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# THE GENERAL BOUNDARY ELEMENT METHOD AND ITS FURTHER GENERALIZATIONS

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## SUMMARY

In this paper, the basic ideas of the general boundary element method (BEM) proposed by Liao [in *Boundary Elements XVII*, Computational Mechanics Publications, Southampton, MA, 1995, pp. 67–74; *Int. J. Numer. Methods Fluids*, **23**, 739–751 (1996), **24**, 863–873 (1997); *Comput. Mech.*, **20**, 397–406 (1997)] and Liao and Chwang [*Int. J. Numer. Methods Fluids*, **23**, 467–483 (1996)] are further generalized by introducing a non-zero parameter $\hbar$. Some related mathematical theorems are proposed. This general BEM contains the traditional BEM in logic, but is valid for non-linear problems, including those whose governing equations and boundary conditions have no linear terms. Furthermore, the general BEM can solve non-linear differential equations by means of no iterations. This disturbs the absolutely governing place of iterative methodology of the BEM for non-linear problems. The general BEM can greatly enlarge application areas of the BEM as a kind of numerical technique. Two non-linear problems are used to illustrate the validity and potential of the further generalized BEM. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: general boundary element method; strongly non-linear problems; homotopy; non-iterative approach

## 1. INTRODUCTION

Although the boundary element method (BEM) is in principle based on the linear superposition of the fundamental solution of a linear operator, many researchers [6–8] applied it to solve non-linear problems. Let

$$\mathscr{A}(u) = f(\vec{r}) \tag{1.1}$$

denote a differential equation in general, where $\mathscr{A}$ may be a non-linear operator, $u$ is a dependent variable and $f(\vec{r})$ is a known function of the independent position vector $\vec{r}$. The basic ideas of the traditional BEM for non-linear problems are first of all to move all non-linear terms of Equation (1.1) to its right-hand-side and then to find the corresponding fundamental solution of the linear term still remaining on the left-hand-side. This implies, however, the following two assumptions:

1. The non-linear operator $\mathscr{A}$ can be divided into two parts, say $\mathscr{A} = \mathscr{L}_0 + N_0$, where $\mathscr{L}_0$ and $N_0$ are linear and non-linear operators respectively;

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2. Also, the fundamental solution of the linear operator $\mathscr{L}_0$ must be exist and be known *a priori*.

If above two assumptions are satisfied, (1.1) can be rewritten in the form

$$c(\vec{r})u(\vec{r}) = \int_\Gamma [u\mathscr{B}_0(\omega_0) - \omega_0\mathscr{B}_0(u)]\,\mathrm{d}\Gamma + \int_\Omega [f(\vec{r}) - N_0(u)]\omega_0\,\mathrm{d}\Omega, \tag{1.2}$$

where $\omega_0$ is the fundamental solution of the adjoint operator of $\mathscr{L}_0$, $\mathscr{B}_0$ is its corresponding boundary operator, $\Gamma$ denotes the boundary of domain $\Omega$. The geometric coefficient $c(\vec{r})$ in (1.2) depends on the location of the position vector $\vec{r}$. However, these two assumptions unfortunately cannot always be satisfied. First, there obviously exists a possibility such that, if (1.1) does not contain any linear terms, nothing is left on its left-hand-side after moving all non-linear terms to its right-hand-side. Second, even if there exists such a linear term that contains a linear operator $\mathscr{L}_0$, this linear operator might be either too simple to satisfy all boundary conditions, or too complicated to find out its fundamental solution. In all the above mentioned cases, the traditional BEM for non-linear problems does not work at all. Therefore, the foregoing two assumptions greatly restrict applications of the traditional BEM.

Liao [1–4] and Liao and Chwang [5] proposed a new kind of BEM, namely the general boundary element method, to overcome the above mentioned restrictions of the traditional BEM. Based on a new kind of analytical technique for non-linear problems proposed by Liao [9–12], namely the 'homotopy analysis method' (HAM), the general boundary element method has such an advantage that its validity is independent of the above mentioned two assumptions of the traditional BEM. Therefore, it can be applied to solve most of the non-linear problems, including those whose governing equations and boundary conditions do not contain any linear terms at all, as reported by Liao [1–4] and Liao and Chwang [5].

This paper is the continuation of the foregoing work of Liao [1–4] and Liao and Chwang [5]. Here, a non-zero parameter $\hbar$ is introduced to construct the so-called zeroth-order deformation equations so as to make the BEM even more general and its applications more flexible. As a result, the convergence radius of the related Taylor series of approximations is now not a constant, but dependent on $\hbar$. Therefore, if the value of $\hbar$ is properly selected and the order of approximation is high enough, we can obtain accurate enough approximations even by means of no iteration! This disturbs the absolutely governing place of the iterative methodology of the BEM for non-linear problems. On the other hand, this also verifies the validity and great potential of the further generalized BEM. Furthermore, some related mathematical theorems are proposed about the mechanism of determining the order $M$ of approximation and selecting the value of $\hbar$. Two non-linear problems are used to illustrate the validity of this further generalized boundary element method.

## 2. BASIC IDEAS

Let $\mathscr{A}$ and $\mathscr{B}$ be differential operators in general. Consider a governing equation

$$\mathscr{A}(u) = f(\vec{r}), \quad \vec{r}\in\Omega, \tag{2.1}$$

with the boundary condition

$$\mathscr{B}(u) = g(\vec{r}), \quad \vec{r}\in\Gamma, \tag{2.2}$$

 

where $f(\vec{r})$ and $g(\vec{r})$ are known functions of the position vector $\vec{r}$, $\Gamma$ denotes the boundary of domain $\Omega$. Selecting two proper familiar auxiliary linear operators $\mathscr{L}$ and $\mathscr{L}_{\mathrm{B}}$, whose fixed points are zero, say $\mathscr{L}(0)=0$ and $\mathscr{L}_{\mathrm{B}}(0)=0$, we construct a homotopy $U(\vec{r}, p, \hbar): \Omega \times [0, 1] \times \Re_0 \rightarrow \Re$, satisfying

$$(1-p)\{\mathscr{L}[U(\vec{r}, p, \hbar)] - \mathscr{L}[u_0(\vec{r})]\} = p\hbar\{\mathscr{A}[U(\vec{r}, p, \hbar)] - f(\vec{r})\}, \quad \vec{r}\in\Omega, \ p\in[0, 1], \ \hbar \neq 0, \tag{2.3}$$

with the boundary condition

$$(1-p)\{\mathscr{L}_B[U(\vec{r}, p, \hbar)] - \mathscr{L}_B[u_0(\vec{r})]\} = p\hbar\{\mathscr{B}[U(\vec{r}, p, \hbar)] - g(\vec{r})\}, \quad \vec{r}\in\Omega, \ p\in[0, 1], \ \hbar \neq 0, \tag{2.4}$$

where $\Re_0 = (-\infty, 0)\cup(0, +\infty)$, $u_0(\vec{r})$ is an initial approximation, $p\in[0, 1]$ is an embedding parameter, $U(\vec{r}, p, \hbar)$ is a function of the three variables $\vec{r}\in\Omega$, $p\in[0, 1]$ and $\hbar \neq 0$. Owing to (2.3) and (2.4), we have at $p=0$ and $p=1$ that

$$U(\vec{r}, 0, \hbar) = u_0(\vec{r}), \tag{2.5}$$

$$U(\vec{r}, 1, \hbar) = u(\vec{r}) \tag{2.6}$$

respectively, where $u(\vec{r})$ is the solution of (2.1) and (2.2). Therefore, $U(\vec{r}, p, \hbar)$ varies continuously from $u_0(\vec{r})$ to $u(\vec{r})$ as the embedding parameter $p$ increases from 0 to 1. It is more precise to say, $u_0(\vec{r})$ and $u(\vec{r})$ are homotopic. In topology, this kind of continuous variation is called deformation. So, we call (2.3) and (2.4) the zeroth-order deformation equations.

Assume that the continuous deformation $U(\vec{r}, p, \hbar)$ is smooth enough about $p$ so that

$$U^{[m]}(\vec{r}, p, \hbar) = \frac{\partial^m U(\vec{r}, p, \hbar)}{\partial p^m}, \quad m = 1, 2, 3, \ldots, \tag{2.7}$$

namely the $m$th-order deformation derivatives, exists. Then, according to the Taylor formula, we have by (2.5) and (2.6) that

$$U(\vec{r}, p, \hbar) = U(\vec{r}, 0, \hbar) + \sum_{m=1}^{\infty} \left.\frac{\partial^m U(\vec{r}, p, \hbar)}{\partial p^m}\right|_{p=0} \left(\frac{p^m}{m!}\right) = u_0(\vec{r}) + \sum_{m=1}^{\infty} \left(\frac{p^m}{m!}\right) u_0^{[m]}(\vec{r}, \hbar), \tag{2.8}$$

where $u_0^{[m]}(\vec{r}, \hbar) = U^{[m]}(\vec{r}, 0, \hbar)$ denotes the $m$th-order deformation derivatives $U^{[m]}(\vec{r}, p, \hbar)$ at $p=0$. For a given initial approximation and auxiliary linear operators, the convergence radius $\rho$ of the Taylor series (2.8) is dependent on $\hbar$. If $\hbar$ is so properly selected that the convergence radius of the series (2.8) is not less than 1, we have by (2.6) that

$$u(\vec{r}) = u_0(\vec{r}) + \sum_{m=1}^{\infty} \left[\frac{u_0^{[m]}(\vec{r}, \hbar)}{m!}\right]. \tag{2.9}$$

Otherwise, it holds only that

$$U(\vec{r}, \lambda, \hbar) = u_0(\vec{r}) + \sum_{m=1}^{\infty} \left[\frac{u_0^{[m]}(\vec{r}, \hbar)}{m!}\right]\lambda^m, \tag{2.10}$$

where $0 < \lambda \leq \rho < 1$. Nevertheless, (2.10) usually gives an approximation better than $u_0(\vec{r})$ so that it provides us with a family of iterative formulae in two parameters $\hbar$ and $\lambda$, say

$$u_{k+1}(\vec{r}) = u_k(\vec{r}) + \sum_{m=1}^{M} \left[\frac{u_0^{[m]}(\vec{r}, \hbar)}{m!}\right]\lambda^m, \quad k = 0, 1, 2, 3, \ldots, \tag{2.11}$$

where $M$ denotes the order of iterative formulas and $u_0^{[m]}(\vec{r}, \hbar)$ $(m = 1, 2, 3, \ldots)$ are dependent upon $u_k(\vec{r})$ and can be determined in what follows.

Differentiating the zeroth-order deformation equations (2.3) and (2.4) $m$ times with respect to the embedding parameter $p$ and then setting $p = 0$, we obtain the $m$th-order deformation equation:

$$\mathscr{L}[u_0^{[m]}(\vec{r}, \hbar)] = f_m(\vec{r}, \hbar), \quad \vec{r} \in \Omega, \ \hbar \neq 0, \ m = 1, 2, 3, \ldots, \tag{2.12}$$

with the boundary condition

$$\mathscr{L}_B[u_0^{[m]}(\vec{r}, \hbar)] = g_m(\vec{r}, \hbar), \quad \vec{r} \in \Gamma, \ \hbar \neq 0, \ m = 1, 2, 3, \ldots, \tag{2.13}$$

where

$$f_1(\vec{r}, \hbar) = \hbar\{\mathscr{A}[u_0(\vec{r})] - f(\vec{r})\}, \quad \vec{r} \in \Omega, \tag{2.14}$$

$$f_m(\vec{r}, \hbar) = m\left\{\mathscr{L}[u_0^{[m-1]}(\vec{r}, \hbar)] + \hbar \left.\frac{\mathrm{d}^{m-1}\mathscr{A}[U(\vec{r}, p, \hbar)]}{\mathrm{d}p^{m-1}}\right|_{p=0}\right\}, \quad \vec{r} \in \Omega, \ m > 1, \tag{2.15}$$

$$g_1(\vec{r}, \hbar) = \hbar\{\mathscr{B}[u_0(\vec{r})] - g(\vec{r})\}, \quad \vec{r} \in \Gamma, \tag{2.16}$$

$$g_m(\vec{r}, \hbar) = m\left\{\mathscr{L}_B[u_0^{[m-1]}(\vec{r}, \hbar)] + \hbar \left.\frac{\mathrm{d}^{m-1}\mathscr{B}[U(\vec{r}, p, \hbar)]}{\mathrm{d}p^{m-1}}\right|_{p=0}\right\}, \quad \vec{r} \in \Gamma, \ m > 1. \tag{2.17}$$

We emphasize that both the $m$th-order $(m \geq 1)$ deformation equation (2.12) and its boundary condition (2.13) are *linear*. Moreover, there is a lot of freedom to select the auxiliary linear operator $\mathscr{L}$ to ensure that its fundamental solution exists and is known *a priori*, no matter if the operator $\mathscr{A}$ contains linear terms. Therefore, the $m$th-order $(m \geq 1)$ deformation equations (2.12) and (2.13) can be rewritten in the forms similar to (1.2) and can be easily solved by the traditional BEM. Obviously, the validity of above approach does *not* need the two assumptions of the traditional BEM mentioned in the first section of this paper. Moreover, if $\mathscr{A} = \mathscr{L}_0 + N_0$ holds, and furthermore, if we select $\mathscr{L} = \mathscr{L}_0$ as the auxiliary linear operator and set $\hbar = -1$, the first-order formula $(M = 1)$ can give the corresponding expression similar to (1.2). Thus, in logic, the above mentioned general boundary element method contains the traditional BEM, as pointed out by Liao [3]. The proposed general BEM can be used to solve most of the non-linear problems in science and engineering, including those whose governing equations and boundary conditions do not contain any linear terms. It can greatly enlarge the application areas of the BEM as a kind of numerical technique.

Note that, in the case of $\hbar = -1$, (2.3) and (2.4) give the same zeroth-order deformation equations as those reported by Liao [1–4] and Liao and Chwang [5]. Therefore, introducing the non-zero parameter $\hbar$ makes the proposed BEM even more general so that it provides us with greater freedom and larger flexibility to construct better zeroth-order deformations equations to ensure that the proposed approach is valid. This can be easily understood: for given auxiliary linear operators and initial approximations, the convergence radius of the Taylor series (2.8) is now a function of the non-zero parameter $\hbar$; and moreover, a 'better' value of $\hbar$ should correspond to a larger convergence radius. If $\hbar$ and the auxiliary linear operators and initial approximations are so properly selected that the convergence radius of series (2.8) is not less than one, and furthermore, if the order $M$ of approximation is large enough, we can obtain by formula (2.9) accurate enough approximations even by means of no iteration! We will especially illustrate this point in Section 4.

 

## 3. SOME MATHEMATICAL DERIVATIONS

Clearly, the convergence radius of the series (2.9) depends on the auxiliary linear operators $\mathscr{L}$ and $\mathscr{L}_{\mathrm{B}}$, the value of $\hbar$ and the initial approximation $u_0(\vec{r})$. For given initial approximation and auxiliary linear operators, the value of $\hbar$ seems critical to ensure that the series (2.9) converges. Although it seems difficult to give general conclusions about how to select the above mentioned factors to ensure that the series (2.9) converges, some mathematical derivations are helpful for better understanding the proposed approach.

First of all, consider the case where both $\mathscr{A}$ and $\mathscr{B}$ are linear operators. In this case, we have by (2.12), (2.14) and (2.15) that

$$\mathscr{L}u_0^{[1]} = \hbar(\mathscr{A}u_0 - f), \tag{3.1}$$

$$\mathscr{L}u_0^{[m]} = m(\mathscr{L}u_0^{[m-1]} + \hbar\mathscr{A}u_0^{[m-1]}) = m(\mathscr{L} + \hbar\mathscr{A})u_0^{[m-1]}, \quad m > 1. \tag{3.2}$$

Due to (3.2), we have

$$u_0^{[m]}(\vec{r}, \hbar) = m(\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A})u_0^{[m-1]}(\vec{r}, \hbar), \quad m > 1, \tag{3.3}$$

which gives

$$u_0^{[2]}(\vec{r}, \hbar) = 2!(\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A})u_0^{[1]}(\vec{r}, \hbar), \tag{3.4}$$

$$u_0^{[3]}(\vec{r}, \hbar) = 3(\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A})u_0^{[2]}(\vec{r}, \hbar) = 3!(\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A})^2 u_0^{[1]}(\vec{r}, \hbar), \tag{3.5}$$

$$u_0^{[4]}(\vec{r}, \hbar) = 4(\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A})u_0^{[3]}(\vec{r}, \hbar) = 4!(\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A})^3 u_0^{[1]}(\vec{r}, \hbar), \tag{3.6}$$

$$\vdots$$

so that it holds that

$$u_0^{[m]}(\vec{r}, \hbar) = m!(\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A})^{m-1} u_0^{[1]}(\vec{r}, \hbar), \tag{3.7}$$

where $\mathbf{I}$ denotes the identity mapping. Therefore, we have by (2.9), that

$$u_0(\vec{r}) + \sum_{k=1}^{+\infty} \frac{u_0^{[k]}(\vec{r}, \hbar)}{k!} = u_0(\vec{r}) + [\mathbf{I} + (\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A}) + (\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A})^2 + \cdots$$
$$+ (\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A})^{m-1} + \cdots]u_0^{[1]}(\vec{r}, \hbar). \tag{3.8}$$

Clearly the right-hand-side of above expression converges if

$$\|\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A}\| < 1. \tag{3.9}$$

Similarly, it should hold that

$$\|\mathbf{I} + \hbar\mathscr{L}_{\mathrm{B}}^{-1}\mathscr{B}\| < 1 \tag{3.10}$$

for the boundary condition (2.2). Moreover, if (3.9) holds, we have by (3.1) and (3.8), that

$$\mathscr{A}\left(u_0 + \sum_{k=1}^{+\infty} \frac{u_0^{[k]}}{k!}\right)$$
$$= \mathscr{A}u_0 + \mathscr{A}[\mathbf{I} + (\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A}) + (\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A})^2 + \cdots + (\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A})^{m-1} + \cdots]u_0^{[1]}$$
$$= \mathscr{A}u_0 + \mathscr{A}[\mathbf{I} - (\mathbf{I} + \hbar\mathscr{L}^{-1}\mathscr{A})]^{-1}\mathscr{L}^{-1}[\hbar(\mathscr{A}u_0 - f)]$$
$$= \mathscr{A}u_0 + \mathscr{A}(-\hbar^{-1})\mathscr{A}^{-1}\mathscr{L}\mathscr{L}^{-1}[\hbar(\mathscr{A}u_0 - f)] = \mathscr{A}u_0 - (\mathscr{A}u_0 - f) = f. \tag{3.11}$$

Similarly, if (3.10) holds, we have

 

$$\mathscr{B}\left(u_0+\sum_{k=1}^{+\infty}\frac{u_0^{[k]}}{k!}\right)=g. \tag{3.12}$$

Thus, if both (3.9) and (3.10) hold, the series

$$u_0+\sum_{k=1}^{+\infty}\frac{u_0^{[k]}}{k!}$$

converges to the solution of (2.1) and (2.2). Therefore, we have

**Theorem 1**
Let $\mathscr{A}$ and $\mathscr{B}$ be linear operators, $\mathscr{L}$ and $\mathscr{L}_{\mathrm{B}}$ be auxiliary linear operators for the governing equation (2.1) and the boundary condition (2.2) respectively. The series (2.9) converges to the solution of (2.1) and (2.2), if both $\|\mathbf{I}+\hbar\mathscr{L}^{-1}\mathscr{A}\|<1$ and $\|\mathbf{I}+\hbar\mathscr{L}_{\mathrm{B}}^{-1}\mathscr{B}\|<1$ hold.

Clearly, when $\mathscr{B}=\mathbf{I}$ is an identity mapping, say, the boundary condition (2.2) is essential, we can select $\mathscr{L}_{\mathrm{B}}=\mathbf{I}$. So, according to theorem 1, we have

**Theorem 2**
Let $\mathscr{A}$ be a linear operator, $\mathscr{B}=\mathbf{I}$ be an identity mapping and $\mathscr{L}$ be an auxiliary linear operator for the governing equation (2.1) whose fixed point is 0. Then, the series (2.9) converges to the solution of (2.1) and (2.2) if both $\|\mathbf{I}+\hbar\mathscr{L}^{-1}\mathscr{A}\|<1$ and $|1+\hbar|<1$ hold.

Furthermore, when $\mathscr{A}$ is a non-linear operator and $\mathscr{B}=\mathbf{I}$ is an identity mapping, it is clear that we must select the value of $\hbar$ in a subset of the region $|1+\hbar|<1$. Thus, we have

**Theorem 3**
Let $\mathscr{A}$ be a non-linear operator, $\mathscr{B}=\mathbf{I}$ be an identity mapping. Then, the series (2.9) converges to the solution of (2.1) and (2.2) when $\hbar$ is in a subset of $|1+\hbar|<1$.

Secondly, let us consider the case where $\mathscr{A}$ and/or $\mathscr{B}$ are non-linear operators. Clearly, the Taylor series of $\mathscr{A}[U(\vec{r},p,\hbar)]$ expanded at $p=0$ is

$$\mathscr{A}u_0+\sum_{m=1}^{+\infty}\frac{p^m}{m!}\left.\frac{\mathrm{d}^m\mathscr{A}[U(\vec{r},p,\hbar)]}{\mathrm{d}p^m}\right|_{p=0}. \tag{3.13}$$

If above series converges at $p=1$, we have

$$\mathscr{A}[U(\vec{r},1,\hbar)]=\mathscr{A}u_0+\sum_{m=1}^{+\infty}\frac{1}{m!}\left.\frac{\mathrm{d}^m\mathscr{A}[U(\vec{r},p,\hbar)]}{\mathrm{d}p^m}\right|_{p=0}. \tag{3.14}$$

Moreover, owing to (2.12), (2.14) and (2.15), we have

$$\mathscr{L}u_0^{[1]}=\hbar(\mathscr{A}u_0-f), \tag{3.15}$$

$$\mathscr{L}u_0^{[2]}=2\left[\mathscr{L}u_0^{[1]}+\hbar\left.\frac{\mathrm{d}\mathscr{A}}{\mathrm{d}p}\right|_{p=0}\right]=2!\hbar\left[\mathscr{A}u_0+\left.\frac{\mathrm{d}\mathscr{A}}{\mathrm{d}p}\right|_{p=0}-f\right], \tag{3.16}$$

$$\mathscr{L}u_0^{[3]}=3\left[\mathscr{L}u_0^{[2]}+\hbar\left.\frac{\mathrm{d}^2A}{\mathrm{d}p^2}\right|_{p=0}\right]=3!\hbar\left[\mathscr{A}u_0+\left.\frac{\mathrm{d}\mathscr{A}}{\mathrm{d}p}\right|_{p=0}+\frac{1}{2!}\left.\frac{\mathrm{d}^2\mathscr{A}}{\mathrm{d}p^2}\right|_{p=0}-f\right], \tag{3.17}$$

$$\vdots$$

therefore, it holds

$$\mathscr{L}u_0^{[m]} = m!\hbar\left[\mathscr{A}u_0 + \frac{\mathrm{d}\mathscr{A}}{\mathrm{d}p}\bigg|_{p=0} + \frac{1}{2!}\frac{\mathrm{d}^2\mathscr{A}}{\mathrm{d}p^2}\bigg|_{p=0} + \cdots + \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}\mathscr{A}}{\mathrm{d}p^{m-1}}\bigg|_{p=0} - f\right], \tag{3.18}$$

which gives

$$\mathscr{L}\left(\frac{u_0^{[m]}}{m!}\right) = \hbar\left[\mathscr{A}u_0 + \frac{\mathrm{d}\mathscr{A}}{\mathrm{d}p}\bigg|_{p=0} + \frac{1}{2!}\frac{\mathrm{d}^2\mathscr{A}}{\mathrm{d}p^2}\bigg|_{p=0} + \cdots + \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}\mathscr{A}}{\mathrm{d}p^{m-1}}\bigg|_{p=0} - f\right]. \tag{3.19}$$

If the series (2.9) converges, say,

$$U(\vec{r}, 1, \hbar) = u_0 + \sum_{m=1}^{+\infty} \frac{u_0^{[m]}}{m!}, \tag{3.20}$$

it holds that

$$\lim_{m\to+\infty} \frac{u_0^{[m]}}{m!} = 0, \tag{3.21}$$

which gives, according to the assumption that zero is the fixed point of the auxiliary linear operators $\mathscr{L}$,

$$\lim_{m\to+\infty} \mathscr{L}\left(\frac{u_0^{[m]}}{m!}\right) = 0. \tag{3.22}$$

Notice that $\hbar$ is assumed to be a non-zero parameter. So, due to (3.19) and (3.22), we have

$$\lim_{m\to+\infty}\left[\mathscr{A}u_0 + \sum_{k=1}^{m-1}\frac{1}{k!}\frac{\mathrm{d}^k\mathscr{A}}{\mathrm{d}p^k}\bigg|_{p=0} - f\right] = 0, \tag{3.23}$$

which means that the series (3.13) is convergent at $p = 1$. Thus, due to (3.14) and (3.20), it holds that

$$\mathscr{A}\left[u_0(\vec{r}) + \sum_{m=1}^{+\infty}\frac{u_0^{[m]}(\vec{r}, \hbar)}{m!}\right] = f(\vec{r}). \tag{3.24}$$

Similarly, we have

$$\mathscr{B}\left[u_0(\vec{r}) + \sum_{m=1}^{+\infty}\frac{u_0^{[m]}(\vec{r}, \hbar)}{m!}\right] = g(\vec{r}). \tag{3.25}$$

Therefore, we have

**Theorem 4**
As long as the series (2.9) converges, it converges to a solution of Equations (2.1) and (2.2), and moreover, it holds

$$\lim_{m\to+\infty}\left(\mathscr{A}[u_0(\vec{r})] + \sum_{k=1}^{m-1}\frac{1}{k!}\frac{\mathrm{d}^k\mathscr{A}[U(\vec{r}, p, \hbar)]}{\mathrm{d}p^k}\bigg|_{p=0} - f(r)\right) = 0,$$

and

$$\lim_{m\to+\infty}\left(\mathscr{B}[u_0(\vec{r})] + \sum_{k=1}^{m-1}\frac{1}{k!}\frac{\mathrm{d}^k\mathscr{B}[U(\vec{r}, p, \hbar)]}{\mathrm{d}p^k}\bigg|_{p=0} - g(\vec{r})\right) = 0.$$

Besides, comparing (3.18) with (2.12), we have

 

$$f_m(\vec{r}, \hbar) = m!\hbar\left[\mathscr{A}u_0 + \sum_{k=1}^{m-1} \frac{1}{k!} \frac{\mathrm{d}^k\mathscr{A}}{\mathrm{d}p^k}\bigg|_{p=0} - f(\vec{r})\right], \tag{3.26}$$

which gives

$$\frac{f_m(\vec{r}, \hbar)}{m!\hbar} = \mathscr{A}[u_0(\vec{r})] + \sum_{k=1}^{m-1} \frac{1}{k!} \frac{\mathrm{d}^k\mathscr{A}[U(\vec{r}, p, \hbar)]}{\mathrm{d}p^k}\bigg|_{p=0} - f(\vec{r}), \quad m \geq 1. \tag{3.27}$$

Similarly, we have

$$\frac{g_m(\vec{r}, \hbar)}{m!\hbar} = \mathscr{B}[u_0(\vec{r})] + \sum_{k=1}^{m-1} \frac{1}{k!} \frac{\mathrm{d}^k\mathscr{B}[U(\vec{r}, p, \hbar)]}{\mathrm{d}p^k}\bigg|_{p=0} - g(\vec{r}), \quad m \geq 1. \tag{3.28}$$

Define

$$\delta_{m-1}(\vec{r}, \hbar) = \frac{f_m(\vec{r}, \hbar)}{m!\hbar} = \mathscr{B}[u_0(\vec{r})] + \sum_{k=1}^{m-1} \frac{1}{k!} \frac{\mathrm{d}^k\mathscr{A}[U(\vec{r}, p, \hbar)]}{\mathrm{d}p^k}\bigg|_{p=0} - f(\vec{r}), \quad m \geq 1 \tag{3.29}$$

and

$$\Delta_{m-1}(\vec{r}, \hbar) = \frac{g_m(\vec{r}, \hbar)}{m!\hbar} = \mathscr{B}[u_0(\vec{r})] + \sum_{k=1}^{m-1} \frac{1}{k!} \frac{\mathrm{d}^k\mathscr{B}[U(\vec{r}, p, \hbar)]}{\mathrm{d}p^k}\bigg|_{p=0} - g(\vec{r}), \quad m \geq 1. \tag{3.30}$$

Clearly, if the series (2.9) is convergent, then owing to theorem 4 and foregoing derivations, the series

$$\delta_0(\vec{r}, \hbar),\ \delta_1(\vec{r}, \hbar),\ \delta_2(\vec{r}, \hbar), \ldots,\ \delta_m(\vec{r}, \hbar), \ldots \tag{3.31}$$

and

$$\Delta_0(\vec{r}, \hbar),\ \Delta_1(\vec{r}, \hbar),\ \Delta_2(\vec{r}, \hbar), \ldots,\ \Delta_m(\vec{r}, \hbar), \ldots \tag{3.32}$$

converge. Thus, we have

**Theorem 5**
If the series (2.9) converges, then the series

$$\frac{f_1(\vec{r}, \hbar)}{\hbar}, \frac{f_2(\vec{r}, \hbar)}{2!\hbar}, \ldots, \frac{f_m(\vec{r}, \hbar)}{m!\hbar}$$

and

$$\frac{g_1(\vec{r}, \hbar)}{\hbar}, \frac{g_2(\vec{r}, \hbar)}{2!\hbar}, \ldots, \frac{g_m(\vec{r}, \hbar)}{m!\hbar},$$

also converge, where $f_m(\vec{r}, \hbar)$ and $g_m(\vec{r}, \hbar)$, defined by (2.14)–(2.17), are respectively inhomogeneous terms of the $m$th-order deformation equations (2.12) and (2.13) ($m \geq 1$).

What are the meanings of the terms $\delta_m(\vec{r}, \hbar)$ and $\Delta_m(\vec{r}, \hbar)$? According to (3.14),

$$\mathscr{A}[u_0(\vec{r})] + \sum_{k=1}^{m-1} \frac{1}{k!} \frac{\mathrm{d}^k\mathscr{A}[U(\vec{r}, p, \hbar)]}{\mathrm{d}p^k}\bigg|_{p=0} = f(\vec{r}) \tag{3.33}$$

can be seen as the $(m-1)$th-order Maclaurin expansion about $p$ of the governing equation

$$\mathscr{A}[U(\vec{r}, p, \hbar)] = f(\vec{r}) \tag{3.34}$$

at $p = 1$, where $U(\vec{r}, p, \hbar)$ is governed by the zeroth-order deformation equations (2.3) and (2.4). If the series (2.9) converges, then according to theorem 4 and the related derivations, the

$m$th-order Maclaurin expansion (3.33) of the governing equation (2.1) becomes closer and closer to the exact equation (2.1) as the order of $m$ increases. Clearly, $\delta_0(\vec{r}, \hbar) = \mathscr{A} u_0 - f(\vec{r})$ gives residual errors of the governing equation (2.1) under the initial approximation $u_0(\vec{r})$. Moreover, if the series (2.9) converges, we have by theorem 4 and (3.14) and (3.24) that $\delta_{+\infty}(\vec{r}, \hbar) = \mathscr{A}[U(\vec{r}, 1, \hbar)] - f(\vec{r}) = 0$, say, the residual error $\delta_{+\infty}(\vec{r}, \hbar)$ of (2.1) under the convergent series

$$u(\vec{r}) = u_0(\vec{r}) + \sum_{k=1}^{+\infty} \frac{u_0^{[k]}(\vec{r}, \hbar)}{k!}$$

is zero. On the other hand, if the series (2.9) is divergent, we have certainly $|\delta_{+\infty}(\vec{r}, \hbar)| \to +\infty$, say, the residual error $\delta_{+\infty}(\vec{r}, \hbar)$ of (2.1) tends to infinity when the series (2.9) diverges. Thus, in general,

$$\delta_{M-1}(\vec{r}, \hbar) = \frac{f_M(\vec{r}, \hbar)}{M!\hbar}$$

denotes the residual errors of the $(M-1)$th-order Maclaurin expansion of the governing equation (2.1) under the $(M-1)$th-order of approximation

$$u_{M-1}(\vec{r}) = u_0(\vec{r}) + \sum_{k=1}^{M-1} \frac{u_0^{[k]}(\vec{r}, \hbar)}{k!}.$$

Similarly, the residual errors of the $(M-1)$th-order Maclaurin expansion of the boundary condition (2.2) under the $(M-1)$th-order of approximation are given by

$$\Delta_{M-1}(\vec{r}, \hbar) = \frac{g_M(\vec{r}, \hbar)}{M!\hbar}.$$

If the series (2.9) is convergent and the order $M$ of the approximation is high enough, $\delta_M(\vec{r}, \hbar)$ and $\Delta_M(\vec{r}, \hbar)$ are usually rather close to the residual errors of the governing equation (2.1) and boundary condition (2.2) respectively. Therefore, although for finite integers $m$ ($m \geq 1$) we have

$$\delta_m(\vec{r}, \hbar) \neq \mathscr{A}\left[u_0(\vec{r}) + \sum_{k=1}^{m} \frac{u_0^{[k]}(\vec{r}, \hbar)}{k!}\right] - f(\vec{r}) \tag{3.35}$$

and

$$\Delta_m(\vec{r}, \hbar) \neq \mathscr{B}\left[u_0(\vec{r}) + \sum_{k=1}^{m} \frac{u_0^{[k]}(\vec{r}, \hbar)}{k!}\right] - g(\vec{r}), \tag{3.36}$$

say, $\delta_m(\vec{r}, \hbar)$ and $\Delta_m(\vec{r}, \hbar)$ are not exactly equal to the residual errors of the governing equation (2.1) and boundary condition (2.2) under the $m$th-order approximation respectively. Nevertheless, the terms $\delta_m(\vec{r}, \hbar)$ and $\Delta_m(\vec{r}, \hbar)$ still report the accuracy of the $m$th-order approximation. Notice that $\delta_0(\vec{r}, \hbar)$ and $\Delta_0(\vec{r}, \hbar)$ denote the residual errors of the governing equation (2.1) and boundary condition (2.2) under the initial approximation $u_0(\vec{r})$, i.e. the zeroth-order approximation. So, we can generalize this concept and call $\delta_m(\vec{r}, \hbar)$ and $\Delta_m(\vec{r}, \hbar)$ *the mth-order residual error* of the governing equation (2.1) and the boundary condition (2.2) respectively. Thus, we have

**Theorem 6**

The $(M-1)$th-order residual errors of the governing equation (2.1) and the boundary condition (2.2) under the $(M-1)$th-order approximation

 

$$u_{M-1}(\vec{r}) = u_0(\vec{r}) + \sum_{k=1}^{M-1} \frac{u_0^{[k]}(\vec{r}, \hbar)}{k!}$$

are given by

$$\delta_{M-1}(\vec{r}, \hbar) = \frac{f_M(\vec{r}, \hbar)}{M!\hbar}$$

and

$$\Delta_{M-1}(\vec{r}, \hbar) = \frac{g_M(\vec{r}, \hbar)}{M!\hbar}$$

respectively, where $f_M(\vec{r}, \hbar)$ and $g_M(\vec{r}, \hbar)$, defined by (2.22)–(2.25), are inhomogeneous terms of the $M$th-order deformation equations (2.12) and (2.13) ($M \geq 1$).

If the initial approximation $u_0(\vec{r})$, the non-zero parameter $\hbar$, the auxiliary linear operators $\mathscr{L}$ and $\mathscr{L}_{\mathrm{B}}$ are so properly selected that the series (2.9) is convergent, then by theorem 5, the $M$th-order residual errors $|\delta_M(\vec{r}, \hbar)|$ and $|\Delta_M(\vec{r}, \hbar)|$ should decrease, although not always monotonously, for large enough values of $M$. So, the order $M$ of the approximation should be increased until some convergence criterion *a priori* are satisfied. This can be easily done in a computer program, because the terms $f_M(\vec{r}, \hbar)$ and $g_M(\vec{r}, \hbar)$ are known before calculating the $M$th-order approximation. Suppose the convergence criterion are

$$\sqrt{\frac{\iint_\Omega |\delta_{M-1}(\vec{r}, \hbar)|^2 \, \mathrm{d}\Omega}{\iint_\Omega \mathrm{d}\Omega}} < \alpha, \qquad \sqrt{\frac{\oint_\Gamma |\Delta_{M-1}(\vec{r}, \hbar)|^2 \, \mathrm{d}\Gamma}{\oint_\Gamma \mathrm{d}\Gamma}} < \beta. \tag{3.37}$$

Then, owing to (3.29) and (3.30), we have

$$\frac{1}{|\hbar| M!} \sqrt{\frac{\iint_\Omega |f_M(\vec{r}, \hbar)|^2 \, \mathrm{d}\Omega}{\iint_\Omega \mathrm{d}\Omega}} < \alpha, \qquad \frac{1}{|\hbar| M!} \sqrt{\frac{\oint_\Gamma |g_M(\vec{r}, \hbar)|^2 \, \mathrm{d}\Gamma}{\oint_\Gamma \mathrm{d}\Gamma}} < \beta, \tag{3.38}$$

which determine the minimum value of $M$, the order of the approximation. So, we can simply use the known terms $f_M(\vec{r}, \hbar)$ and $g_M(\vec{r}, \hbar)$, defined by (2.14)–(2.17), to examine the accuracy of the approximations and then to determine if we should further go on to get higher-order approximations. This provides us with a simple way of determining the order $M$ of the approximation.

Finally, we point out that the $m$th-order deformation equations (2.12) and (2.13) can be rewritten in other forms more convenient for numerical calculations. Writing

$$\bar{u}_0^{[k]}(\vec{r}, \hbar) = \frac{u_0^{[k]}(\vec{r}, \hbar)}{k!}, \tag{3.39}$$

we have the corresponding $m$th-order deformation equation

$$\mathscr{L}[\bar{u}_0^{[m]}(\vec{r}, \hbar)] = F_m(\vec{r}, \hbar), \quad \vec{r} \in \Omega, \ \hbar \neq 0, \ m = 1, 2, 3, \ldots, \tag{3.40}$$

with the boundary condition

$$\mathscr{L}_{\mathrm{B}}[\bar{u}_0^{[m]}(\vec{r}, \hbar)] = G_m(\vec{r}, \hbar), \quad \vec{r} \in \Gamma, \ \hbar \neq 0, \ m = 1, 2, 3, \ldots, \tag{3.41}$$

 

where

$$F_1(\vec{r}, \hbar) = \hbar\{\mathscr{A}[u_0(\vec{r})] - f(\vec{r})\}, \quad \vec{r}\in\Omega, \tag{3.42}$$

$$F_m(\vec{r}, \hbar) = \mathscr{L}[\bar{u}_0^{[m-1]}(\vec{r}, \hbar)] + \frac{\hbar}{(m-1)!}\left.\frac{\mathrm{d}^{m-1}\mathscr{A}[U(\vec{r}, p, \hbar)]}{\mathrm{d}p^{m-1}}\right|_{p=0}$$

$$= F_{m-1}(\vec{r}, \hbar) + \frac{\hbar}{(m-1)!}\left.\frac{\mathrm{d}^{m-1}\mathscr{A}[U(\vec{r}, p, \hbar)]}{\mathrm{d}p^{m-1}}\right|_{p=0}, \quad \vec{r}\in\Omega, \ m>1, \tag{3.43}$$

$$G_1(\vec{r}, \hbar) = \hbar\{\mathscr{B}[u_0(\vec{r})] - g(\vec{r})\}, \quad \vec{r}\in\Gamma, \tag{3.44}$$

$$G_m(\vec{r}, \hbar) = \mathscr{L}_{\mathrm{B}}[u_0^{[m-1]}(\vec{r}, \hbar)] + \frac{\hbar}{(m-1)!}\left.\frac{\mathrm{d}^{m-1}\mathscr{B}[U(\vec{r}, p, \hbar)]}{\mathrm{d}p^{m-1}}\right|_{p=0}$$

$$= G_{m-1}(\vec{r}, \hbar) + \frac{\hbar}{(m-1)!}\left.\frac{\mathrm{d}^{m-1}\mathscr{B}[U(\vec{r}, p, \hbar)]}{\mathrm{d}p^{m-1}}\right|_{p=0}, \quad \vec{r}\in\Gamma, \ m>1. \tag{3.45}$$

The corresponding $M$th-order approximation is now

$$u_M(\vec{r}) = u_0(\vec{r}) + \sum_{k=1}^{M} \bar{u}_0^{[k]}(\vec{r}, \hbar). \tag{3.46}$$

The corresponding $(M-1)$th-order residual errors of the governing equation (2.1) and boundary condition (2.2) are

$$\delta_{M-1}(\vec{r}, \hbar) = \frac{F_M(\vec{r}, \hbar)}{\hbar} \tag{3.47}$$

and

$$\Delta_{M-1}(\vec{r}, \hbar) = \frac{G_M(\vec{r}, \hbar)}{\hbar}$$

respectively.

## 4. NUMERICAL EXAMPLES

### *4.1. Example 1*

Firstly, let us consider a two-dimensional non-linear partial differential equation:

$$\nabla[\sin(x+y)\nabla u(x, y)] + u^2(x, y) = f(x, y), \quad (x, y)\in\Omega, \tag{4.1}$$

with the essential boundary condition

$$u(x, y) = \sin(x+y), \quad (x, y)\in\Gamma, \tag{4.2}$$

where $\Omega = [0, 1]\times[0, 1]$ and $\Gamma$ is its boundary. To clearly show the validity of the proposed approach, we set $f(x, y) = 2\cos^2(x+y) - \sin^2(x+y)$ so that $\sin(x+y)$ is a solution of (4.1) and (4.2).

Although (4.1) has a linear term

$$\mathscr{L}_0(u) = \nabla[\sin(x+y)\nabla u(x, y)], \tag{4.3}$$

the fundamental solution of its related linear operator $\mathscr{L}_0 = \nabla[\sin(x+y)\nabla]$ is unfortunately unknown (we do not even know if there exists a fundamental solution of it). Therefore, the traditional BEM is invalid for this non-linear problem. However, the proposed general BEM can be easily applied to solve it. In order to show this, we select the familiar two-dimensional Laplace operator,

$$\mathscr{L} = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{4.4}$$

as our auxiliary linear operator to construct the corresponding zeroth-order deformation equation

$$\begin{aligned}&(1-p)\nabla^2[U(x, y, p, \hbar) - u_0(x, y)]\\&= p\hbar\{\nabla[\sin(x+y)\nabla U(x, y, p, \hbar)] + U^2(x, y, p, \hbar) - f(x, y)\},\\&\quad (x, y)\in[0, 1]\times[0, 1],\ p\in[0, 1],\ \hbar \neq 0,\end{aligned} \tag{4.5}$$

with the boundary condition

$$\begin{aligned}&(1-p)[U(x, y, p, \hbar) - u_0(x, y)] = \hbar p[U(x, y, p, \hbar) - \sin(x+y)],\\&\quad (x, y)\in\Gamma,\ p\in[0, 1],\ \hbar \neq 0,\end{aligned} \tag{4.6}$$

where $u_0(x, y)$ is an initial approximation and we define here

$$\nabla U(x, y, p, \hbar) = \frac{\partial U(x, y, p, \hbar)}{\partial x}\vec{i} + \frac{\partial U(x, y, p, \hbar)}{\partial y}\vec{j} \tag{4.7}$$

and

$$\nabla^2 U(x, y, p, \hbar) = \frac{\partial^2 U(x, y, p, \hbar)}{\partial x^2} + \frac{\partial U^2(x, y, p, \hbar)}{\partial y^2}. \tag{4.8}$$

According to (2.8), we have

$$U(x, y, p, \hbar) = u_0(x, y) + \sum_{k=1}^{+\infty}\left[\frac{u_0^{[k]}(x, y, \hbar)}{k!}\right]p^k, \tag{4.9}$$

where

$$u_0^{[k]}(x, y, \hbar) = \left.\frac{\partial^k U(x, y, p, \hbar)}{\partial p^k}\right|_{p=0} \tag{4.10}$$

is the corresponding $k$th-order deformation derivatives of $U(x, y, p, \hbar)$ at $p = 0$. If the value of $\hbar$ is properly selected that, for every point $(x, y)\in\Omega$, the convergence radius $\rho$ of the Taylor series (4.9) is not less than 1, we have by (2.9) that

$$u(x, y) = u_0(x, y) + \sum_{k=1}^{+\infty}\frac{u_0^{[k]}(x, y, \hbar)}{k!}, \tag{4.11}$$

where, by (2.12)–(2.17), $u_0^{[k]}(x, y, \hbar)$ $(k \geq 1)$ are determined by the linear equations

$$\nabla^2 u_0^{[k]}(x, y, \hbar) = f_k(x, y, \hbar), \quad (x, y)\in\Omega,\ \hbar \neq 0,\ k \geq 1, \tag{4.12}$$

with boundary condition

$$u_0^{[k]}(x, y, \hbar) = \begin{cases}\hbar[u_0(x, y) - \sin(x+y)], & k = 1,\ (x, y)\in\Gamma,\ \hbar \neq 0\\(1+\hbar)u_0^{[k-1]}(x, y, \hbar), & k \geq 1,\ (x, y)\in\Gamma,\ \hbar \neq 0\end{cases}, \tag{4.13}$$

where

$$f_1(x, y, \hbar) = \hbar\{\nabla[\sin(x+y)\nabla u_0(x, y)] + [u_0(x, y)]^2 - f(x, y)\}, \tag{4.14}$$

$$f_k(x, y, \hbar) = k\left\{\nabla^2 u_0^{[k-1]}(x, y, \hbar) + \hbar\nabla[\sin(x+y)\nabla u_0^{[k-1]}(x, y, \hbar)] + \hbar\sum_{n=0}^{k-1}\binom{k-1}{n} u_0^{[n]}(x, y, \hbar)u_0^{[k-1-n]}(x, y, \hbar)\right\}, \quad k \geq 2. \tag{4.15}$$

Here, we define $u_0^{[0]}(x, y, \hbar) = u_0(x, y)$. The above $k$th-order deformation equations are obtained by differentiating the zeroth-order deformation equations (4.5) and (4.6) $k$ times with respect to $p$ and then setting $p = 0$. We emphasize that (4.12) is now a typical Poisson equation and the boundary condition (4.13) is linear. The Poisson equation (4.12) can be easily solved by the traditional BEM. Therefore, if the value of $\hbar$ is properly selected so that the convergence radius $\rho$ of the series (4.9) is not less than 1, we have by (4.11) the $M$th-order approximation

$$u_M(x, y) = u_0(x, y) + \sum_{k=1}^{M} \frac{u_0^{[k]}(x, y, \hbar)}{k!}. \tag{4.16}$$

If so, the larger the value of $M$, the more accurate the approximation of $u_M(x, y)$. Therefore, if $M$ is large enough, we can obtain an accurate enough approximation by means of solving $M$ linear equations (4.12) one after another, so that *no* iteration is necessary.

In this example, we simply select $u_0(x, y) = 0$ as our initial approximation. We divide each side of the boundary $\Gamma$ into $N$ ($N = 30$) equal elements and the domain $\Omega$ into $N \times N$ ($N = 30$) equal subdomains. At four corners of the boundary, two points, which are very close to each other but respectively belong to two different sides, are used to treat the discontinuation of the solutions there. Among each element, we use a linear distribution function. Thus, there exists $4(N+1)$ unknowns. Moreover, the Gauss integral is applied to calculate related volume integrals. We emphasize that the corresponding coefficient matrix $\mathbf{M}$ is the same for all $k$th-order ($k \geq 1$) deformation equations (4.12), so that its inverse matrix $\mathbf{M}^{-1}$, as long as we get it, can be repeatedly used. Thus, if direct techniques are used to solve the $k$th-order ($k \geq 1$) deformation equations, the proposed general BEM has a rather high efficiency.

Notice that we introduce here a new non-zero parameter $\hbar$ whose value is critical for the convergence of the series (4.11). According to theorem 3, the series (4.11) converges if $\hbar$ is in a subset of the region $-2 < \hbar < 0$. Our numerical experiments confirm this conclusion: the series (4.11) indeed converges when $-1.5 \leq \hbar < 0$. Without loss of generality, we illustrate here only three different cases, say $\hbar = -0.25$, $\hbar = -0.75$, $\hbar - 1.25$ respectively. Note that $\sin(x+y)$ is an exact solution of (4.1) and (4.2). So, the accuracy of the $M$th-order approximation given by (4.16) can be reported by the root-mean-square absolute error

$$\tilde{R}_M = \frac{\sqrt{\sum_{i=0}^{N}\sum_{j=0}^{N} |\sin(x_i + y_j) - u_M(x_i, y_j)|^2}}{(N+1)}. \tag{4.17}$$

The values of the root-mean-square absolute error $\tilde{R}_M$ of the $M$th-order approximations ($0 \leq M \leq 50$) are shown in Figure 1 respectively for $\hbar = -0.25$, $\hbar = -0.75$, $\hbar = -1.25$. It shows that, in all three cases under consideration, the higher the order of $M$, the more accurate the related approximation, so that the series (4.11) converges to the exact solution $\sin(x+y)$. Thus, no iteration is necessary, if the order $M$ of the approximation is high enough. We emphasize that iteration is absolutely necessary when the traditional BEM is applied to solve non-linear problems. However, by means of the proposed general BEM, we can get accurate approximations even by means of no iterations! This shakes the absolutely governing place of the iterative methodology of the BEM for non-linear problems and also shows the validity and potential of the further generalized BEM.

On the other hand, we can use the root-mean-square residual errors

$$R_M = \frac{\sqrt{\sum_{i=0}^{N} \sum_{j=0}^{N} |\nabla[\sin(x_i + y_j)\nabla u_M(x_i, y_j)] + u_M^2(x_i, y_j) - f(x_i, y_j)|^2}}{(N+1)} \tag{4.18}$$

of Equation (4.1) under the $M$th-order approximation $u_M(x, y)$ to show its accuracy. Clearly, the smaller the value of $R_M$, the more accurate the corresponding approximation. The values of $R_M$ under the three considered cases are given in Figure 2, which confirms once again that the series (4.11) under three considered cases, say $\hbar = -0.25$, $\hbar = -0.75$, $\hbar = -1.25$, is indeed convergent. Note that, the term $\delta_{M-1}(x, y, \hbar) = f_M(x, y, \hbar)/(\hbar M!)$, where $f_M(x, y, \hbar)$ is now defined by (4.14) or (4.15), is very close to the corresponding root-mean-square residual error $R_{M-1}$, as shown in Figure 2. Therefore, we can use the term $\delta_{M-1}(x, y, \hbar) = f_M(x, y, \hbar)/\hbar M!$ to determine if we need go on to get higher-order approximations. It indeed provides us with a very simple way to determine $M$, the order of approximation.

The approximations at low-order are usually not satisfactory. Nevertheless, one can use the unsatisfactory approximations to renew the initial one, say $u_0(x, y)$. This gives a kind of iterative procedure, as pointed out by Liao [1–4] and Liao and Chwang [5] in details. What we would like to emphasize here is that non-linear problems can be solved by the general BEM without iterations. It means that we can obtain, even by means of no iterations, accurate enough approximations of the non-linear Equations (4.1) and (4.2) only by solving $M$ ($M$ is large enough) linear equations one after another! This example infers that iteration is not absolutely necessary for solving non-linear problems even by the BEM. In other words, the iterative methodology might lose its absolute governing place to the BEM for non-linear problems. This should impel us to think deeply about the essence of the BEM for non-linear

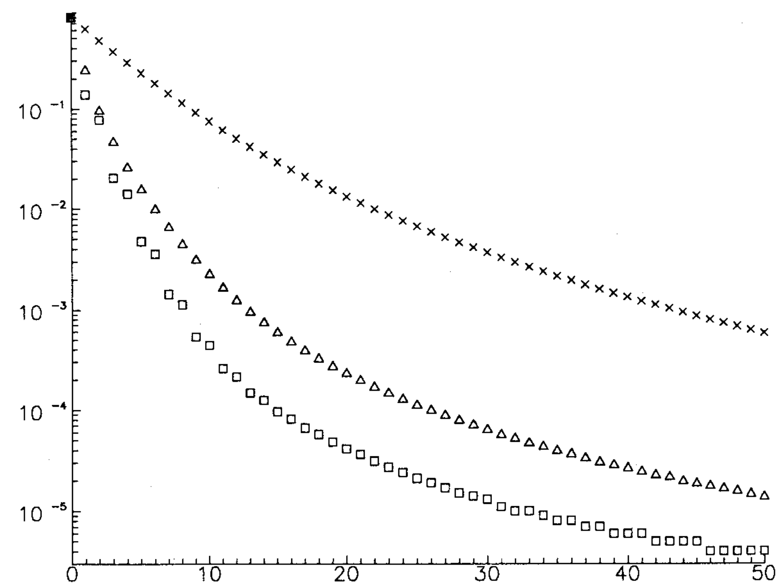


Figure 1. Root-mean-square absolute errors $\tilde{R}_M$ of example 1. Horizontal axis: $M$, the order of approximation (4.12); vertical axis: $\tilde{R}_M$ defined by (4.17). $\times$, $\hbar = -0.25$; $\triangle$, $\hbar = -0.75$; $\square$, $\hbar = -1.25$.

 

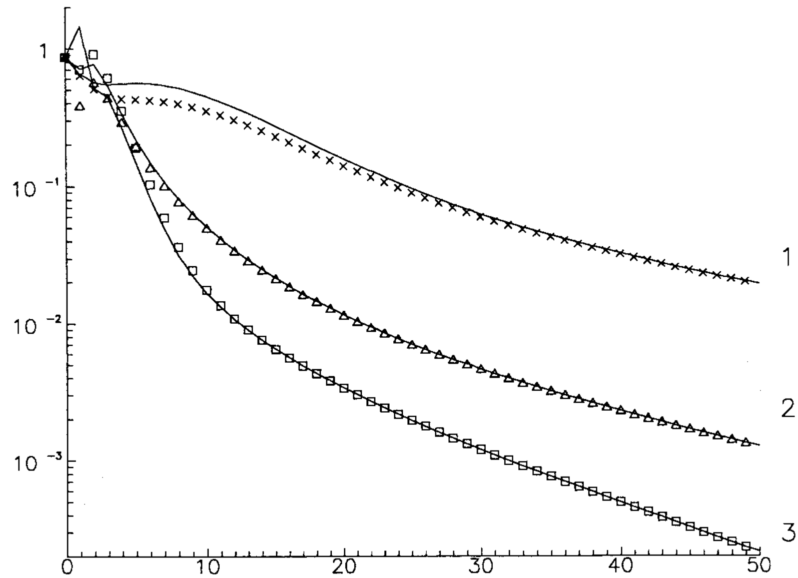


Figure 2. Root-mean-square residual errors $R_M$ of example 1. Horizontal axis: $M$, the order of approximation (4.12), vertical axis: $R_M$ defined by (4.18). —, $R_M$ ($0 \leq M \leq 50$); symbols: root-mean-square of $f_M(x, y, \hbar)/(\hbar M!)$ ($1 \leq M \leq 50$); $\times$, $\hbar = -0.25$; $\triangle$, $\hbar = -0.75$; $\square$, $\hbar = -1.25$.

problems. Moreover, it confirms that introducing the new non-zero parameter $\hbar$ into the so-called zeroth-order deformation equations can really provide us with greater freedom and larger flexibility.

Notice that we can construct the zeroth-order deformation equations in different ways. For example, we can construct the same zeroth-order deformation equation as (4.3), say

$$\begin{aligned}&(1-p)\nabla^2[U(x, y, p, \hbar) - u_0(x, y)]\\&= p\hbar\{\nabla[\sin(x+y)\nabla U(x, y, p, \hbar)] + U^2(x, y, p, \hbar) - f(x, y)\},\\&\quad (x, y)\in[0, 1]\times[0, 1],\ p\in[0, 1],\ \hbar\neq 0,\end{aligned} \tag{4.19}$$

but with a different boundary condition

$$U(x, y, p, \hbar) = (1-p)u_0(x, y) + p\sin(x+y), \quad (x, y)\in\Gamma,\ p\in[0, 1]. \tag{4.20}$$

The corresponding $k$th-order deformation equation about the $k$th-order deformation derivative $u_0^{[k]}(x, y, \hbar)$ is

$$\nabla^2 u_0^{[k]}(x, y, \hbar) = f_k(x, y, \hbar), \quad (x, y)\in\Omega,\ \hbar\neq 0,\ k\geq 1, \tag{4.21}$$

with the boundary condition

$$u_0^{[k]}(x, y, \hbar) = \begin{cases} -[u_0(x, y) - \sin(x+y)], & k=1,\ (x, y)\in\Gamma,\ \hbar\neq 0\\ 0, & k\geq 1,\ (x, y)\in\Gamma,\ \hbar\neq 0\end{cases} \tag{4.22}$$

where $f_k(x, y, \hbar)$ is defined by (4.14) or (4.15). Notice that (4.22) is different from (4.13). In this case, our numerical experiments indicate that the series (4.11) converges when $-2 < \hbar < 0$, a region containing $-1.5 \leq \hbar < 0$. The root-mean-square absolute errors $\widetilde{R}_M$ in cases of $\hbar = -0.25$, $\hbar = -1.00$, $\hbar = -1.75$ are given in Figure 3. Moreover, the corresponding root-mean-square $M$th-order residual errors $R_M$ are given in Figure 4. They confirm once again that the series (4.11) converges under properly selected values of $\hbar$, so that if the order $M$ is high enough, no iteration is necessary to get an accurate enough approximation. Moreover, as shown in Figure 4, the root-mean-square residual errors $R_{M-1}$ of the

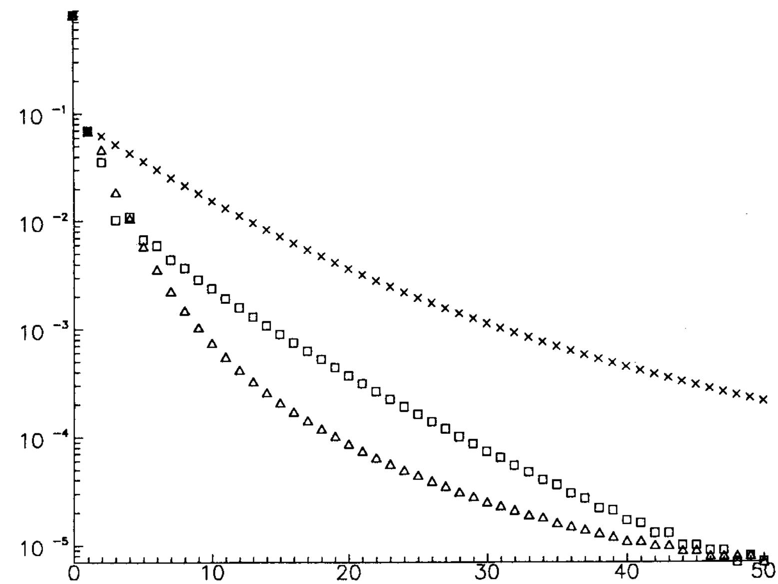


Figure 3. Root-mean-square absolute errors $\tilde{R}_M$ of example 1. Horizontal axis: $M$, the order of approximation (4.12), vertical axis: $\tilde{R}_M$ defined by (4.17). $\times$, $\hbar = -0.25$; $\triangle$, $\hbar = -1.00$; $\square$, $\hbar = -1.75$.

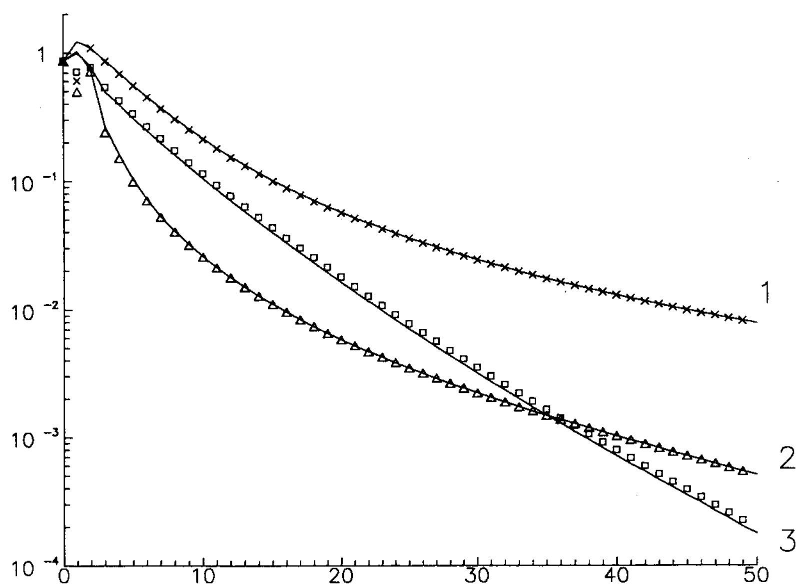


Figure 4. Root-mean-square residual errors $R_M$ of example 1. Horizontal axis: $M$, the order of approximation (4.12), vertical axis: $R_M$ defined by (4.18). —, $R_M$ $(0 \le M \le 50)$; symbols: root-mean-square of $f_M(x, y, \hbar)/(\hbar M!)$ $(1 \le M \le 50)$; $\times$, $\hbar = -0.25$; $\triangle$, $\hbar = -1.00$; $\square$, $\hbar = -1.75$.

$(M-1)$th-order approximation $u_{M-1}(x, y)$ are rather close to the related terms $\delta_{M-1}(x, y, \hbar) = f_M(x, y, \hbar)/(\hbar M!)$. This confirms once again our mathematical derivations described in the Section 2, say the term $\delta_{M-1}(x, y, \hbar) = f_M(x, y, \hbar)/(\hbar M!)$ can be simply used to determine if we need get approximations at higher-order.

### 4.2. *Example 2*

Secondly, let us consider the non-linear differential equation

$$\tau(1-\tau)^5\left[\phi\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}\right)+\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right]+[\tau^4-(1+2\tau)(1-\tau)^4\phi]\frac{\mathrm{d}\phi}{\mathrm{d}\tau}=0, \quad \tau\in[0, 1], \tag{4.23}$$

with boundary conditions

$$\phi(0)=0, \qquad \phi(1)=1, \tag{4.24}$$

which come from the equations governing the Blasius' viscous flow over a two-dimensional semi-infinite plate, say

$$u'''(\eta)+\frac{1}{2}u(\eta)u''(\eta)=0, \quad \eta\in[0, +\infty), \tag{4.25}$$

with boundary conditions

$$u(0)=u'(0)=0, \qquad u'(+\infty)=1, \tag{4.26}$$

under the transformations

$$\phi=u'(\eta), \qquad \tau=\frac{\sqrt{u}}{1+\sqrt{u}}. \tag{4.27}$$

Note that (4.23) has one linear term $\tau^4(\mathrm{d}\phi/\mathrm{d}\tau)$, related to a first-order linear differential operator that needs only one boundary condition. Thus, this linear operator is useless, because it cannot satisfy the two boundary conditions (4.24) at the same time. Therefore, the traditional BEM is invalid for this non-linear problem also.

Defining

$$\mathscr{A}\phi=\tau(1-\tau)^5\left[\phi\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}\right)+\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right]+[\tau^4-(1+2\tau)(1-\tau)^4\phi]\frac{\mathrm{d}\phi}{\mathrm{d}\tau}, \quad \tau\in[0, 1], \tag{4.28}$$

the related zeroth-order deformation equation is

$$(1-p)\{\mathscr{L}[\Phi(\tau, p, \hbar)]-\mathscr{L}[\phi_0(\tau)]\}=p\hbar\mathscr{A}[\Phi(\tau, p, \hbar)], \quad \tau\in[0, 1],\ p\in[0, 1],\ \hbar\neq 0, \tag{4.29}$$

with boundary conditions

$$\Phi(0, p, \hbar)=0, \qquad \Phi(1, p, \hbar)=1, \tag{4.30}$$

where the auxiliary operator $\mathscr{L}$ is a linear second-order differential operator, $\phi_0(\tau)$ is an initial approximation satisfying the boundary conditions (4.24). Suppose $\phi_0(\tau)$, $\hbar$ and the auxiliary linear operator $\mathscr{L}$ are properly selected. Then, we have by (3.39) the $M$th-order approximation of $\phi(\tau)$, say

$$\phi(\tau)=\phi_0(\tau)+\sum_{n=1}^{M}\bar{\phi}_0^{[n]}(\tau, \hbar), \quad \tau\in[0, 1], \tag{4.31}$$

where

 

Table I. Root-mean-square residual errors of (4.23) under the auxiliary linear operator $\mathscr{L}\varphi = \varphi'' - \varphi$ and approximations at differential orders

| $\hbar$ | Root-mean-square residual errors of (4.23) | | | | |
|---|---|---|---|---|---|
| | 1st-order appr. | 50th-order appr. | 100th-order appr. | 150th-order appr. | 200th-order appr. |
| −0.01 | 0.3192 | 0.3100 | 0.2979 | 0.2863 | 0.2752 |
| −1 | 0.2975 | 0.1262 | 0.1208 | 9.91e−2 | 9.52e−2 |
| −2 | 0.2734 | 0.1274 | 0.1008 | 8.51e−2 | 6.82e−2 |
| −3 | 0.2503 | 0.1109 | 9.04e−2 | 6.95e−2 | 5.96e−2 |
| −4 | 0.2288 | 0.1138 | 7.68e−2 | 6.34e−2 | 5.27e−2 |
| −5 | 0.2090 | 0.1018 | 6.63e−2 | 5.51e−2 | 4.48e−2 |
| −6 | 0.1917 | 0.1095 | 7.20e−2 | 4.85e−2 | 3.55e−2 |
| −7 | 0.1775 | 8.87e−2 | 6.22e−2 | 5.01e−2 | 3.10e−2 |
| −8 | 0.1673 | 9.84e−2 | 6.94e−2 | 4.03e−2 | 3.08e−2 |
| −9 | 0.1618 | 9.72e−2 | 5.88e−2 | 3.95e−2 | 2.52e−2 |
| −10 | 0.1614 | 8.67e−2 | 6.37e−2 | 3.45e−2 | 2.58e−2 |
| −11 | 0.1662 | 9.44e−2 | 5.75e−2 | 3.45e−2 | 2.25e−2 |
| −12 | 0.1758 | 0.1068 | 5.25e−2 | 3.04e−2 | 2.09e−2 |
| −13 | 0.1895 | 0.1053 | 5.66e−2 | 3.35e−2 | 1.84e−2 |
| −14 | 0.2064 | 9.86e−2 | 5.58e−2 | 2.80e−2 | 1.68e−2 |
| −15 | 0.2259 | 0.1107 | 4.70e−2 | 3.03e−2 | 1.40e−2 |
| −16 | 0.2472 | 0.1250 | 4.98e−2 | 2.69e−2 | 1.42e−2 |
| −17 | 0.2701 | 0.1205 | 4.43e−2 | 2.33e−2 | 1.11e−2 |
| −18 | 0.2940 | 0.1098 | 4.42e−2 | 2.51e−2 | 1.21e−2 |
| −19 | 0.3189 | 0.1184 | 5.08e−2 | 1.97e−2 | 1.11e−2 |
| −20 | 0.3444 | 0.1402 | 5.12e−2 | 1.88e−2 | 9.72e−3 |
| −21 | 0.3704 | 0.1447 | 4.56e−2 | 1.97e−2 | 1.07e−2 |
| −22 | 0.3969 | 0.1230 | 4.95e−2 | 1.59e−2 | 8.24e−3 |
| −23 | 0.4238 | 0.1115 | 5.43e−2 | 1.51e−2 | 7.42e−3 |
| −24 | 0.4509 | 0.1274 | 4.54e−2 | 1.70e−2 | 7.15e−3 |
| −25 | 0.4783 | 0.1465 | 3.87e−2 | 1.72e−2 | 4.73e−3 |
| −26 | 0.5059 | 0.1608 | 4.37e−2 | 1.52e−2 | 4.51e−3 |
| −27 | 0.5337 | 0.1370 | 4.58e−2 | 1.58e−2 | 4.69e−3 |
| −28 | 0.5616 | 0.1077 | 3.24e−2 | 1.78e−2 | 3.97e−3 |

$$\bar{\phi}_0^{[n]}(\tau, \hbar) = \frac{1}{n!} \frac{\partial^n \Phi(\tau, p, \hbar)}{\partial p^n}\bigg|_{p=0}, \quad n \geq 1,$$

is governed, according to (3.40)–(3.43), by the linear differential equations

$$\mathscr{L}\bar{\phi}_0^{[n]} = F_n(\tau, \hbar), \quad \tau \in [0, 1],\ \hbar \neq 0,\ n \geq 1, \tag{4.32}$$

with linear boundary conditions

$$\bar{\phi}_0^{[n]}(0, \hbar) = 0, \qquad \bar{\phi}_0^{[n]}(1, \hbar) = 0, \tag{4.33}$$

where

$$F_1(\tau, \hbar) = \hbar \mathscr{A} \phi_0, \tag{4.34}$$

$$F_n(\tau, \hbar) = F_{n-1}(\tau, \hbar) + \frac{\hbar}{(n-1)!} \frac{\mathrm{d}^{n-1} \mathscr{A}[\Phi(\tau, p, \hbar)]}{\mathrm{d}p^{n-1}}\bigg|_{p=0}, \quad n \geq 2. \tag{4.35}$$

According to definition (4.28), we have here for $n \geq 2$ that

$$\frac{1}{(n-1)!}\frac{\mathrm{d}^{n-1}\mathscr{A}[\Phi(\tau,p,\hbar)]}{\mathrm{d}p^{n-1}}\bigg|_{p=0} = \tau(1-\tau)^5\sum_{k=0}^{n-1}\left[\phi_0^{[k]}\frac{\partial^2\phi_0^{[n-1-k]}}{\partial\tau^2}+\frac{\partial\phi_0^{[k]}}{\partial\tau}\frac{\partial\phi_0^{[n-1-k]}}{\partial\tau}\right] + \tau^4\frac{\partial\phi_0^{[n-1]}}{\partial\tau} - (1+2\tau)(1-\tau)^4\sum_{k=0}^{n-1}\phi_0^{[k]}\frac{\partial\phi_0^{[n-1-k]}}{\partial\tau}. \tag{4.36}$$

For simplicity, we select the following auxiliary linear operator

$$\mathscr{L} = \frac{\partial^2}{\partial\tau^2} + \beta, \tag{4.37}$$

and set our initial approximation $\phi_0(\tau) = \tau$ that satisfies the boundary conditions (4.24). The fundamental solution of the above auxiliary linear operator is well-known and the linear deformation equations (4.32) and (4.33) can be easily solved by the traditional BEM, as pointed out by Liao and Chwang [5]. For the domain integral, we divide the domain [0, 1] into $N$ ($N = 100$) equal subdomains. Without loss of generality, we consider here only three cases of $\beta$, say $\beta = -1$, $\beta = 0$ and $\beta = 1$ respectively. For a given $\beta$, the root-mean-square residual errors $\bar{R}_M$ of the governing equation (4.23) under the $M$th-order approximation (4.31) are

Table II. Root-mean-square residual errors of (4.23) under the auxiliary linear operator $\mathscr{L}\varphi = \varphi''$ and approximations at differential orders

| $\hbar$ | Root-mean-square residual errors of (4.23) | | | | |
|---|---|---|---|---|---|
| | 1st-order appr. | 50th-order appr. | 100th-order appr. | 150th-order appr. | 200th-order appr. |
| −0.01 | 0.3221 | 0.3093 | 0.2966 | 0.2845 | 0.2728 |
| −1 | 0.2961 | 0.1221 | 0.1172 | 9.68e−2 | 9.19e−2 |
| −2 | 0.2706 | 0.1234 | 9.72e−2 | 8.26e−2 | 6.71e−2 |
| −3 | 0.2464 | 0.1083 | 8.78e−2 | 6.81e−2 | 5.86e−2 |
| −4 | 0.2031 | 0.1093 | 7.55e−2 | 6.24e−2 | 5.18e−2 |
| −5 | 0.1853 | 9.94e−2 | 6.51e−2 | 5.43e−2 | 4.41e−2 |
| −6 | 0.1711 | 0.1059 | 7.08e−2 | 4.75e−2 | 3.50e−2 |
| −7 | 0.1615 | 8.66e−2 | 6.10e−2 | 4.91e−2 | 3.05e−2 |
| −8 | 0.1573 | 9.66e−2 | 6.80e−2 | 3.97e−2 | 2.47e−2 |
| −9 | 0.1590 | 9.54e−2 | 5.75e−2 | 3.88e−2 | 2.50e−2 |
| −10 | 0.1660 | 8.53e−2 | 6.26e−2 | 3.41e−2 | 2.55e−2 |
| −11 | 0.1664 | 9.32e−2 | 5.62e−2 | 3.41e−2 | 2.23e−2 |
| −12 | 0.1788 | 0.1050 | 5.17e−2 | 3.00e−2 | 2.06e−2 |
| −13 | 0.1953 | 0.1025 | 5.57e−2 | 3.30e−2 | 1.82e−2 |
| −14 | 0.2148 | 9.63e−2 | 4.51e−2 | 2.75e−2 | 1.66e−2 |
| −15 | 0.2367 | 0.1087 | 4.65e−2 | 2.99e−2 | 1.39e−2 |
| −16 | 0.2603 | 0.1223 | 4.92e−2 | 2.65e−2 | 1.40e−2 |
| −17 | 0.2853 | 0.1175 | 4.38e−2 | 2.30e−2 | 1.10e−2 |
| −18 | 0.3113 | 0.1070 | 4.38e−2 | 2.48e−2 | 1.20e−2 |
| −19 | 0.3380 | 0.1161 | 5.02e−2 | 1.95e−2 | 1.10e−2 |
| −20 | 0.3654 | 0.1368 | 5.03e−2 | 1.87e−2 | 9.63e−3 |
| −21 | 0.3932 | 0.1399 | 4.47e−2 | 1.95e−2 | 1.06e−2 |
| −22 | 0.4214 | 0.1199 | 4.88e−2 | 1.58e−2 | 8.11e−3 |
| −23 | 0.4499 | 0.1099 | 5.32e−2 | 1.51e−2 | 7.35e−3 |
| −24 | 0.4757 | 0.1260 | 4.44e−2 | 1.69e−2 | 7.07e−3 |
| −25 | 0.5076 | 0.1449 | 3.84e−2 | 1.70e−2 | 4.70e−3 |
| −26 | 0.5368 | 0.1505 | 4.43e−2 | 1.43e−2 | 1.32e−2 |

 

Table III. Root-mean-square residual errors of (4.23) under the auxiliary linear operator $\mathscr{L}\varphi = \varphi'' + \varphi$ and approximations at differential orders

| $\hbar$ | Root-mean-square residual errors of (4.23) | | | | |
|---|---|---|---|---|---|
| | 1st-order appr. | 50th-order appr. | 100th-order appr. | 150th-order appr. | 200th-order appr. |
| −0.01 | 0.3221 | 0.3085 | 0.2951 | 0.2823 | 0.2699 |
| −1 | 0.2945 | 0.1177 | 0.1133 | 9.45e−2 | 8.86e−2 |
| −2 | 0.2675 | 0.1192 | 9.37e−2 | 8.04e−2 | 6.59e−2 |
| −3 | 0.2418 | 0.1056 | 8.53e−2 | 6.67e−2 | 5.76e−2 |
| −4 | 0.2178 | 0.1051 | 7.41e−2 | 6.13e−2 | 5.08e−2 |
| −5 | 0.1962 | 9.66e−2 | 6.39e−2 | 5.34e−2 | 3.34e−2 |
| −6 | 0.1779 | 0.1025 | 6.96e−2 | 4.65e−2 | 3.45e−2 |
| −7 | 0.1638 | 8.47e−2 | 5.98e−2 | 4.81e−2 | 3.01e−2 |
| −8 | 0.1553 | 9.48e−2 | 6.67e−2 | 3.92e−2 | 3.01e−2 |
| −9 | 0.1532 | 9.37e−2 | 5.63e−2 | 3.82e−2 | 2.47e−2 |
| −10 | 0.1578 | 8.39e−2 | 6.15e−2 | 3.38e−2 | 2.51e−2 |
| −11 | 0.1685 | 9.20e−2 | 5.50e−2 | 3.36e−2 | 2.20e−2 |
| −12 | 0.1843 | 0.1032 | 5.10e−2 | 2.97e−2 | 2.03e−2 |
| −13 | 0.2040 | 9.98e−2 | 5.47e−2 | 3.26e−2 | 1.80e−2 |
| −14 | 0.2266 | 9.41e−2 | 4.49e−2 | 2.71e−2 | 1.63e−2 |
| −15 | 0.2513 | 0.1068 | 4.60e−2 | 2.96e−2 | 1.37e−2 |
| −16 | 0.2776 | 0.1196 | 4.87e−2 | 2.60e−2 | 1.39e−2 |
| −17 | 0.3049 | 0.1144 | 4.33e−2 | 2.28e−2 | 1.09e−2 |
| −18 | 0.3332 | 0.1044 | 4.34e−2 | 2.45e−2 | 1.19e−2 |
| −19 | 0.3621 | 0.1139 | 4.97e−2 | 1.92e−2 | 1.09e−2 |
| −20 | 0.3915 | 0.1336 | 4.94e−2 | 1.85e−2 | 9.55e−3 |
| −21 | 0.4214 | 0.1357 | 4.39e−2 | 1.93e−2 | 1.05e−2 |
| −22 | 0.4516 | 0.1169 | 4.81e−2 | 1.57e−2 | 7.99e−3 |
| −23 | 0.4820 | 0.1082 | 5.24e−2 | 1.51e−2 | 7.68e−3 |

dependent upon the value of $\hbar$, as shown in Table I for $\beta = -1$, Table II for $\beta = 0$ and Table III for $\beta = 1$. Our calculations show that, for the above mentioned initial approximation and auxiliary linear operators, the corresponding series (2.9) converges in a large region of $\hbar$, say $-28 \leq \hbar < 0$ for $\beta = -1$, $-26 \leq \hbar < 0$ for $\beta = 0$ and $-23 \leq \hbar < 0$ for $\beta = 1$. In the region $-1 < \hbar < 0$, the series of approximations converges rather slow. However, for $\hbar \leq -5$, the 50th-order approximations are satisfactory and the 100th-, 150th- and 200th-order approximations agree very well with the result given by iterative techniques, as shown in Figures 5–10. Therefore, the approximation (4.31) at a high enough order can give accurate enough results. It confirms once again that we can get accurate enough approximations of non-linear problems by means of no iterations!

The series of approximations (4.31) converges fastest when $\hbar = -28$ for $\beta = -1$, or $\hbar = -25$ for $\beta = 0$, or $\hbar = -23$ for $\beta = 1$. The root-mean-square residual errors $\bar{R}_m$ of the governing equation (4.23) under the $m$th-order approximation, together with the corresponding root-mean-square of the related terms $\delta_m(\tau, \hbar) = F_{m+1}(\tau, \hbar)/\hbar$ are given in Figures 11–13 respectively for the cases of $\hbar = -5$ and $\hbar = -23$ under $\beta = -1, 0, 1$. Notice that the series (4.31) converges faster when $\hbar = -23$ than when $\hbar = -5$. However, the root-mean-square residual error of the first-order approximation when $\hbar = -5$ is less than that when $\hbar = -23$. Therefore, it is not certain that a value of $\hbar$ that gives the better first-order approximation corresponds to the faster convergent series. This fact shows the difficulty to find out the 'best' value of $\hbar$. Fortunately, the series of approximation (4.31) converges fast enough in a large region of $\hbar$, say $-28 \leq \hbar \leq -5$ for $\beta = -1$, $-26 \leq \hbar \leq -5$ for $\beta = 0$ and $-23 \leq \hbar \leq -5$ for $\beta = 1$. Moreover, the values of $\hbar$ that give the best first-order approximation are in the above mentioned regions, as shown in Tables I, II and III.

Notice that, the root-mean-square of $\delta_m(\vec{r}, \hbar) = F_{m+1}(\tau, \hbar)/\hbar$ is usually very close to the root-mean-square residual errors $\bar{R}_m$ of the governing equation (4.23) under the $m$th-order approximation, as shown in Figures 11–13, where

$$\bar{R}_M = \sqrt{\frac{\sum_{i=1}^{N-1} |\mathscr{A}[\phi_M(\tau_i, \hbar)]|^2}{N-1}}$$

This confirms once again that $\delta_m(\vec{r}, \hbar) = F_{m+1}(\tau, \hbar)/\hbar$ can indeed report the accuracy of the approximations, and therefore can be used to determine if we need calculate higher-order approximations.

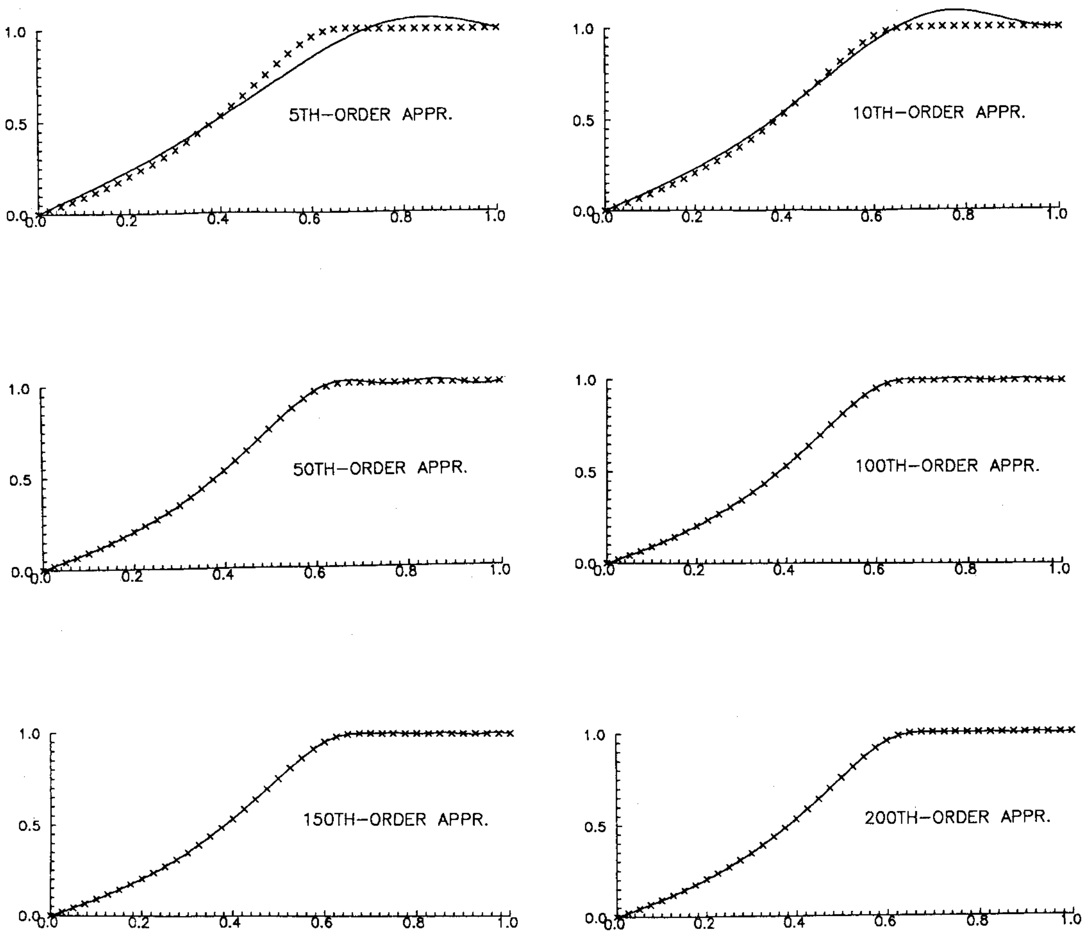


Figure 5. Comparisons of approximations at different orders with the exact solution in the case of $\hbar = -5$ and $\mathscr{L}\varphi = \varphi'' - \varphi$ (example 2). —, Approximate solutions; centered symbols: exact solution.

 

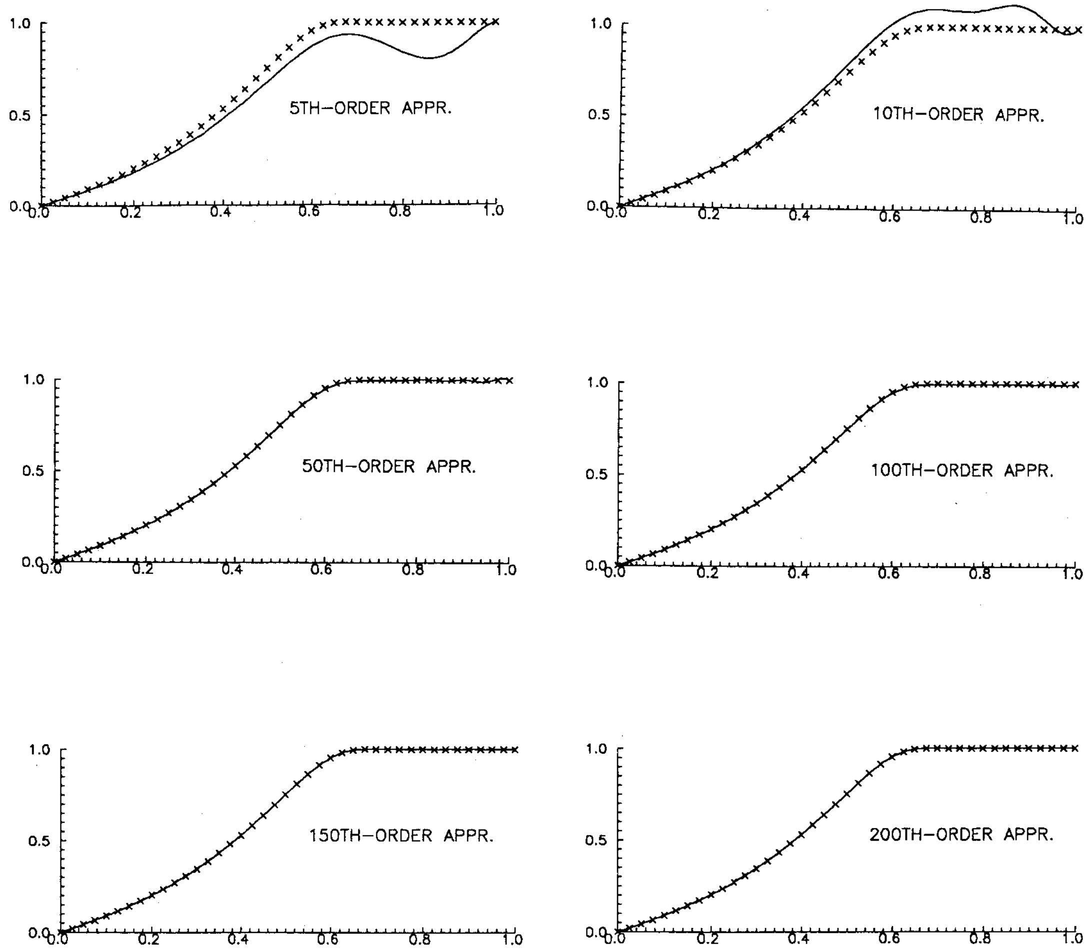


Figure 6. Comparisons of approximations at different orders with the exact solution in the case of $\hbar = -23$ and $\mathscr{L}\varphi = \varphi'' - \varphi$ (example 2). —, Approximate solutions; centered symbols, exact solution.

Among the three auxiliary linear operators under consideration, it seems difficult to clearly point out which one is better than the others, as shown in Figures 11–13. However, we are not sure if this conclusion is right in general. Besides, the coefficient matrix $\mathbf{M}$ of the $n$th-order deformation equation (4.32) is the same for all $n \geq 1$, so that if direct techniques are applied to give its inverse matrix $\mathbf{M}^{-1}$, all $n$th-order deformation equations (4.32) as a whole can be rather efficiently solved.

Our second example confirms once again that, applying the proposed general BEM, we can solve strongly non-linear problems even by means of no iterations. This, as pointed out in the first example, disturbs the absolute governing place of the iterative methodology of the BEM for non-linear problems and indicates the validity and great potential of the proposed general BEM.

## 5. CONCLUSIONS AND DISCUSSIONS

In this paper, we further generalize the general BEM proposed by Liao [1–4] and Liao and Chwang [5] by means of introducing a new non-zero parameter $\hbar$ to construct the so-called zeroth-order deformation equations. This new parameter $\hbar$ can provide us with larger freedom and greater flexibility so that it makes the proposed BEM approach even more general. The two examples considered in this paper verify that, if the value of $\hbar$, the initial approximation and the auxiliary linear operators are properly selected, and besides, the order $M$ of approximation is high enough, accurate enough approximations can be obtained by the proposed general BEM. Therefore, no iterations are needed even for non-linear problems. This disturbs the absolute governing place of the iterative methodology of the BEM for non-linear problems and verifies the validity and great potential of the general BEM.

The convergence radius $\rho$ of the series (2.9) of the further generalized BEM depends on auxiliary linear operators, initial approximations and the value of $\hbar$. The introduced non-zero

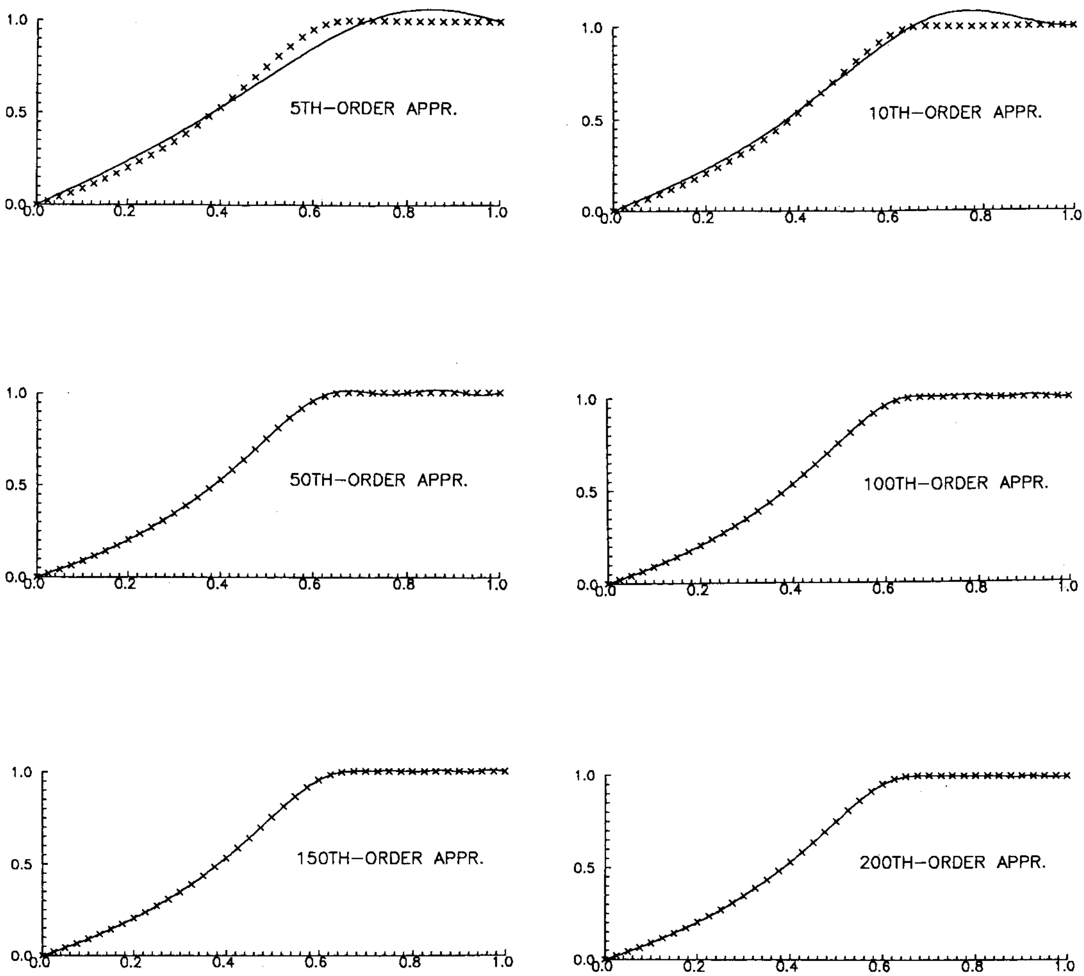


Figure 7. Comparisons of approximations at different orders with the exact solution in the case of $\hbar = -5$ and $\mathscr{L}\varphi = \varphi''$ (example 2). —, Approximate solutions; centered symbols, exact solution.

 

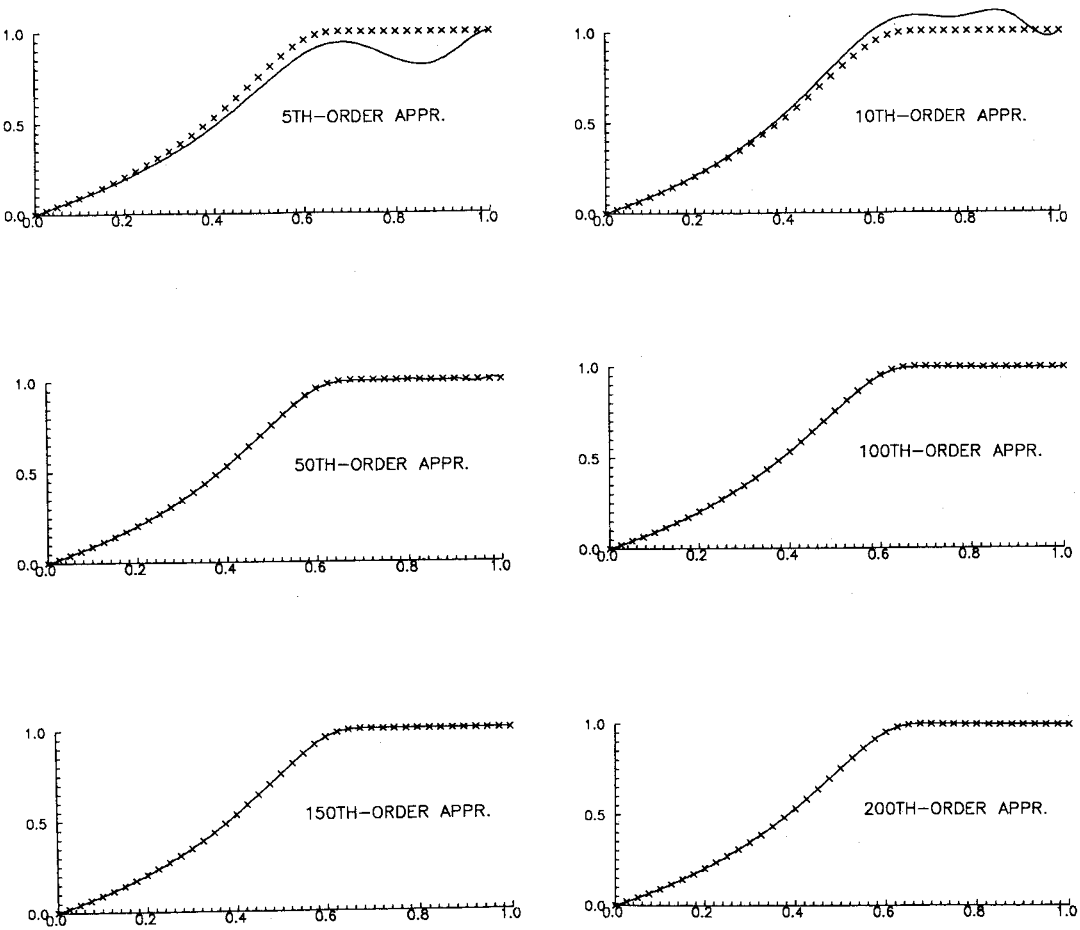


Figure 8. Comparisons of approximations at different orders with the exact solution in the case of $\hbar = -23$ and $\mathscr{L}\varphi = \varphi''$ (example 2). —, Approximate solutions; centered symbols, exact solution.

parameter $\hbar$ provides us with a new degree of freedom and greater flexibility to construct 'satisfactory' or 'better' zeroth-order deformation equations. Thus, the convergence radius $\rho$ of the series (2.9) now becomes a function of $\hbar$. Therefore, if the auxiliary linear operators, the initial approximation and the value of $\hbar$ are properly selected, the convergence radius $\rho$ of the series (2.9) may be not less than one. If so, approximations at considerably high-order may be so accurate that no iterations are necessary, as illustrated in this paper. Notice that the iterative methodology is, in tradition, absolutely necessary when applying the BEM to solve non-linear problems. However, this paper seems to make this kind of absoluteness obsolete.

In Section 3, we propose some mathematical derivations for the general BEM. In some special cases, we give the corresponding criterion for the validity of the general BEM. Moreover, theorem 4 ensures that if the series (2.9) converges, it must be a solution of the problem under consideration. Furthermore, theorems 5 and 6 provide us with a simple way to examine the accuracy of approximations and to determine if we need to get other approximations at higher-order. These rational derivations are helpful for applications of the general BEM in other problems. Introducing the new non-zero parameter $\hbar$ indeed provides us with a new degree of freedom. Maybe this kind of new freedom might provide us with some new fields of researches. For example, it seems worthwhile to further investigate the factors that determine the convergence radius $\rho$ of the series (2.9), such as the auxiliary linear operators, values of $\hbar$, the initial approximations and even the ways to construct the zeroth-order deformation equations. All of these might greatly enrich our numerical techniques for non-linear problems.

As pointed out by Liao [1–4] and Liao and Chwang [5], domain integrals appear when the general BEM is applied to solve non-linear problems. This disadvantage can be overcome by some developed or developing numerical techniques. One of them is the so-called dual reciprocity boundary element method (DRBEM) [13,14], which can avoid the domain integration by transforming it to a surface integration. Another numerical technique, namely the parallel fast multipole method (FMM), whose basic ideas were first proposed by Greengard [15–18], is currently still in development. The parallel fast multipole method can greatly increase the efficiency of calculating coefficient matrix related to the BEM. Moreover, it is rather suitable for parallel calculations. Thus, it seems to have a bright future. Therefore, it is

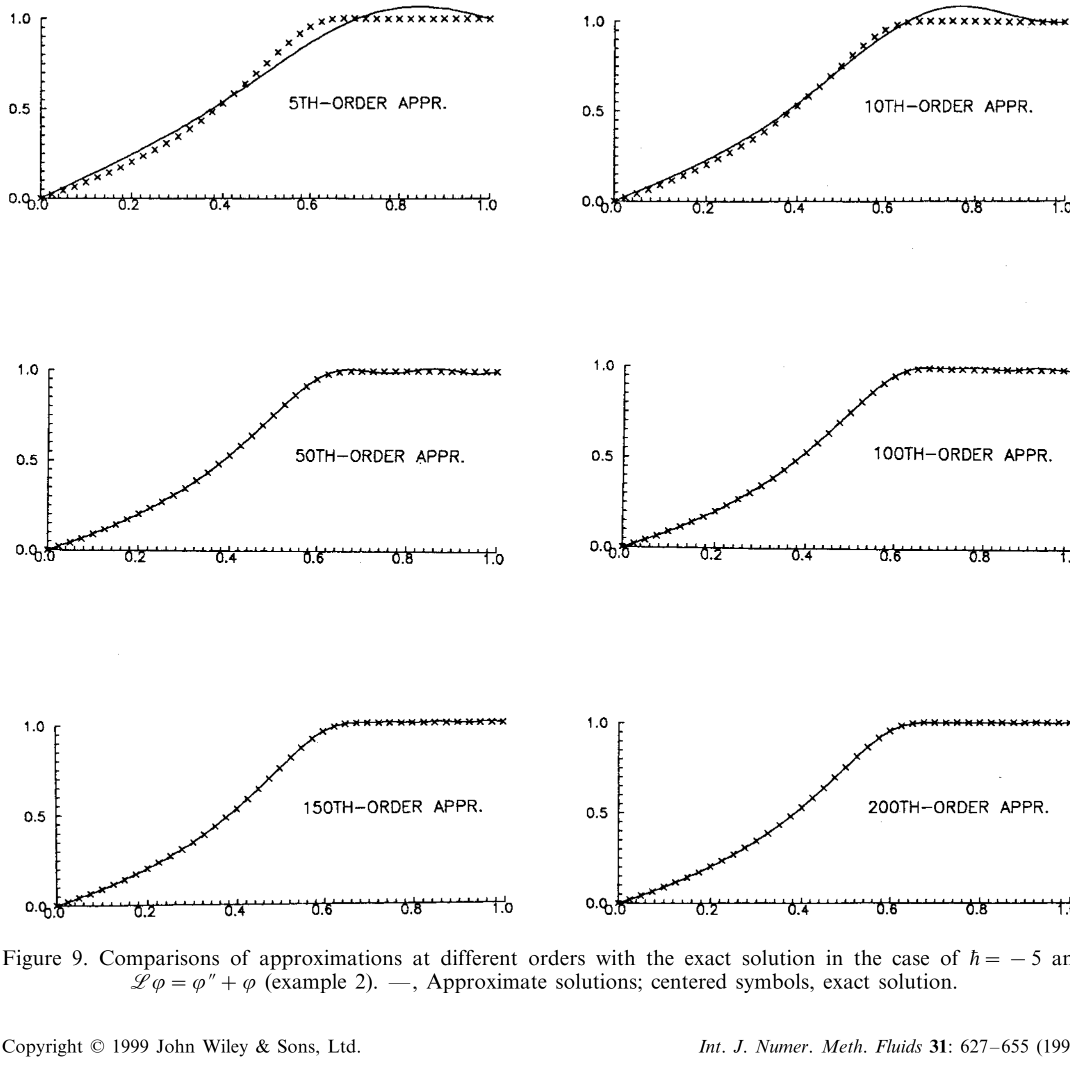


Figure 9. Comparisons of approximations at different orders with the exact solution in the case of $\hbar = -5$ and $\mathscr{L}\varphi = \varphi'' + \varphi$ (example 2). —, Approximate solutions; centered symbols, exact solution.

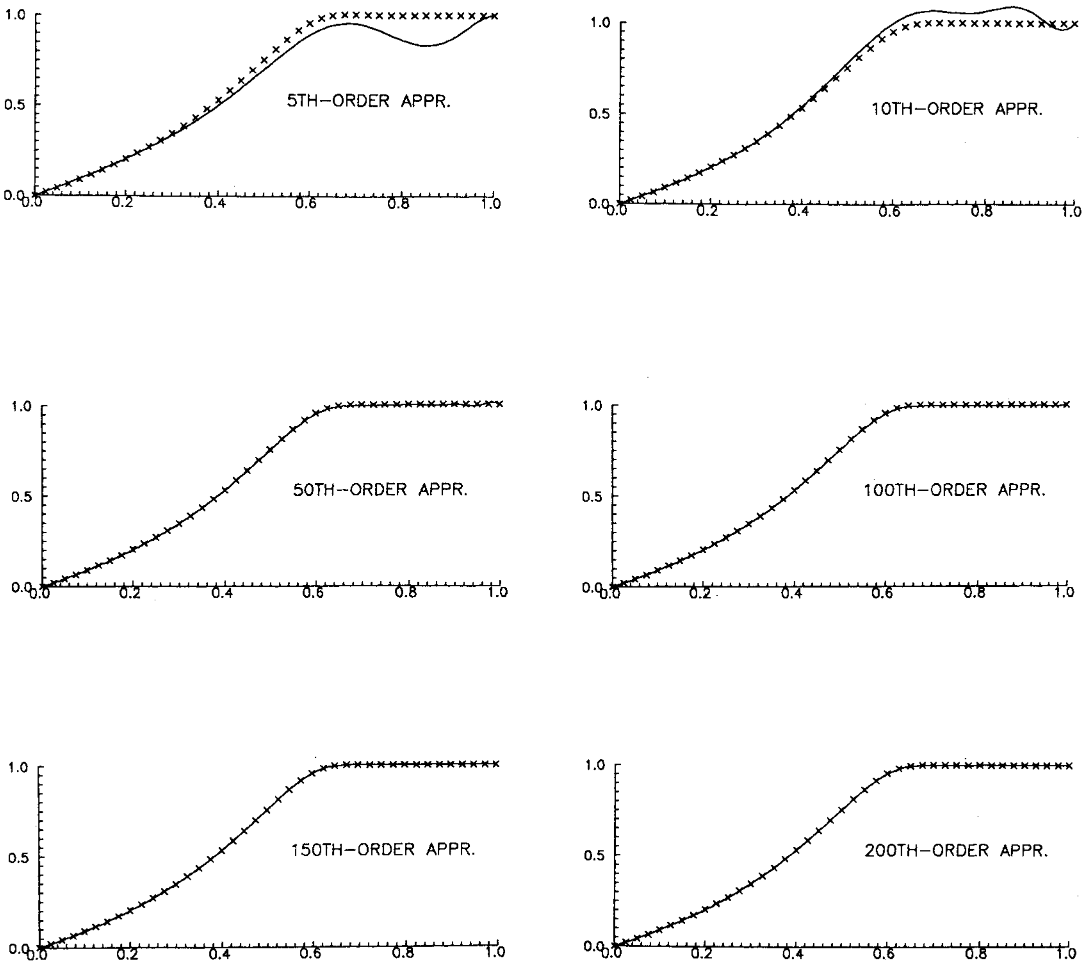


Figure 10. Comparisons of approximations at different orders with the exact solution in the case of $\hbar = -23$ and $\mathscr{L}\varphi = \varphi'' + \varphi$ (example 2). —, Approximate solutions; centered symbols, exact solution.

hoped to make the proposed general BEM numerically more efficient by combining it with some parallel and accelerated techniques, such as FMM, DRBEM and so on. Besides, let us point out that the coefficient matrix $\mathbf{M}$ related to the $k$th-order ($k \geq 1$) deformation equations (2.12) and (2.13) are the same for any $k \geq 1$. Thus, if its inverse matrix $\mathbf{M}^{-1}$ can be given by direct techniques, all of the $k$th-order deformation equations (2.12) and (2.13) can be rather efficiently solved as a whole.

Finally, we simply point out that, the currently proposed general BEM described in this paper is only a simple application of a new analytical technique for non-linear problems, namely the homotopy analysis method (HAM). The HAM was first proposed by Liao [9–12] to overcome restrictions of perturbation methods. In [12], Liao applied the HAM to obtain a family of power series for a typical non-linear problem in fluid mechanics, whose convergence radius can be greatly enlarged if $\hbar$ is properly selected. This provides us with an indirect but rational support to the validity of the general BEM and our foregoing conclusions. Finally, let us point out that the basic ideas proposed in this paper are quite general. For example, the linear $k$th-order ($k \geq 1$) deformation equations (2.12) and (2.13) can be easily solved by lots of other numerical techniques, such as the finite difference method, the finite element method, the

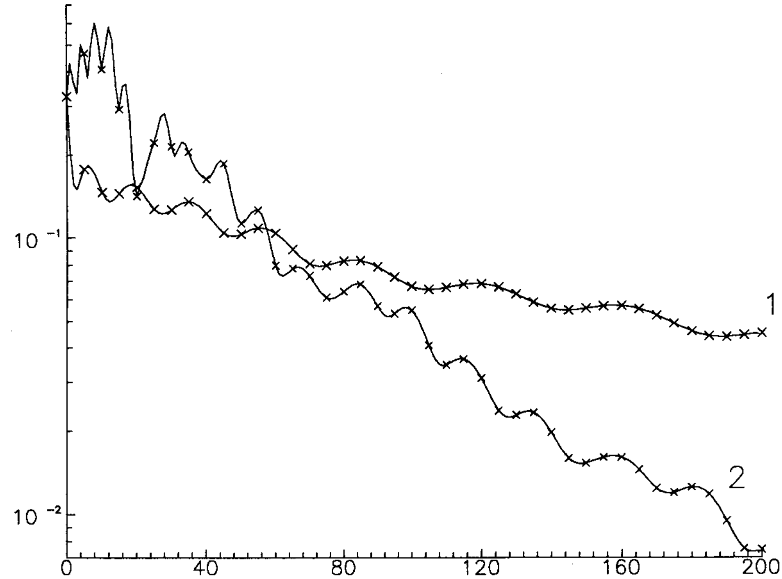


Figure 11. Root-mean-square residual errors $\bar{R}_M$ of (4.23) in the case of $\mathscr{L}\varphi = \varphi'' + \varphi$ under $\hbar = -5$ (curve 1) or $\hbar = -23$ (curve 2). Horizontal axis: $M$, the order of approximation; —, $\bar{R}_M$ of (4.23) under the $M$th-order approximation; centered symbols, root-mean-square of $F_m(\tau, \hbar)/|\hbar|$.

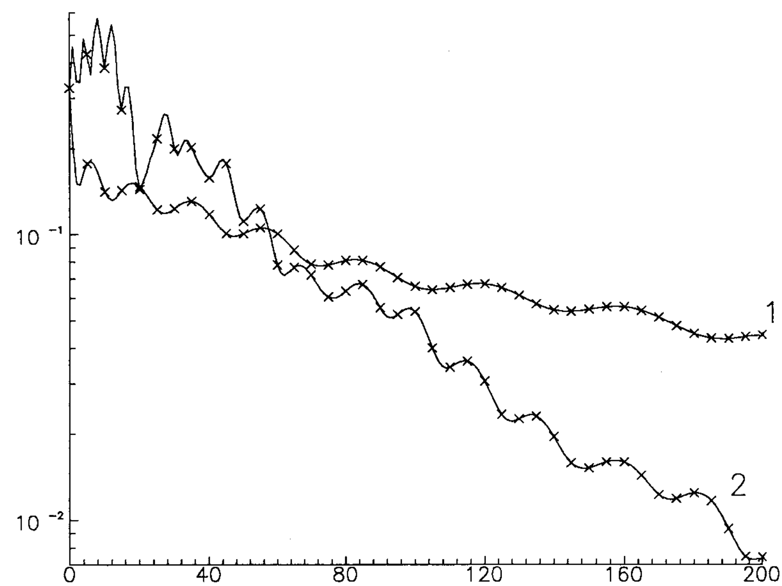


Figure 12. Root-mean-square residual errors $\bar{R}_M$ of (4.23) in the case of $\mathscr{L}\varphi = \varphi''$ under $\hbar = -5$ (curve 1) or $\hbar = -23$ (curve 2). Horizontal axis: $M$, the order of approximation; —, $\bar{R}_M$ of (4.23) under the $M$th-order approximation; centered symbols, root-mean-square of $F_m(\tau, \hbar)/|\hbar|$.

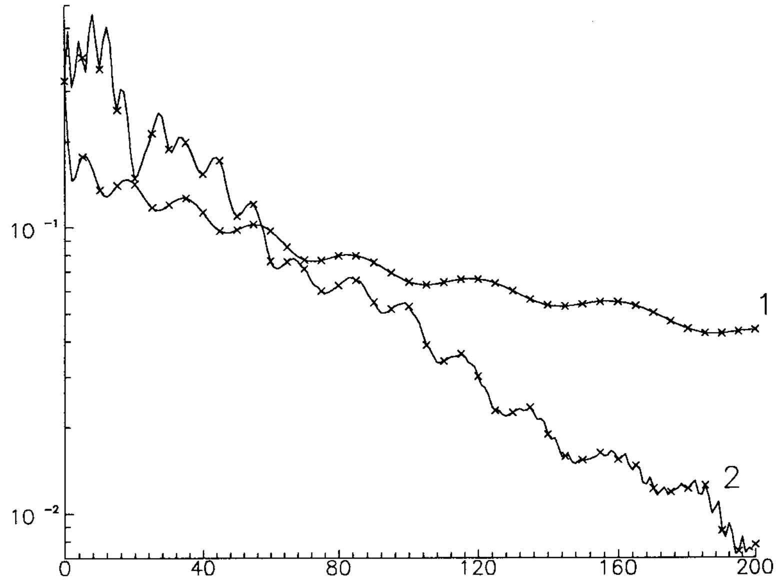


Figure 13. Root-mean-square residual errors $\bar{R}_M$ of (4.23) in the case of $\mathscr{L}\varphi = \varphi'' + \varphi$ under $\hbar = -5$ (curve 1) or $\hbar = -23$ (curve 2). Horizontal axis: $M$, the order of approximation; —, $\bar{R}_M$ of (4.23) under the $M$th-order approximation; centered symbols, root-mean-square of $F_m(\tau, \hbar)/|\hbar|$.

finite volume method and so on. All of these might provide us with some new fields of researches.

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## APPENDIX A. NOMENCLATURE

| | |
|---|---|
| $\mathscr{A}$ | non-linear differential operator for governing equation |
| $\mathscr{B}$ | non-linear differential operator for boundary condition |
| $c(\vec{r})$ | geometric coefficient |
| $f(\vec{r})$ | known function |
| $g(\vec{r})$ | known function |
| $\hbar$ | non-zero parameter |
| $\mathscr{L}_0$ | linear differential operator |
| $\mathscr{L}$, $\mathscr{L}_B$ | auxiliary linear differential operator |
| $M$ | order of approximation |
| $N_0$ | non-linear differential operator |
| $p$ | embedding parameter |
| $\vec{r}$ | independent position vector |
| $u(\vec{r})$ | dependent variable |

 

| | |
|---|---|
| $u_M(\vec{r})$ | $M$th-order approximation |
| $U(\vec{r}, p, \hbar)$ | homotopy of $u(\vec{r})$ |
| $x, y$ | co-ordinates |

*Greek letters*

| | |
|---|---|
| $\delta_m(\vec{r}, \hbar)$ | residual errors of governing equation for $m$th-order approximation |
| $\rho$ | convergence radius of Taylor series |
| $\Delta_m(\vec{r}, \hbar)$ | residual errors of boundary condition for $m$th-order approximation |
| $\Gamma$ | boundary of domain $\Omega$ |
| $\Omega$ | domain |

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