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Unsteady non-similarity boundary-layer flows caused by an impulsively stretching flat sheet

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A B S T R A C T

The unsteady non-similarity boundary-layer flows caused by an impulsively stretching flat sheet have been investigated. The partial differential equations governing the flows have been solved analytically by means of an analytic technique for strongly non-linear problems, namely the homotopy analysis method (HAM). This analytic approach gives us the convergent series solution uniformly valid for all dimensionless time in the whole spatial region $0 \le x < \infty$ and $0 \le y < \infty$. To the best of our knowledge, such a kind of series solution has never been obtained. The skin friction coefficient and the boundary-layer thickness are also analyzed for different dimensionless time τ.

1. Introduction

Boundary-layer flows over a moving or stretching plate are of great importance in view of their relevance to a wide variety of technical applications, particularly in the manufacture of fibers in glass and polymer industries. The first and foremost work regarding the boundary-layer behavior in moving surfaces in quiescent fluid was considered by Sakiadis [1]. Subsequently, many researchers [2–9] worked on the problem of moving or stretching plates under different situations.

In the boundary-layer theory, similarity solutions are found to be useful in the interpretation of certain fluid motions at large Reynolds numbers. Similarity solutions often exist for the flow over semi-infinite plates and stagnation point flow for two dimensional, axi-symmetric and three dimensional bodies. In some special cases, when there is no similarity solution, one has to solve a system of non-linear partial differential equations (PDEs). For similarity boundary-layer flows, velocity profiles are similar. But, this kind of similarity is lost for non-similarity flows [10–14]. Obviously, the non-similarity boundary-layer flows are more general in nature and more important not only in theory but also in applications.

The unsteady boundary-layer flows are important in a wide range of applications. Examples include the design and analysis of water supply systems, flows in natural gas pipelines and flows of blood in arteries. Pump shutdowns or rapid changes in value settings are known to generate unsteady flows in hydraulics devices or water supply systems. The pressure in unsteady flows can cause cavitation, pitting and corrosion [15]. The unsteadiness of the flow, so long as the fluid is incompressible, may not directly be of much influence to the performance of the machines, but indirectly it will influence the performance by changing the boundary-layer formed on the solid surface. In particular, the flow unsteadiness will induce a change in the separation characteristics of the boundary-layer, which in turn will change the overall flow pattern with the time.

The boundary-layer development occurs in two stages. If we introduce dimensionless time $\tau = U_w t/x$, then for small τ (small time or large x) the flow is dominated by the viscous and pressure gradient forces and the unsteady acceleration.

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During this stage the convective acceleration plays only a minor role in the flow development. For the large τ (large time or small x) the flow is dominated by the viscous forces, pressure gradient and the convective acceleration while the unsteady acceleration has negligible role in the flow development. This change in character of the flow manifest itself mathematically as a change in the character of the equations which describe the fluid motion. Unsteady boundary-layer flows due to an impulsively stretching surface in a viscous fluid was considered by some researchers [16–21].

As mentioned before, when the similarity does not exist, one had to solve a system of non-linear PDEs. In the investigation of non-similarity boundary-layer flows, numerical methods are widely applied as a tool. However, by using numerical methods some additional errors and uncertainty can be found in the results, because, for numerical computation one has to replace the infinite domain with finite one. However, PDEs can be solved in the infinite domain by using the analytic methods. However, by the traditional analytic techniques, such as perturbation technique, it is hard to get analytic approximations that are valid and accurate for all physical parameters due to the reason that these techniques often depend upon the small physical parameters. Among the analytic methods, the method of *local non-similarity* for thermal boundary-layer problems is frequently used. Sparrow et al. [12] introduced this method, but the results given by it are of *uncertain accuracy*, as pointed out by Sparrow and Yu [13], and valid only for small ξ in general. Currently, Anwar et al. [22] applied the method of local non-similarity to solve the non-similar steady-state electrically-conducting forced convection liquid metal boundary-layer flow with induced magnetic field effects. Anwar et al. [22] gave numerical results which are valid only for small ξ $(0.25 < \xi < 2)$. Eswara and Nath [23] obtained numerical solutions for unsteady non-similarity forced convection laminar boundary-layer flow over a moving longitudinal cylinder by using an implicit finite-difference scheme in combination with a quasilinearization technique. However, it is pity that they gave only numerical results which seem inaccurate for large values of ξ and thus valid only for small ξ $(\xi \ll 1)$ in the finite time domain.

It seems hard to obtain analytic solution of unsteady non-similarity boundary-layer flows valid for all time. To the best of our knowledge, no one has reported any analytic solutions of unsteady non-similarity boundary layer flows due to an impulsively stretching flat sheet that are valid and accurate for all time. In 1992, a kind of analytic method, namely the homotopy analysis method (HAM) [24,25], was developed to solve highly nonlinear problems. Different from perturbation techniques [26], the homotopy analysis method does not depend upon any small or large physical parameters and thus is valid for most nonlinear problems in science and engineering. Besides, it logically contains other non-perturbation techniques such as Lyapunov's small parameter method [27], the δ-expansion method [28], and Adomian's decomposition method [29], as proved by Liao in his book [25]. The homotopy analysis method has been successfully applied to many nonlinear problems [30–40]. In this paper, by means of the homotopy analysis method, the explicit series solution for unsteady non-similarity boundary-layer flows caused by an impulsively stretching flat sheet is given.

2. Mathematical formulations

Liao [41] solved the steady non-similarity boundary-layer flows caused by a stretching flat sheets by means of homotopy analysis method. Here, we further consider the unsteady non-similarity boundary-layer flows caused by impulsively stretching flat sheets. Two equal but opposite forces are applied along the sheet so as to fix the origin of the coordinate axes. The x and y axes are along and perpendicular to the sheet, respectively. The fluid far from the sheet is at rest. The velocity variation across the flow direction is much larger than that in the flow direction, so that there exist a thin boundary-layer near the sheet. The unsteady viscous boundary-layer flows developed by a stretching flat sheet is governed by the equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

subject to the boundary conditions:

$$t \geqslant 0: \quad u = U_w(x), \qquad v = 0 \quad \text{at } y = 0 \quad \text{and} \quad u \to 0 \quad \text{as } y \to \infty \tag{3}$$

and the initial conditions

$$t = 0: \quad u = v = 0 \quad \text{for any } x, y, \tag{4}$$

where t denotes the time, ν the kinematic viscosity coefficient of the fluid, u and v are the velocity components of the fluid in x and y directions, respectively. When $t < 0$, both the fluid and plate are at rest. At $t = 0$, the plate suddenly has the velocity $u = U_w(x)$. Let ψ denote the stream function for all time, satisfying:

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \tag{5}$$

Following Liao [41], we use the transformations:

$$\psi = \sqrt{\nu\zeta}\,\sigma(x) f(x, \eta, \zeta)$$

and

$$\eta = \frac{y}{\sqrt{\nu\zeta}\sigma(x)}, \qquad \zeta = \frac{\tau}{1+\tau} \quad \text{and} \quad \tau = \frac{U_w(x)}{x}t,$$

where $\sigma(x) > 0$ is a real function to be chosen later. Then the governing equations (1) and (2) become

$$\frac{\partial^3 f}{\partial \eta^3} + \zeta\sigma'(x)\sigma(x)f\frac{\partial^2 f}{\partial \eta^2} + \zeta\sigma^2(x)\left[\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta\partial x}\right] + \frac{\zeta(1-\zeta)}{2}\sigma^2(x)\left(\frac{U_w'}{U_w} - \frac{1}{x}\right)$$
$$\times\left[f\frac{\partial^2 f}{\partial \eta^2} + 2\zeta\frac{\partial f}{\partial \zeta}\frac{\partial^2 f}{\partial \eta^2} - 2\zeta\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \zeta\partial \eta}\right] - \frac{(1-\zeta)^2 U_w\sigma^2(x)}{2x}$$
$$\times\left[2\zeta\frac{\partial^2 f}{\partial \zeta\partial \eta} - \eta\frac{\partial^2 f}{\partial \eta^2}\right] = 0, \quad \zeta \geq 0 \tag{6}$$

subject to the boundary conditions

$$f(x,0,\zeta) = 0, \qquad f_\eta(x,0,\zeta) = U_w(x), \qquad f_\eta(x,+\infty,\zeta) = 0. \tag{7}$$

Following Liao [41], let us consider the case $U_w(x) = U_w(\xi)$, where $\xi = \Gamma(x)$ is a given real function of x. By means of the transformation

$$\xi = \Gamma(x), \tag{8}$$

we have

$$\frac{\partial^3 f}{\partial \eta^3} + \zeta\sigma_1(\xi)f\frac{\partial^2 f}{\partial \eta^2} + \zeta\sigma_2(\xi)\left[\frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta\partial \xi}\right] + \frac{\zeta(1-\zeta)}{2}\sigma_3(\xi)$$
$$\times\left[f\frac{\partial^2 f}{\partial \eta^2} + 2\zeta\frac{\partial f}{\partial \zeta}\frac{\partial^2 f}{\partial \eta^2} - 2\zeta\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \zeta\partial \eta}\right] + \frac{(1-\zeta)^2\sigma_4(\xi)}{2}$$
$$\times\left[\eta\frac{\partial^2 f}{\partial \eta^2} - 2\zeta\frac{\partial^2 f}{\partial \eta\partial \zeta}\right] = 0 \quad \zeta \geq 0 \tag{9}$$

where

$$\sigma_1(\xi) = \frac{[\sigma^2(x)]'}{2}, \qquad \sigma_2(\xi) = \Gamma'(x)\sigma^2(x), \tag{10}$$

$$\sigma_3(\xi) = \sigma^2(x)\left(\frac{U_w'}{U_w} - \frac{1}{x}\right) \quad \text{and} \quad \sigma_4(\xi) = \frac{U_w\sigma^2(x)}{x} \tag{11}$$

subject to the boundary conditions

$$f(\xi,0,\zeta) = 0, \qquad f_\eta(\xi,0,\zeta) = U_w(\xi), \qquad f_\eta(\xi,+\infty,\zeta) = 0. \tag{12}$$

3. The analytic approach based on the HAM

3.1. Initial unsteady solution at $\zeta = 0$

When $\zeta = 0$ (corresponding to the $\tau = 0$) Eq. (9) becomes the Rayleigh type of equation

$$\frac{\partial^3 f}{\partial \eta^3} + \frac{\sigma_4(\xi)}{2}\eta\frac{\partial^2 f}{\partial \eta^2} = 0, \tag{13}$$

subject to

$$f(\xi,0,0) = 0, \qquad f_\eta(\xi,0,0) = U_w(\xi), \qquad f_\eta(\xi,+\infty,0) = 0. \tag{14}$$

The exact solution of Eq. (13), by means of $\sigma_4(\xi)$ as given in (45), is

$$f(\xi,\eta,0) = U_w\left[\frac{2}{\sqrt{\pi}} - \frac{2\exp(-\eta^2/4)}{\sqrt{\pi}} + \eta\left(1 - \text{erf}\left(\frac{\eta}{2}\right)\right)\right], \tag{15}$$

where $\text{erf}(\eta)$ is the Gaussian integral or probability integral, given by

$$\text{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\infty \exp(-\eta^2)\text{d}\eta.$$

Thus, when $\zeta = 0$, we have

$$\left.\frac{\partial^2 f}{\partial \eta^2}\right|_{\eta=0,\zeta=0} = -\frac{\xi}{\sqrt{\pi}}. \tag{16}$$

3.2. Steady-state solution at $\zeta = 1$

When $\zeta = 1$ (corresponding to the $\tau \to +\infty$), we have from Eq. (9) that

$$\frac{\partial^3 f}{\partial \eta^3} + \sigma_1(\xi) f \frac{\partial^2 f}{\partial \eta^2} + \sigma_2(\xi)\left[\frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta \partial \xi}\right] = 0 \tag{17}$$

subject to the boundary conditions

$$f(\xi, 0, 1) = 0, \qquad f_\eta(\xi, 0, 1) = U_w(\xi), \qquad f_\eta(\xi, +\infty, 1) = 0. \tag{18}$$

For details, please refer to Liao [41] for the steady-state flows corresponding to $\zeta = 1$.

3.3. Series solution $0 \le \zeta \le 1$

Many previous studies show that the boundary-layer flows decay exponentially at infinity [42,43]. Further, Xu and Pop [44] studied the unsteady boundary-layer flow started impulsively from rest along a symmetric wedge. Very recently, Liao [41] gave the series solution on non-similarity boundary-layer flows caused by a stretching flat sheet. Following Liao [41], Andersson [42], and Xu and Pop [44], we express $f(\xi, \eta, \zeta)$ by a set of base functions

$$\{\xi^n \zeta^k \mathrm{e}^{-m\eta} \mid n \ge 0, k \ge 0, m \ge 0\} \tag{19}$$

in the form

$$f(\xi, \eta, \zeta) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n}^k \xi^n \zeta^k \mathrm{e}^{-m\eta} \tag{20}$$

where $a_{m,n}^k$ is the coefficient to be determined. Our boundary conditions (12) and solution expression defined by Eq. (20) suggest us to choose the initial guess as

$$f_o(\xi, \eta, \zeta) = U_w(\xi)(1 - \mathrm{e}^{-\eta}) \tag{21}$$

with the linear operator

$$\mathcal{L} f = \frac{\partial^3 f}{\partial \eta^3} - \frac{\partial f}{\partial \eta} \tag{22}$$

which satisfies the property

$$\mathcal{L}[C_1 + C_2 \mathrm{e}^{-\eta} + C_3 \mathrm{e}^{\eta}] = 0. \tag{23}$$

The zeroth-order deformation equation

From Eq. (9) we define a nonlinear operator N as

$$\begin{aligned}\mathcal{N} F = {} & \frac{\partial^3 F}{\partial \eta^3} + \zeta \sigma_1(\xi) F \frac{\partial^2 F}{\partial \eta^2} - \zeta \sigma_2(\xi)\left[\frac{\partial F}{\partial \eta}\frac{\partial^2 F}{\partial \eta \partial \xi} + \frac{\partial F}{\partial \xi}\frac{\partial^2 F}{\partial \eta^2}\right] - \frac{\zeta(1-\zeta)}{2}\sigma_3(\xi) \\ & \times \left[F\frac{\partial^2 F}{\partial \eta^2} - 2\zeta \frac{\partial F}{\partial \zeta}\frac{\partial^2 F}{\partial \eta^2} + 2\zeta \frac{\partial F}{\partial \eta}\frac{\partial^2 F}{\partial \zeta \partial \eta}\right] + \frac{(1-\zeta)^2 \sigma_4(\xi)}{2}\left[\eta \frac{\partial^2 F}{\partial \eta^2} - 2\zeta \frac{\partial^2 F}{\partial \eta \partial \zeta}\right].\end{aligned} \tag{24}$$

Let $q \in [0, 1]$ denote the homotopy-parameter and $\hbar \neq 0$ the convergence-control parameter [45]. We construct the so-called zeroth-order deformation equation

$$(1 - q)\mathcal{L}[F(\xi, \eta, \zeta; q) - f_0(\xi, \eta, \zeta)] = q\hbar \mathcal{N}[F(\xi, \eta, \zeta; q)], \tag{25}$$

subject to the boundary conditions on the sheet

$$F(\xi, 0, \zeta; q) = 0, \qquad F_\eta(\xi, 0, \zeta; q) = U_w(\xi) \tag{26}$$

and the boundary condition at infinity

$$F_\eta(\xi, +\infty, \zeta; q) \to 0. \tag{27}$$

Note that $f_0(\xi, \eta, \zeta)$ satisfies the boundary condition equation (12). Clearly, when $q = 0$ and $q = 1$, we have from Eq. (25) that

$$F(\xi, \eta, \zeta; 0) = f_0(\xi, \eta, \zeta) \tag{28}$$

and

$$F(\xi, \eta, \zeta; 1) = f(\xi, \eta, \zeta). \tag{29}$$

Thus, as the homotopy-parameter q increases from 0 to 1, $F(\xi,\eta,\zeta)$ varies continuously from the initial guess $f_0(\xi,\eta,\zeta)$ to the exact solution $f(\xi,\eta,\zeta)$ of Eqs. (9) and (12). Assume that the convergence-control parameter $\hbar$ is chosen properly that the Taylor series of $F(\xi,\eta,\zeta)$ expanded with respect to the homotopy-parameter q, i.e.

$$F(\xi,\eta,\zeta;q)=f_0(\xi,\eta,\zeta)+\sum_{m=1}^{+\infty}f_m(\xi,\eta,\zeta)q^m, \tag{30}$$

where

$$f_m(\xi,\eta,\zeta)=\frac{1}{m!}\frac{\partial^m F(\xi,\eta,\zeta;q)}{\partial q^m}\bigg|_{q=0}, \tag{31}$$

converges at $q=1$. Then, we have from (28) and (29) that

$$f(\xi,\eta,\zeta)=f_0(\xi,\eta,\zeta)+\sum_{m=1}^{+\infty}f_m(\xi,\eta,\zeta). \tag{32}$$

Here, (30) is called homotopy-series and (32) is called homotopy-series solution, respectively.

The high-order deformation equation

Differentiating the zeroth-order deformation equation (25) m-times with respect to the homotopy-parameter q and dividing the resulting expression by $m!$ and then setting $q=0$, we have the mth-order deformation equation

$$\mathcal{L}[f_m(\xi,\eta,\zeta)-\chi_m f_{m-1}(\xi,\eta,\zeta)]=\hbar R_m(\vec{f}_{m-1}), \tag{33}$$

subject to the boundary conditions on the sheet

$$f_m=0,\qquad \frac{\partial f_m}{\partial\eta}=0,\quad \text{at } \eta=0 \tag{34}$$

and boundary condition at infinity

$$\frac{\partial f_m}{\partial\eta}\to 0,\quad \text{at } \eta\to\infty, \tag{35}$$

where

$$\begin{aligned}R_m(\vec{f}_{m-1}) ={}& \frac{\partial^3 f_{m-1}}{\partial\eta^3}+\zeta\sigma_1(\xi)\sum_{n=0}^{m-1}f_{m-1-n}\frac{\partial^2 f_n}{\partial\eta^2}+\frac{(1-\zeta)^2\sigma_4(\xi)}{2}\eta\frac{\partial^2 f_{m-1}}{\partial\eta^2}\\ &+\zeta\sigma_2(\xi)\sum_{n=0}^{m-1}\left[\frac{\partial f_n}{\partial\xi}\frac{\partial^2 f_{m-1-n}}{\partial\eta^2}-\frac{\partial f_n}{\partial\eta}\frac{\partial^2 f_{m-1-n}}{\partial\xi\partial\eta}\right]-\frac{(1-\zeta)^2\sigma_4(\xi)}{2}2\zeta\frac{\partial^2 f_{m-1}}{\partial\eta\partial\zeta}\\ &-\frac{\zeta(1-\zeta)}{2}\sigma_3(\xi)\sum_{n=0}^{m-1}\left[f_{m-1-n}\frac{\partial^2 f_n}{\partial\eta^2}+2\zeta\frac{\partial f_n}{\partial\zeta}\frac{\partial^2 f_{m-1-n}}{\partial\eta^2}-2\zeta\frac{\partial^2 f_{m-1-n}}{\partial\zeta\partial\eta}\right]\end{aligned} \tag{36}$$

and

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases} \tag{37}$$

Let $f_m^*(\xi,\eta,\zeta)$ denote a special solution of Eq. (33). We have

$$f_m^*(\xi,\eta,\zeta)=\chi_m f_{m-1}(\xi,\eta,\zeta)+\hbar\mathcal{L}^{-1}[R_m(\vec{f}_{m-1})], \tag{38}$$

where L^{-1} denotes the inverse operator of L. Thus the solution of the high-order deformation equation (33) reads

$$f_m(\xi,\eta,\zeta)=f_m^*(\xi,\eta,\zeta)+C_1+C_2e^{-\eta}+C_3e^{\eta}, \tag{39}$$

where C_1, C_2 and C_3 are determined by the boundary conditions (34)–(35). In this way, it is easy to get the successive solution of the problem by means of symbolic software such as Mathematica, Maple and so on. The skin friction at the wall is given by

$$C_f=\frac{\tau_w}{\frac{1}{2}\rho U_w^2(x)}=\frac{2\sqrt{\nu}}{\sigma(x)U_w^2(x)\sqrt{\zeta}}\frac{\partial^2 f}{\partial\eta^2}\bigg|_{\eta=0} \tag{40}$$

and the corresponding boundary-layer thickness $\delta(x)$ is defined by

$$\delta(x)=\frac{1}{U_w(x)}\int_0^\infty u(x,y)dy. \tag{41}$$

4. Result analysis

As Liao [41] mentioned that similarity solutions exist only in some special cases of $U_w(x)$. When a similarity solution exists, the system of two coupled non-linear PDEs becomes an ODE. Following Schlichting and Gersten [46], the criterion function for the existence of a similarity solution is

$$\Lambda(x) = \frac{U_w'(x)}{U_w(x)} \int_0^x U_w(z)\mathrm{d}z. \tag{42}$$

There is a similarity solution if $\Lambda(x)$ is constant. There are an infinite number of sheet velocities $U_w(x)$ that do not satisfy the similarity criteria (42) and thus lead to the non-similarity boundary-layer flows.

Liao [25] proved that, whenever a homotopy-series solution (32) converges, it will be one of the solutions of the considered problem. The convergence of the homotopy-series solution (32) strongly depends on the convergence-control parameter $\hbar$. The admissible value of $\hbar$ for which the homotopy-series converges can be determined by plotting the so-called $\hbar$-curves [25] or by plotting the square residual error versus $\hbar$. The square residual error of the original equation is

$$E(\hbar) = \int_0^1 \int_0^1 \int_0^\infty [N(f)]^2 \mathrm{d}\eta \mathrm{d}\xi \mathrm{d}\zeta.$$

Enforcing

$$\frac{\mathrm{d}E(\hbar)}{\mathrm{d}\hbar} = 0, \tag{43}$$

we have an optimal value of $\hbar$ corresponding to the least error. In this way, we get the best value of $\hbar$ corresponding to the minimum residual error of the original equation.

Without loss of generality, let us consider here the sheet stretching velocity $U_w = x/(1+x)$, which tends to 1 as $x \to \infty$. The stretching velocity $U_w(x)$ increases monotonously from 0 to 1 along the sheet. Note that $U_w \to x$ as $x \to 0$ and $U_w \to 1$ as $x \to +\infty$, respectively. Physically, the flows near $x = 0$ should be close to the similarity ones with $U_w = x$ and also the flows as $x \to +\infty$ should be close to the similarity ones with $U_w = 1$, respectively. For similarity flows with $U_w = x$ and $U_w = 1$, the similarity variables are $y/\sqrt{\nu\zeta}$ and $y/\sqrt{\nu\zeta x}$, respectively. Therefore, according to the definition of the variable η, we choose $\sigma(x) = \sqrt{1+x}$, so that η tends to $y/\sqrt{\nu\zeta}$ as $x \to 0$ and to $y/\sqrt{\nu\zeta x}$ as $x \to +\infty$, respectively. For the sake of simplicity, we define

$$\xi = \Gamma(x) = \frac{x}{1+x} \tag{44}$$

which gives

$$U_w = \xi, \qquad \sigma_1(\xi) = \frac{1}{2}, \qquad \sigma_2(\xi) = 1 - \xi, \qquad \sigma_3(\xi) = -1, \quad \text{and} \quad \sigma_4(\xi) = 1. \tag{45}$$

Note that our homotopy-series solution (32) contains convergence-control parameter $\hbar$ and the physical parameters ξ and ζ (corresponding to τ), "Regarding $\hbar$ as un unknown parameter", we plot curves of the square residual error versus $\hbar$, as shown in Fig. 1. It is clear from Fig. 1 that the region for the convergence of the homotopy-series solution (32) is about $-3/2 \le \hbar \le 0$. We can find a proper value of $\hbar$ from this range to ensure that the homotopy-series solution is convergent for all the parameters in the whole domain $0 \le \zeta < 1$, $0 \le \xi < 1$ and $0 \le \eta < \infty$ corresponding to $0 \le \tau < \infty$, $0 \le x < \infty$ and $0 \le y < \infty$, respectively.

Firstly, we check the validity and correctness of the analytic approximations obtained by the above-mentioned technique. Besides, the convergence can be greatly accelerated by means of the homotopy-Padé technique. Excellent agreement is found between the exact solution (15), homotopy-series solution (32) and the homotopy-Padé approximations. It is found that when $\hbar = -1/2$ and $\zeta = 0$ (or $\tau = 0$), our 15th order homotopy-series solutions and the [2, 2] homotopy-Padé approximations agree very-well with the exact solution (15) in the whole domain $0 \le \xi < 1$ and $0 \le \eta < \infty$, as shown in the Fig. 2. We then consider $f''(\xi, 0, 0)$, which relates to the local skin friction coefficient C_f and thus has an important physical meanings. It is found that $f''(\xi, 0, 0) = -\xi/\sqrt{\pi}$, which gives the results for unsteady similarity-flow for particular value of $\xi = 1$.

It is found that the local coefficient of skin friction for steady-state similarity flows tends to $-2\sqrt{\nu}/x$ as $x \to 0$ and $-0.8875\sqrt{\nu/x}$ as $x \to \infty$. The boundary-layer thickness for steady-state similarity flows is $\sqrt{\nu}$ in case of the $U_w(x) = x$ and $1.61613\sqrt{\nu x}$ in case of $U_w(x) = 1$. It is seen from Figs. 3 and 4 that our homotopy-series solutions at $\tau = 10$ by means of $\hbar = -1/2$ agree very well with steady-state similarity boundary-layer flows in the case of $U_w(x) = x$ and $U_w(x) = 1$. The skin friction coefficient and boundary-layer thickness for the final steady-state solution ($\zeta = 1$) is also shown in Figs. 3 and 4. It is seen that there is a good agreement between the results when we solve the full unsteady non-similarity boundary-layer equations at $\tau = 10$ and with Liao's [41] steady-state solutions at $\zeta = 1$. This agreement is taken as a verification of the present analytic approach.

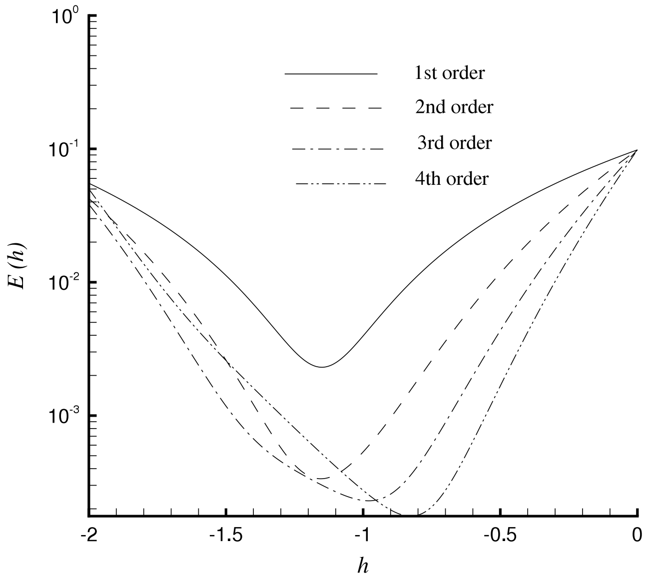

Fig. 1. Square residual error versus convergence-control parameter $\hbar$.

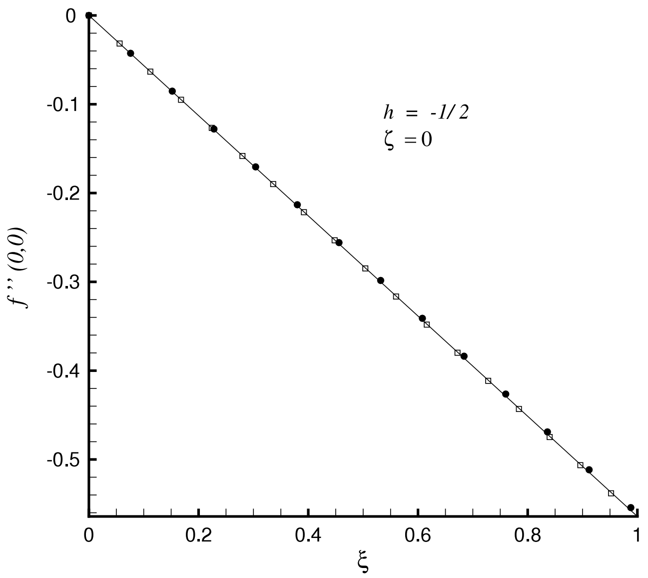

Fig. 2. Comparison of the exact solution (15) with homotopy-series solution (32) and homotopy-Padé approximations when $\zeta = 0$ by means of $\hbar = -1/2$. Solid line: exact solution (15); Squares: 15th order homotopy-series solution (32); Circles: [2, 2] homotopy-Padé approximations.

Table 1
Square residual error of Eq. (9) when $\hbar = -1/2$.

Order of approximation	Residual error
1st	0.033080
3rd	0.004338
5th	0.000666

In general, we can substitute the series solutions into the governing equations and evaluate the square residual error so as to check the convergence of the solutions. Table 1 shows the square residual error of Eq. (9). It is seen that by increasing the order of approximation the square residual error is decreasing. This indicates that our HAM series solution is convergent.

Fig. 5 represents the variation of the skin friction coefficient with τ. It is noticed that the values of $-C_f/\sqrt{\nu}$ decrease as the τ increase. It is also noticed that due to impulsive motion, the skin friction has large magnitude for small time $\zeta = 0$ ($\tau \to 0$) after the start of motion, and it decreases monotonically and reaches steady-state values at $\zeta = 1$ ($\tau \to \infty$). Fig. 6 shows the effect of τ on the boundary-layer thickness. It is clear from Fig. 6 that the effect of time on the boundary-layer thickness is just opposite. Therefore, there is a smooth transition from the small time to the large time solution. The homotopy-Padé technique is used to accelerate the convergence of the homotopy-series solutions. It is found that our 15th order homotopy-series results have excellent agreement with the [6, 6] homotopy-Padé approximations. Hence, we get

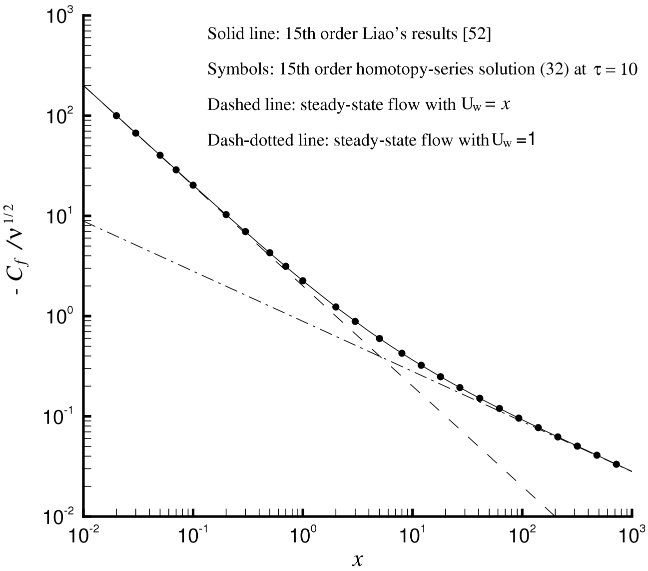

Fig. 3. Comparison of skin friction coefficient of the homotopy-series solution (32) at $\tau = 10$ when $\hbar = -1/2$ with Liao's steady-state results [41]. Solid line: 15th-order Liao's results [41]; Symbols: 15th-order homotopy-series solution (32) at $\tau = 10$; Dashed-line: $C_f = -2\sqrt{\nu}/x$ for steady-state similarity flow with $U_w = x$; Dash–dotted line: $C_f = -0.8875\sqrt{\nu/x}$ for steady-state similarity flow with $U_w = 1$.

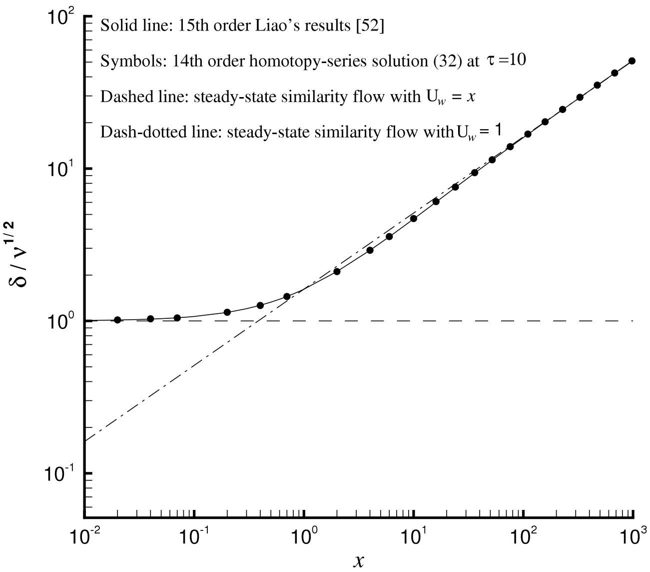

Fig. 4. Comparison of boundary-layer thickness of the homotopy-series solution (32) at $\tau = 10$ when $\hbar = -1/2$ with Liao's steady-state results [41]. Solid line: 15th-order Liao's results [41]; Symbols: 15th-order homotopy-series solution (32) at $\tau = 10$; Dashed-line: $\delta(x) = \sqrt{\nu}$ for steady-state similarity flow with $U_w = x$; Dash–dotted line: $\delta(x) = 1.61613\sqrt{\nu x}$ for steady-state similarity flow with $U_w = 1$.

purely analytic solutions of the unsteady non-similarity boundary-layer flows caused by an impulsively stretching flat sheet which are accurate and uniformly valid for all dimensionless times $0 \le \tau < \infty$ in the whole spatial region $0 \le x < \infty$ and $0 \le y < \infty$ by means of the homotopy analysis method (HAM).

5. Conclusion

In this paper, a system of non-linear partial differential equations is solved analytically by means of the HAM. An auxiliary artificial parameter is used to ensure the convergence of the homotopy-series solution. We use new time scaling to the impulsively stretching flat sheet as compared to the Williams and Rhyne [47] time scaling. The new time scaling avoids the appearance of the singularity resulting from the logarithmic function $\ln(1 - \xi)$ as discussed by Xu and Pop [44]. Different from the previous analytic results, our homotopy-series solutions are convergent and valid for all dimensionless time $0 \le \tau < \infty$ in the whole domain $0 \le x < \infty$ and $0 \le y < \infty$. The effect of dimensionless time τ on the skin friction coefficient and boundary-layer thickness is analyzed. To the best of our knowledge, such a kind of analytic solution has never been reported in the literature. This approach is general and thus can be applied to get the accurate analytic solutions of the other unsteady non-similarity boundary-layer flows which are uniformly valid for all time.

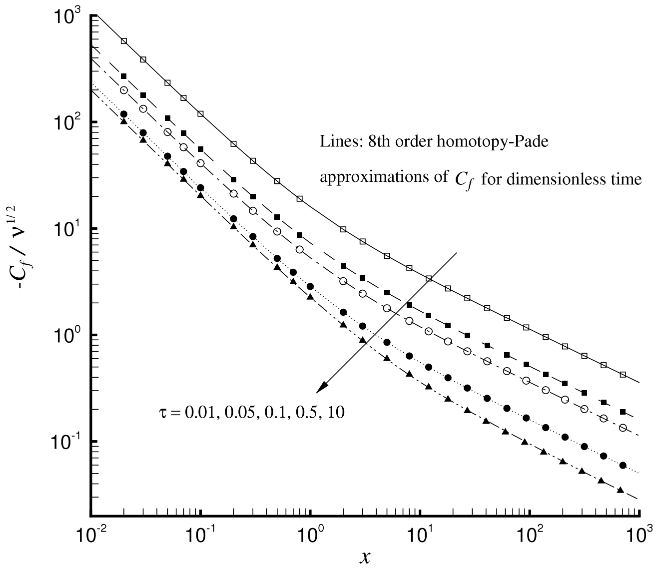

Fig. 5. The 15th order approximation of skin friction coefficient for different time by means of $\hbar = -1/2$. Squares: $\tau = 0.01$; Filled squares: $\tau = 0.05$; Circles: $\tau = 0.1$; Filled circles: $\tau = 0.5$ and Filled triangles: $\tau = 10$.

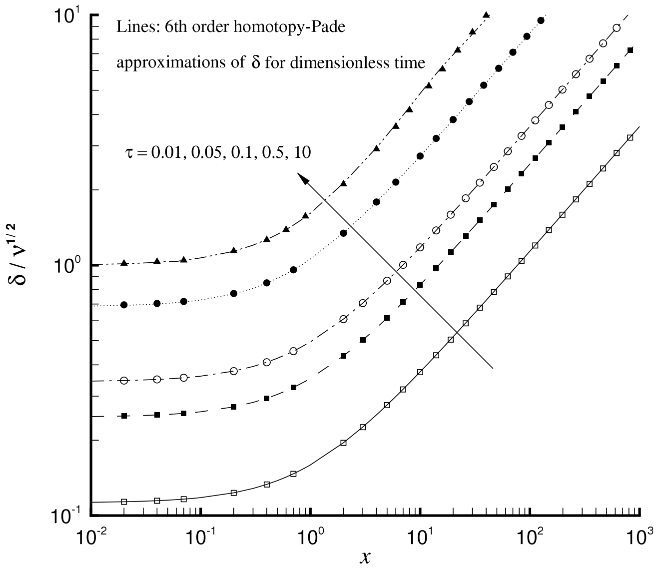

Fig. 6. Displacement thickness for different values of τ by means of $\hbar = -1/2$. Squares: 10th-order approximation of $\delta(x)$ when $\tau = 0.01$; Filled squares: 10th-order approximation of $\delta(x)$ when $\tau = 0.05$; Circles: 10th-order approximation of $\delta(x)$ when $\tau = 0.1$; Filled circles: 14th-order approximation of $\delta(x)$ when $\tau = 0.5$; Filled triangles: 14th-order approximation of $\delta(x)$ when $\tau = 10$.

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References

[1] B.C. Sakiadis, Boundary-layer behavior on continuous solid surfaces. I. Boundary-layer equations for two-dimensional and axisymmetric flow, AIChE J. 7 (1961) 26–28;
Boundary-layer behavior on continuous solid surfaces. II. The boundary-layer on a continuous flat surface, AIChE J. 7 (1961) 221–225.
[2] I.J. Crane, Flow past a stretching plate, Z. Angew. Math. Phys. 21 (56) (1970) 1–37.
[3] P.S. Gupta, A.S. Gupta, Heat and mass transfer on a stretching sheet with suction and blowing, Can. J. Chem. Eng. 55 (1977) 744–746.
[4] P. Carragher, I.J. Crane, Heat transfer on a continuous stretching sheet, Z. Angew. Math. Mech. 62 (1982) 564–565.
[5] J.E. Danberg, K.S. Fansler, A nonsimilar moving wall boundary-layer problem, Quart. Appl. Math. 34 (1976) 305–309.
[6] A. Chakrabarti, A.S. Gupta, Hydromagnetic flow and heat transfer over a stretching sheet, Quart. Appl. Math. 37 (1979) 73–78.
[7] K. Vajravelu, Hydromagnetic flow and heat transfer over a continuous moving porous, flat surface, Acta Mech. 64 (1986) 179–185.
[8] B.K. Dutta, Heat transfer from a stretching sheet with uniform suction and blowing, Acta Mech. 78 (1986) 255–262.
[9] S.L. Lee, J.S. Tsai, Cooling of a continuous moving sheet of finit thickness in the presence of natural convection, Int. J. Heat Mass Transfer 33 (1990) 457–464.

[10] K.J. Wanous, E.M. Sparrow, Heat transfer for flow longitudinal to a cylinder with surface mass transfer, J. Heat Transf. ASME. Ser. C 87 (1) (1965) 317–319.
[11] D. Catherall, Williams. Stewartson, Viscous flow past a flat plate with uniform injection, Proc. R. Soc. A. 284 (1965) 370–396.
[12] E.M. Sparrow, H. Quack, C.J. Boerner, Local non-similarity boundary layer solutions, J. AIAA 8 (11) (1970) 1936–1942.
[13] E.M. Sparrow, H.S. Yu, Local non-similarity thermal boundary-layer solutions, J. Heat Transf. ASME (1971) 328–334.
[14] M. Massoudi, Local non-similarity solutions for the flow of a non-Newtonian fluid over a wedge, Internat. J. Non-Linear Mech. 36 (2001) 961–976.
[15] B. Brunone, B. Karney, M. Mecarelli, M. Ferrante, Velocity profiles and unsteady pipe friction in transient flow, J. Water Resour. Plann. Manage. 126 (2000) 236–244.
[16] K. Stewartson, On the impulsive motion of a flat plate in a viscous fluid (part I), Quart. J. Mech. 4 (1951) 182–198.
[17] K. Stewartson, On the impulsive motion of a flat plate in a viscous fluid (part II), Quart. J. Mech. Appl. Math. 22 (1973) 143–152.
[18] M.G. Hall, The boundary layer over an impulsively started flat plate, Proc. R. Soc. A. 310 (1969) 401–414.
[19] S.C.R. Dennis, The motion of a viscous fluid past an impulsively started semi-infinite flat plate, J. Inst. Math. Appl. 10 (1972) 105–117.
[20] C.B. Watkins, Heat transfer in the boundary layer over an impulsively started flat plate, J. Heat Transfer 97 (1975) 282–484.
[21] R. Seshadri, N. Sreeshylan, G. Nath, Unsteady mixed convection flow in the stagnation region of a heated vertical plate due to impulsive motion, Int. J. Heat Mass Transfer 45 (2002) 1345–1352.
[22] O.B. Anwar, A.Y. Bakier, V.R. Prasad, J. Zueco, S.K. Ghosh, Nonsimilar laminar steady electrically-conducting forced convection liquid metal boundary layer flow with induced magnetic field effects, Int. J. Therm. Sci. (2008) doi:10.1016/j.ijthermalsci.12.007.
[23] A.T. Eswara, G. Nath, Unsteady forced convection laminar boundary layer flow over a moving longitudinal cylinder, Acta Mech. 93 (1992) 13–28.
[24] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, Ph.D. Thesis, Shanghai Jiao Tong University, 1992.
[25] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall, CRC Press, Boca Raton, 2003.
[26] A.H. Nayfeh, Perturbation Methods, John Wiley and Sons, New York, 2000.
[27] A.M. Lyapunov, General Problem on Stability of Motion, Taylor and Francis, London, 1992, (English Translation, Original Work Published 1892).
[28] A.V. Karmishin, A.I. Zhukov, V.G. Kolosov, Methods of Dynamics Calculation and Testing for Thin-Walled Structures, Mashinostroyenie, Moscow, 1990 (in Russian).
[29] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston, London, 1994.
[30] S.J. Liao, A uniformaly valid analytic solution of 2D viscous flow past a semi-infinite flat plate, J. Fluid Mech. 385 (1999) 101–128.
[31] S.J. Liao, On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet, J. Fluid Mech. 488 (2003) 189–212.
[32] S.J. Liao, I. Pop, Explicit analytic solution for similarity boundary layer equations, Int. J. Heat Mass Transfer 47 (1) (2004) 75–85.
[33] M. Yamashita, K. Yabushita, K. Tsuboi, An analytic solution of projectile motion with the quadratic resistance law using the homotopy analysis method, J. Phys. A. 40 (2007) 8403–8416.
[34] Y. Bouremel, Explicit series solution for the Glauert-jet problem by means of the homotopy analysis method, Commun. Nonlinear Sci. Numer. Simul. 12 (5) (2007) 714–724.
[35] S. Abbasbandy, The application of the homotopy analysis method to solve a generalized Hirota–Satsuma coupled KdV equation, Phys. Lett. A 361 (2007) 478–483.
[36] T. Hayat, M. Sajid, Analytic solution for axisymmetric flow and heat transfer of a second grade fluid past a stretching sheet, Int. J. Heat Mass Transfer 50 (2007) 75–84.
[37] F.M. Allan, Derivation of the Adomian decomposition method using the homotopy analysis method, Appl. Math. Comput. 190 (2007) 6–14.
[38] M. Sajid, T. Hayat, S. Asghar, On the analytic solution of the steady flow of a fourth grade fluid, Phys. Lett. A 355 (2006) 18–26.
[39] S.P. Zhu, A closed-form analytical solution for the valuation of convertible bonds with constant dividend yield, ANZIAM J. 47 (2006) 477–494.
[40] S.P. Zhu, An exact and explicit solution for the valuation of American put options, Quant. Finance 6 (2006) 229–242.
[41] S.J. Liao, A general approach to get series solutions of non-similarity boundary-layer flows, Commun. Nonlinear Sci. Numer. Simul. (2008).
[42] H.I. Andersson, MHD flow of a viscous fluid past a stretching surface, Acta Mech. 95 (1992) 227–230.
[43] S.J. Liao, Series solutions of unsteady boundary-layer flows over a stretching flat plate, Stud. Appl. Math. 117 (3) (2006) 2529–2539.
[44] H. Xu, I. Pop, Homotopy analysis of unsteady boundary-layer flow started impulsivley from rest along a symmetric wedge, J. Appl. Math. Mech. 88 (6) (2008) 507–514.
[45] S.J. Liao, Notes on the homotopy analysis method: some definitions and theorems, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 983–997.
[46] H. Schlichting, K. Gersten, Boundary Layer Theory, Springer, Berlin, 2000.
[47] J.C. Williams, T.H. Rhyne, Boundary layer development on a wedge impulsively set into motion, SIAM J. Appl. Math. 38 (1980) 215–224.