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HOMOTOPY ANALYSIS METHOD: A NEW ANALYTIC METHOD FOR NONLINEAR PROBLEMS

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Abstract

In this paper, the basic ideas of a new analytic technique, namely the Homotopy Analysis Method (HAM), are described. Different from perturbation methods, the validity of the HAM is independent on whether or not there exist small parameters in considered nonlinear equations. Therefore, it provides us with a powerful analytic tool for strongly nonlinear problems. A typical nonlinear problem is used as an example to verify the validity and the great potential of the HAM.

Key words nonlinear analytic technique, strong nonlinearity, homotopy, topology

I. Introduction

Although the rapid development of digital computers makes it easier and easier to numerically solve nonlinear problems, it is still rather difficult to give their analytic approximations. Currently, most of our nonlinear analytic techniques are unsatisfactory. For instance, although perturbation techniques are widely applied to analyze nonlinear problems in science and engineering, they are however so strongly dependent on small parameters appeared in equations under consideration that they are restricted only to weakly nonlinear problems. For strongly nonlinear problems which don't contain any small parameters, perturbation techniques are invalid. So, it seems necessary and worthwhile developing a new kind of analytic technique independent of small parameters.

Liao[1~6] has made some attempts in this direction. Liao[1,2] proposed a new analytic technique in his Ph. D. dissertation, namely the Homotopy Analysis Method (HAM). Based on homotopy of topology, the validity of the HAM is independent of whether or not there exist small parameters in considered equations. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides us with a powerful tool to analyze strongly nonlinear problems. Liao[3~6] successfully applied the HAM to solve some nonlinear problems while he has been making unremitting efforts to improve it step by step. Moreover, Liao[7,8] and Liao & Chwang[9] applied the basic ideas of the HAM to propose the so-called "general boundary element method" (GBEM), which is valid even for those strongly nonlinear problems whose governing equations and boundary conditions don't contain any linear terms so that it greatly generalizes the traditional BEM. All of these verify the validity and great potential of the HAM.

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In this paper, we use a simple but typical example to simply introduce the basic ideas of the HAM so as to further show its validity and great potential.

II. Basic Ideas of the Homotopy Analysis Method

Consider the following equation

$$u'(t) + 2tu^2(t) = 0, \; u(0) = 1 \tag{2.1}$$

whose exact solution is

$$u(t) = \frac{1}{1 + t^2} \tag{2.2}$$

Assuming t is small, we can obtain by perturbation techniques the following power series

$$u(t) = 1 - t^2 + t^4 - t^6 + \cdots = \sum_{k=0}^{+\infty} (-1)^k t^{2k} \tag{2.3}$$

whose convergence radius is however rather small, say, $\rho = 1$, so that the above perturbation approximation is nearly worthless.

Now we apply the HAM to solve Eq. (2.1). Firstly, we construct such a continuous mapping $U(t,p,\hbar):[0,+\infty) \times [0,1] \times \mathscr{R}_0 \to \mathscr{R}$, governed by

$$(1-p)\frac{\partial U(t,p,\hbar)}{\partial t} = p\hbar\left[\frac{\partial U(t,p,\hbar)}{\partial t} + 2tU^2(t,p,\hbar)\right] \quad t \geqslant 0, p \in [0,1], \hbar \neq 0 \tag{2.4}$$

with boundary condition

$$U(0,p,\hbar) = 1, \qquad p \in [0,1], \hbar \neq 0 \tag{2.5}$$

so that

$$u(t,0,\hbar) = 1, \; U(t,1,\hbar) = u(t) = \frac{1}{1 + t^2} \tag{2.6}$$

where $\mathscr{R} = (-\infty, +\infty)$ and $\mathscr{R}_0 = (-\infty, 0) \cup (0, +\infty)$. Note that we let here $\hbar$ be a nonzero real number. Clearly, as p increases from 0 to 1, $U(t,p,\hbar)$ varies continuously from $U(t,0,\hbar) = 1$ to the exact solution $u(t) = 1/(1+t^2)$. This kind of continuous variation is called deformation in topology so that we call Eqs. (2.4),(2.5) the zeroth-order deformation equations. Secondly, assuming that $U(t,p,\hbar)$ is so smooth that

$$u_0^{[k]}(t,\hbar) = \left.\frac{\partial^k U(t,p,\hbar)}{\partial p^k}\right|_{p=0} \qquad (k \geqslant 1)$$

namely the kth-order deformation derivatives, exists, and moreover, the corresponding Maclaurin series of $U(t,p,\hbar)$ at $p = 0$, say,

$$U(t,0,\hbar) + \sum_{k=1}^{+\infty}\left[\frac{u_0^{[k]}(t,\hbar)}{k!}\right]p^k \tag{2.7}$$

converges at $p = 1$, then, we have by (2.6) that

$$u(t) = 1 + \sum_{k=1}^{+\infty}\frac{u_0^{[k]}(t,\hbar)}{k!} \tag{2.8}$$

Here, $u_0^{[k]}(t,\hbar)(k \geqslant 1)$ is governed by the kth-order deformation equations

$$\frac{\partial u_0^{[k]}(t,\hbar)}{\partial t}\begin{cases} 2\hbar t & (k=1) \\ k\left[(1+\hbar)\dfrac{\partial u_0^{[k-1]}(t,\hbar)}{\partial t}+2t\hbar\displaystyle\sum_{m=0}^{k-1}\binom{k-1}{m}u_0^{[m]}(t,\hbar)u_0^{[k-1-m]}(t,\hbar)\right] & (k \geqslant 2)\end{cases} \tag{2.9}$$

under boundary condition

$$u_0^{[k]}(0,\hbar) = 0, \qquad \hbar \neq 0, k \geqslant 1 \tag{2.10}$$

Eqs. (2.9) and (2.10) are obtained by differentiating the zeroth-order deformation equations (2.4),(2.5) k times $(k \geqslant 1)$ with respect to p and then setting $p=0$. The above linear first-order differential equation can be rather easily solved, especially by software such as MATHEMATICA or MAPLE. Then, substituting these solutions into (2.8), we obtain

$$u(t) = \lim_{m\to+\infty}\sum_{k=0}^{m}[(-1)^k t^{2k}]\Phi_{m,k}(\hbar) \tag{2.11}$$

where $\Phi_{m,n}(\hbar)$ is defined by

$$\Phi_{m,n}(\hbar) = \begin{cases} 0 & (n > m) \\ (-\hbar)^n\displaystyle\sum_{k=0}^{m-n}\binom{m}{m-n-k}\binom{n+k-1}{k}\hbar^k & (1 \leqslant n \leqslant m) \\ 1 & (n \leqslant 0)\end{cases} \tag{2.12}$$

namely the approaching function.

The approaching function $\Phi_{m,k}(\hbar)$ has very general meanings. Liao[10] rigorously proved that $\Phi_{m,n}(\hbar)$ has the following properties:

Lemma 1 For complex number ζ and positive integers m, n $(1 \leqslant n \leqslant m)$, it holds

$$\Phi'_{m,n}(\zeta) = (-1)^n n\binom{m}{n}\zeta^{n-1}(1+\zeta)^{m-n} \tag{2.13}$$

Lemma 2 For complex number ζ and positive integers $m, n(1 \leqslant n \leqslant m)$, it holds

$$\Phi_{m+1,n}(\zeta) - \Phi_{m,n}(\zeta) = \binom{m}{n-1}(-\zeta)^n(1+\zeta)^{m-n+1} \tag{2.14}$$

Lemma 3 For positive integers m, n $(n \leqslant m)$, it holds

$$\Phi_{m,n}(-1) = 1 \tag{2.15}$$

Lemma 4 For complex number ζ and finite positive integer $n(n \geqslant 1)$, it holds

$$\lim_{m\to+\infty}\Phi_{m,n}(\zeta) = 1 \qquad (|1+\zeta| < 1) \tag{2.16}$$

Furthermore, Liao[10] rigorously proved the so-called General Taylor Theorem which is simply cited in the Appendix. We emphasize that the General Taylor Theorem contains in logic the traditional Taylor theorem. Moreover, it may greatly enlarge the convergence region of a traditional Taylor series by means of the approaching function $\Phi_{m,n}(\zeta)$ if a proper ζ is selected. The example considered in this paper is a good case to show this point. Notice that the exact solution of Eq. (2.1) is $1/(1+t^2)$ and the related complex function $f(z) = 1/(1+z^2)$ has two singularities $\xi_1 = i$ and $\xi_2 = -i$. So, according to the so-called General Taylor Theorem, series (2.11) is valid in the region

$$|t| < \sqrt{\frac{2}{|\hbar|} - 1} \qquad (-2 < \hbar < 0) \tag{2.17}$$

III. Discussions and Conclusions

Now, let us compare the perturbation approximation (2.3) with the series (2.11). The former converges only in the region $|t| < 1$, but the latter converges in a region which is however a function of $\hbar$. Firstly, according to (2.15) and (2.17), the series (2.11) gives in case of $\hbar = -1$ the same result as (2.3), so that the series (2.3) is just a special member of the family of the series (2.11). It means that (2.11) contains the perturbation approximation (2.3) in logic. This kind of logical continuation has been proved rather important in science and mathematics. Second, as $\hbar$ increases from -1 to 0, the convergence region of the power series (2.11) becomes larger and larger. In limit $\hbar \to 0$, the series (2.11) is valid even in the whole real axis! However, as $\hbar$ decreases from -1 to -2, its convergence region becomes smaller and smaller. So, the perturbation approximation (2.3) is neither the best nor the worst: it is just one special but common member among the family of the series (2.11). At last, we emphasize that the whole approach mentioned above needs not the small parameter assumption. In other words, the validity of the HAM is independent on whether or not there exist small parameters in considered problems. In form, the HAM seems able to be applied to any nonlinear problems, although there should certainly exist some restrictions for its applications which we currently don't know very clearly.

Different from the perturbation approximation (2.3), the series (2.11) provides us with a family of approximations. The convergence region of the series (2.11) is determined by values of $\hbar$ which is introduced to construct the so-called zeroth-order deformation equations (2.4) and (2.5). Clearly, different values of $\hbar$ correspond to different continuous mappings, or more precisely to say, different homotopies. In fact, Eqs. (2.4) and (2.5) construct a family of homotopies $U(t,p,\hbar)$ dependent on the variable $\hbar$. Certainly, some among them are 'better', some are worse. This can well explain why the convergence region of the series (2.11) is a function of $\hbar$. The introducing of the real variable $\hbar$ really provides us with larger freedom to get more and even better approximations. In fact, if we substitute the real variable $\hbar$ in (2.4) and (2.5) by a complex variable ζ, we can get in the same way such a family

$$u(t) = \lim_{m \to +\infty} \sum_{k=0}^{m} [(-1)^k t^{2k}] \Phi_{m,k}(\zeta) \tag{3.1}$$

which converges to the exact solution $u(t) = 1/(1+t^2)$ in the region determined by

$$|1 + \zeta \pm it\zeta| < 1 \qquad (|1 + \zeta| < 1) \tag{3.2}$$

Note that setting ζ to be a real variable gives the formula (2.17). Therefore, the series (3.1) contains (2.11) and is then even more general than (2.11).

The foregoing example is indeed very simple and its exact solution is also known. However, it still verifies the validity and great potential of the HAM. In fact, the HAM can be applied to solve more complicated nonlinear problems. For example, the author successfully applied the HAM to solve the 2D viscous flow over a semi-infinite plain in fluid mechanics, governed by

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0, \qquad \eta \in [0, +\infty) \tag{3.3}$$

with boundary conditions

$$f(0) = f'(0) = 0, f'(+\infty) = 1 \tag{3.4}$$

where the prime denotes the derivatives with respect to η. It is well-known that Blasius[11] gave a solution of this problem in the form of a power series

$$f(\eta) = \sum_{k=0}^{+\infty}\left(-\frac{1}{2}\right)^k \frac{A_k\sigma^{k+1}}{(3k+2)!}\eta^{3k+2} \tag{3.5}$$

where

$$A_k = \begin{cases} 1, & k = 0,1 \\ \sum_{r=0}^{k-1}\binom{3k-1}{3r}A_rA_{k-r-1}, & k \geq 2 \end{cases} \tag{3.6}$$

Here $\sigma = f''(0) = 0.33206$ was given numerically by Howarth[12, 13]. The power series (3.5) can also be obtained by perturbation method. However, it converges only in the region $|\eta| \leqslant 5.690$. Liao[6] successfully applied the HAM to obtain such a family of power series

$$f(\eta) = \lim_{m\to+\infty}\sum_{k=0}^{m}\left[\left(-\frac{1}{2}\right)^k \frac{A_k\sigma^{k+1}}{(3k+2)!}\eta^{3k+2}\right]\Phi_{m,k}(\hbar), \quad \eta \in [0, +\infty), -2 < \hbar < 0 \tag{3.7}$$

which is valid in the region

$$-\rho_0 < \eta < \rho_0\left[\frac{2}{|\hbar|} - 1\right]^{1/3} \qquad (-2 < \hbar < 0) \tag{3.8}$$

where $\hbar$ is a real variable. We emphasize that, as $\hbar(-2 < \hbar < 0)$ tends to zero, the power series (3.7) converges in the whole positive real axis! All of these verify that the HAM is really a powerful analytic tool for nonlinear problems, although it needs more applications and further improvements.

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Appendix

General Taylor Theorem

Let $n \geqslant 0$ be finite integer and ζ, z, z_0 complex numbers. If complex function $f(z)$ is analytic at $z = z_0$, then it holds

$$f(z) = \lim_{m \to +\infty} \sum_{k=0}^{m} \left[\frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] \Phi_{m,k}(\zeta)$$

$$= \lim_{m \to +\infty} \sum_{k=0}^{m+n} \left[\frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] \Phi_{m,k-n}(\zeta)$$

$$= \lim_{m \to +\infty} \sum_{k=0}^{m} \left[\frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] \Phi_{m+n,k+n}(\zeta)$$

in the region

$$\bigcap_{k \in I} \left| 1 + \zeta - \zeta \left(\frac{z - z_0}{\xi_k - z_0} \right) \right| < 1, \qquad | 1 + \zeta | < 1$$

where the complex function $\Phi_{m,k}(\zeta)$ is defined by (2.12) and $\xi_k (k \in I)$ are all singularities of $f(z)$. Moreover, if $\zeta = -1$, it is exactly the classical Taylor series.

General Newtonian Binomial Theorem

Let $t, \hbar$ and α be real numbers. Then, the general Newtonian binomial expression

$$(1 + t)^\alpha = 1 + \lim_{m \to +\infty} \sum_{n=1}^{m} \left[\frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - n + 1)}{n!} t^n \right] \Phi_{m,n}(\hbar) \qquad (\alpha \neq 0, 1, 2, 3, \cdots)$$

holds in the region

$$-1 < t < \frac{2}{|\hbar|} - 1 \qquad (-2 < \hbar < 0)$$

where $\Phi_{m,n}(\hbar)$ is defined by (2.12).