branch of solutions of boundary-layer flows over a stretched impermeable wall indicates that the homotopy analysis method is a useful tool for nonlinear problems, especially for those with multiple solutions which are not easy to find by numerical methods. The homotopy analysis method might be applied to search for new solutions of some nonlinear problems in fluid mechanics, which might deepen our physical understandings about flows and heat transfers.

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