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Physical limit of prediction for chaotic motion of three-body problem

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ABSTRACT

A half century ago, Lorenz found the "butterfly effect" of chaotic dynamic systems and made his famous claim that long-term prediction of chaos is impossible. However, the meaning of the "long-term" in his claim is not very clear. In this article, a new concept, i.e. the physical limit of prediction time, denoted by T_p^{max}, is put forwarded to provide us a time-scale for at most how long mathematically reliable (numerical) simulations of trajectories of a chaotic dynamic system are physically correct. A special case of three-body problem is used as an example to illustrate that, due to the inherent, physical uncertainty of initial positions in the (dimensionless) micro-level of 10^{-60}, the chaotic trajectories are essentially uncertain in physics after $t > T_p^{max}$, where $T_p^{max} \approx 810$ for this special case of the three body problem. Thus, physically, it has no sense to talk about the "accurate, deterministic prediction" of chaotic trajectories of the three body problem after $t > T_p^{max}$. In addition, our mathematically reliable simulations of the chaotic trajectories of the three bodies suggest that, due to the butterfly effect of chaotic dynamic systems, the micro-level physical uncertainty of initial conditions might transfer into macroscopic uncertainty. This suggests that micro-level uncertainty might be an origin of some macroscopic uncertainty. Besides, it might provide us a theoretical explanation about the origin of uncertainty (or randomness) of many macroscopic phenomena such as turbulent flows, the random distribution of stars in the universe, and so on.

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1. Introduction

Why do stars in the sky look like random? Why are velocities of turbulent flows so uncertain? Are there any relationships between microscopic uncertainty and macroscopic uncertainty? What is the origin of the macroscopic uncertainty and/or randomness? Can we give a theoretical explanation for such kind of macroscopic uncertainty and/or randomness? In this paper, we attempt to give such an explanation using chaotic motion of Newtonian three-body problem as an example.

Making mathematically reliable and physically correct prediction is an important scientific object. Laplace claimed that the future can be unerringly predicted, given sufficient knowledge of the present. However, Laplace was wrong even in a classical, non-chaotic universe, as pointed out by Wolpert [1]. In 1890, Poincaré [2] found that trajectories of three-body problem are unintegrable in general. In 1963 Lorenz [3] found that it is impossible to make "long-term" prediction of non-periodic (called chaotic today) solution of a nonlinear dynamic system (called today Lorenz equation), due to the sensitive dependence on initial condition (SDIC), i.e. a very slight variation of initial conditions leads to considerably obvious difference of computer-generated trajectories for a long time. Besides, it was found [4–10] that numerical simulations of

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chaotic trajectories are sensitive not only to initial conditions but also to (traditional) numerical algorithms and digit precision of numerical data, i.e. a slight difference of (traditional) numerical algorithms (such as a tiny variation of time step) and/or digit precision of data might lead to considerably large difference of simulations of chaotic trajectories for a long time. This is easy to understand, since numerical noises, i.e. truncation and round-off errors, are inherent and unavoidable for all numerical simulations, where truncation error is closely related to numerical algorithms and round-off error is due to the limited precision of numerical data. According to the Shadow Lemma [11], for a uniformly hyperbolic dynamic system, there always exists a true trajectory near any computer-generated trajectories, as long as the truncation and round-off errors are small enough. Unfortunately, hardly a nonlinear dynamic system is uniformly hyperbolic in most cases so that the Shadow Lemma [11] often does not wok in practice. Besides, it is found that for some chaotic dynamic systems, computer-generated trajectories can be shadowed only for a short time [12], and in addition it is "virtually impossible to obtain a long trajectory that is even approximately correct" [13]. Furthermore, for some chaotic systems, "there is no fundamental reasons for computer-simulated long-time statistics to be even approximately correct" [14]. As illustrated by Yuan and York [15] using a model, a numerical artifact persists for an arbitrarily high numerical precision. These numerical artifacts "expose an exigent demand of safe numerical simulations" [16]. Thus, it is a huge challenge to give mathematically reliable simulations of chaotic dynamic systems in a long enough interval.

On the other side, although Lorenz's famous claim that "long-term prediction of chaos is impossible" has been widely accepted by scientific community, the word "long-term" is not very clear. Is one day long enough? Or millions of years? Obviously, a time-scale is needed for us to understand Lorenz's claim better.

Currently, a safe numerical approach, namely the "Clean Numerical Simulation" (CNS) [10,17], is proposed to gain mathematically reliable computer-generated trajectories of chaotic dynamic system in a finite but long enough interval. The CNS is based on Taylor series method (TSM) [18,19] at high enough order of approximation and the high precision data with large enough number of significant digits. The Taylor series method [18,19] is one of the oldest method, which can trace back to Newton, Euler, Liouville and Cauchy. It has an advantage that its formula at arbitrarily high order can be easily expressed in the same form. So, from viewpoint of numerical simulations, it is rather easy to use the Taylor series method at very high order so as to deduce the truncation error to a required level. Besides, the round-off error can be reduced to arbitrary level by means of computer algebra system (such as Mathematica) or the multiple precision (MP) library [20]. Thus, using the CNS, the numerical noises can be decreased to such a small level that both truncation and round-off errors are negligible in a given finite but long enough interval. For example, using the CNS ($\Delta t = 0.01$) with the 400th-order Taylor series method and 800-digit precision data, Liao [9] gained, for the first time, the mathematically reliable chaotic simulation of Lorenz equation in the interval $0 \leqslant t \leqslant 1000$ LTU (Lorenz time unit) by means of Mathematica. In 2011, Wang et al. [21] greatly decreased the required CPU times of the CNS by employing a parallel algorithm and the multiple precision (MP) library of C. Using the parallel CNS with the 1000th-order Taylor expansion and the 2100 digit multiple precision, Wang et al. [21] gain a mathematically reliable chaotic simulation of Lorenz equation in the interval [0,2500] within only 30 h, which is validated using a more accurate simulation given by the 1200th-order Taylor expansion and the 2100 digit multiple precision. Their results [21] confirm the correction and reliability of Liao's simulation [9] in the interval $[0, 1000]$. Currently, using 1200 CPUs of the National Supercomputer TH-A1 and the modified parallel integral algorithm based on the CNS with the 3500th-order Taylor expansion and the 4180-digit multiple precision data, Liao and Wang [22] gain a mathematically reliable simulation of chaotic solution of Lorenz equation in a rather long interval $[0, 10000]$. All of these illustrate that, the uncertainty of simulations of chaotic trajectories (in a given, finite but long enough interval) caused by numerical noises can be avoided by means of the CNS. Thus, from *mathematical* viewpoint, given an *exact* initial condition, we can gain *mathematically* reliable trajectories of chaotic dynamic systems in a *finite* but long *enough* interval by means of the CNS, without any observable uncertainty of simulation.

Is such a *mathematically* reliable chaotic trajectory (in a finite but long enough interval) *physically* correct?

Note that the uncertainty of simulations of chaotic trajectories is caused by many factors. Theoretically speaking, given an *exact* initial condition, the uncertainty is completely caused by numerical noises, i.e. truncation and round-off error, where truncation error is determined by numerical algorithms and round-off error is due to the limited precision of numerical data, respectively. However, in practice, initial conditions are *not* exact in practice: they contain both artificial and physical uncertainty. The artificial uncertainty mainly comes from limited precision of measurement. The physical uncertainty is due to the inherently uncertain/random property of nature, caused by such as thermal fluctuation, wave-particle duality of de Broglie's wave, and so on. Generally, the artificial uncertainty is much larger than the physical uncertainty. So, for dynamic systems, physical uncertainty determines a time of physical limit of prediction, denoted by T_p^{max}, beyond which trajectories are essentially uncertain in *physics*, as illustrated in this paper. In order to investigate the time of physical limit of prediction, we *assume* that there is no artificial uncertainty, i.e. measurement can have precision of arbitrary degree. In this way, the uncertainty of initial conditions caused by limited precision of measurement is avoided, too. Thus, we can focus on the case that initial conditions contain micro-level physical uncertainty only, which are often in the (dimensionless) micro-level of 10^{-20} to 10^{-60} or even smaller. It should be emphasized that the uncertainty of initial condition is in the micro-level, but the considered dynamic system is about macroscopic phenomena. Thus, this is a problem with multiple scales. Fortunately, the propagation of such kind of micro-level uncertainty can be reliably and accurately simulated by means of the CNS now, as illustrated by Liao [17].

Without loss of generality, we consider here the famous three-body problem [2,24,23,25] with chaotic motion, governed by Newtonian gravitational law. In Section 2 we briefly describe the numerical algorithms based on the CNS approach.

According to the string theory [26], the Planck length is the order of magnitude of oscillating strings that form elementary particles, and shorter length do not make physical senses. Thus, as pointed out in Section 3, the physical uncertainty of initial position of the three bodies might be in the (dimensionless) level of 10^{-60}. The propagation of such kind of micro-level uncertainty for the three-body problem in a finite but long enough interval, together with the mathematically reliable chaotic trajectories, can be accurately simulated by means of the CNS, as illustrated by Liao [17]. It should be emphasized that such a micro-level uncertainty of initial position is inherent and unavoidable in physics, i.e. objective. Our reliable simulations given in Section 4 suggest that such tiny physical uncertainty might lead to the macroscopic uncertainty of chaotic trajectories beyond a limit time T_p^{max}, where $T_p^{max} \approx 810$, although all of these chaotic trajectories are mathematically reliable in the interval [0,1000]. In other word, considering the physical uncertainty of initial position at the micro-level 10^{-60}, we can gain chaotic trajectories in the interval $[0, T_p^{max}]$ without obvious difference, but beyond it these mathematically reliable chaotic trajectories have obvious difference and thus the system becomes essentially uncertain in physics. This is mainly because the micro-level physical uncertainty of initial condition transfers into the macroscopic uncertainty when $t > T_p^{max}$, so that it is impossible to make a *physically* correct prediction beyond it. Thus, $T_p^{max} \approx 810$ is the time of the *physical limit of prediction* of chaotic trajectories for the special case of three-body problem considered in this paper, which provides us a time-scale for at most how long mathematically reliable simulations of chaotic motion of the three bodies are physically correct. Finally, concluding remarks and some discussions about the concept "physical limit of prediction" are given in Section 5.

2. Numerical algorithms of the CNS

Applications of the CNS to the Hamiltonian Hénon-Heiles system for motion of stars orbiting in a plane suggest that there exist a physical limit of prediction time, beyond which trajectories of chaotic dynamic systems become essentially uncertain in physics [17]. Note that the Hamiltonian Hénon-Heiles system is a simplified model for motion of stars orbiting in a plane. However, orbits of stars are three dimensional in practice. Thus, it is necessary to investigate some more accurate physical models, such as the famous three-body problem [2,24,23,25] governed by the Newtonian gravitation law. In fact, non-periodic results were first found by Poincaré [2] for three-body problem.

Let us consider the famous three-body problem, say, the motion of three celestial bodies under their mutual gravitational attraction. Let x_1, x_2, x_3 denote the three orthogonal axises. The position vector of the i body is expressed by $\mathbf{r}_i = (x_{1,i}, x_{2,i}, x_{3,i})$. Let T and L denote the characteristic time and length scales, and m_i the mass of the ith body, respectively. Using Newtonian gravitation law, the motion of the three bodies are governed by the corresponding non-dimensional equations

$$\ddot{x}_{k,i} = \sum_{j=1, j\neq i}^{3} \rho_j \frac{(x_{k,j} - x_{k,i})}{R_{i,j}^3}, \quad k = 1, 2, 3, \tag{1}$$

where

$$R_{i,j} = \left[\sum_{k=1}^{3} (x_{k,j} - x_{k,i})^2 \right]^{1/2} \tag{2}$$

and

$$\rho_i = \frac{m_i}{m_1}, \quad i = 1, 2, 3 \tag{3}$$

denotes the ratio of the mass.

In the frame of the CNS, we use the M-order Taylor series method

$$x_{k,i}(t) \approx \sum_{m=0}^{M} \alpha_m^{k,i} (t - t_0)^m \tag{4}$$

to accurately calculate the orbits of the three bodies, where the coefficient $\alpha_m^{k,i}$ is only dependent upon the time t_0. Note that the position $x_{k,i}(t)$ and velocity $\dot{x}_{k,i}(t)$ at $t = t_0$ are known, i.e.

$$\alpha_0^{k,i} = x_{k,i}(t_0), \quad \alpha_1^{k,i} = \dot{x}_{k,i}(t_0). \tag{5}$$

The recursion formula of $\alpha_m^{k,i}$ for $m \geqslant 2$ is derived from (1), as described below.

To apply the high-order Taylor series method efficiently, we should give explicit formulas for the coefficient $\alpha_m^{k,i}$. Write $1/R_{i,j}^3$ in the form

$$f_{i,j} = \frac{1}{R_{i,j}^3} \approx \sum_{m=0}^{M} \beta_m^{i,j} (t - t_0)^m \tag{6}$$

with the symmetry property $\beta_m^{i,j} = \beta_m^{j,i}$, where $\beta_m^{i,j}$ is determined later. Substituting (4) and (6) into (1) and comparing the like-power of $(t - t_0)$, we have the recursion formula

$$\alpha_{m+2}^{k,i} = \frac{1}{(m+1)(m+2)} \sum_{j=1, j\neq i}^{3} \rho_j \sum_{n=0}^{m} (\alpha_n^{k,j} - \alpha_n^{k,i})\, \beta_{m-n}^{i,j}, \quad m \geqslant 0. \tag{7}$$

Thus, the positions and velocities of the three bodies at the next time-step $t_0 + \Delta t$ read

$$x_{k,i}(t_0 + \Delta t) \approx \sum_{m=0}^{M} \alpha_m^{k,i} (\Delta t)^m, \tag{8}$$

$$\dot{x}_{k,i}(t_0 + \Delta t) \approx \sum_{m=1}^{M} m\, \alpha_m^{k,i} (\Delta t)^{m-1}. \tag{9}$$

Write

$$S_{i,j} = R_{i,j}^6 \approx \sum_{m=0}^{M} \gamma_m^{i,j} (t - t_0)^m, \quad f_{i,j}^2 = \sum_{m=0}^{M} \sigma_m^{i,j} (t - t_0)^m, \tag{10}$$

with the symmetry property $\gamma_m^{i,j} = \gamma_m^{j,i}$ and $\sigma_m^{i,j} = \sigma_m^{j,i}$. Substituting (2), (4) and (6) into the above definitions and comparing the like-power of $(t - t_0)$, we have

$$\gamma_m^{i,j} = \sum_{n=0}^{m} \mu_{m-n}^{i,j} \sum_{k=0}^{n} \mu_k^{i,j} \mu_{n-k}^{i,j}, \tag{11}$$

$$\sigma_m^{i,j} = \sum_{n=0}^{m} \beta_n^{i,j}\, \beta_{m-n}^{i,j}, \tag{12}$$

with

$$\mu_m^{i,j} = \sum_{k=1}^{3} \sum_{n=0}^{m} (\alpha_n^{k,j} - \alpha_n^{k,i})\, (\alpha_{m-n}^{k,j} - \alpha_{m-n}^{k,i}), \quad i \neq j,\ m \geqslant 1, \tag{13}$$

and the symmetry $\mu_m^{i,j} = \mu_m^{j,i}$. Using the definition (6), we have

$$S_{i,j} f_{i,j}^2 = 1.$$

Substituting (10) into the above equation and comparing the like-power of $(t - t_0)$, we have

$$\sum_{n=0}^{m} \gamma_n^{i,j}\, \sigma_{m-n}^{i,j} = 0, \quad m \geqslant 1,$$

which gives the recursion formula

$$\beta_m^{i,j} = -\frac{1}{2\beta_0^{i,j}\gamma_0^{i,j}} \left\{ \sum_{n=1}^{m} \gamma_n^{i,j} \sigma_{m-n}^{i,j} + \gamma_0^{i,j} \sum_{k=1}^{m-1} \beta_k^{i,j} \beta_{m-k}^{i,j} \right\}, \quad m \geqslant 1,\ i \neq j. \tag{14}$$

In addition, it is straightforward that

$$\beta_0^{i,j} = \frac{1}{R_{i,j}^3}, \quad \mu_0^{i,j} = R_{i,j}^2, \quad \sigma_0^{i,j} = \left(\beta_0^{i,j}\right)^2, \quad \gamma_0^{i,j} = \left(\mu_0^{i,j}\right)^3, \quad \text{at } t = t_0. \tag{15}$$

It is a common knowledge that numerical methods always contain truncation and round-off errors. To decrease the round-off error, we express the positions, velocities, physical parameters and *all* numerical data in N-digit precision,[1] where N is a large enough positive integer. Obviously, the larger the value of N, the smaller the round-off error. Besides, for a reasonable time step $\Delta t = t - t_0$, the higher the order M of Taylor expansion (4), the smaller the truncation error. Therefore, if the order M of Taylor expansion (4) is high enough and all data are expressed in accuracy of long enough digits, both of truncation and round-off errors of the above-mentioned CNS approach (with reasonable time step Δt) can be so small that numerical noises are negligible in a (given) finite but long enough time interval, say, we can gain mathematically reliable chaotic simulations in a given time interval. In this way, the mathematically reliable chaotic orbits of the three bodies can be gained in a given, finite but long enough interval.

In this article, the computer algebra system Mathematica is employed. By means of the Mathematica, it is rather convenient to express all datas in 300-digit precision, i.e. $N = 300$. In this way, the round-off error is so small that it is almost negligible in the given time interval ($0 \leqslant t \leqslant 1000$). And the accuracy of the CNS simulations increases as the order M of Taylor expansion (4) enlarges, as shown in the next section.

In summary, the uncertainty of chaotic trajectories caused by the numerical noises can be avoided by means of the CNS, so that the propagation of the micro-level physical uncertainty of initial condition can be investigated accurately.

[1] Mathematica is used in this article.

3. Physical uncertainty of initial conditions

It is widely accepted that microscopic phenomena are essentially uncertain/random. Let us first consider some typical length scales of microscopic phenomena which are widely used in modern physics. For example, Bohr radius

$$r = \frac{\hbar^2}{m_e\, e^2} \approx 5.2917720859(36) \times 10^{-11}\ (\mathrm{m})$$

is the approximate size of a hydrogen atom, where $\hbar$ is a reduced Planck's constant, m_e is the electron mass, and e is the elementary charge, respectively. Besides, the so-called Planck length

$$l_P = \sqrt{\frac{\hbar\, G}{c^3}} \approx 1.616252(81) \times 10^{-35}\ (\mathrm{m}) \tag{16}$$

is the length scale at which quantum mechanics, gravity and relativity all interact very strongly, where c is the speed of light in a vacuum, and G is the gravitational constant, respectively. According to the string theory [26], the Planck length is the order of magnitude of oscillating strings that form elementary particles, and *shorter length do not make physical senses*. Especially, in some forms of quantum gravity, it becomes *impossible* to *determine* the difference between two locations less than one Planck length apart. Therefore, in the level of the Planck length, position of a body is *inherently* uncertain. This kind of microscopic physical uncertainty is *inherent* and has nothing to do with Heisenberg uncertainty principle [27], say, it is *objective*.

In addition, according to de Broglie [28], any a body has the so-called wave-particle duality. The de Broglie's wave of a body has non-zero amplitude. Thus, position of a body is uncertain: it could be almost *anywhere* along de Broglie's wave packet. Thus, according to the de Broglie's wave-particle duality, position of a star/planet is *inherent* uncertain, too. Therefore, it is reasonable to assume that, from the physical viewpoint, the micro-level inherent fluctuation of position of a body shorter than the Planck length l_p is essentially uncertain and/or random.

To make the Planck length $l_p \approx 1.62 \times 10^{-35}$ (m) dimensionless, we use the diameter of Milky Way Galaxy as the characteristic length, say, $d_M \approx 10^5$ light year $\approx 9 \times 10^{20}$ m. Obviously, $l_p/d_M \approx 1.8 \times 10^{-56}$ is a rather small dimensionless number. Thus, as mentioned above, two (dimensionless) positions shorter than 10^{-56} do not make physical senses in physics. So, it is reasonable to assume that the inherent uncertainty of the dimensionless position of a star/planet is in the micro-level 10^{-60}. Therefore, the tiny difference $d\mathbf{r}_1 = \pm 10^{-60}(1,0,0)$ of the initial conditions is in the micro-level: the difference is so small that all of these initial conditions can be regarded as the *same* in physics!

Mathematically, 10^{-60} is a tiny number, which is much smaller than truncation and round-off errors of *traditional* numerical algorithms based on 16 or 32-digit precision. So, it is impossible to investigate the influence and evaluation of this inherent micro-level uncertainty of initial conditions by means of these traditional numerical methods. However, the micro-level uncertainty 10^{-60} is much larger than the truncation and round-off errors of the CNS simulations by means of the high-order Taylor expansion and data in 300-digit precision with a reasonable time step Δt, as illustrated below. This is mainly because the CNS provides us a safe tool to study the propagation of such kind of inherent micro-level uncertainty of initial conditions, as currently illustrated in [17].

4. Reliable long-term simulations of chaotic motion

As mentioned before, the initial conditions contain both of artificial and physical uncertainty. For the sake of simplicity, the artificial uncertainty due to limited precision of measurement is assumed to be zero. In this way, we can focus on the propagation of the micro-level physical uncertainty of initial conditions.

Without loss of generality, let us consider the motion of three bodies with the initial positions

$$\mathbf{r}_1 = (0,0,-1) + d\mathbf{r}_1, \quad \mathbf{r}_2 = (0,0,0), \mathbf{r}_3 = -(\mathbf{r}_1 + \mathbf{r}_2), \tag{17}$$

and the initial velocities

$$\dot{\mathbf{r}}_1 = (0,-1,0), \dot{\mathbf{r}}_2 = (1,1,0), \dot{\mathbf{r}}_3 = -(\dot{\mathbf{r}}_1 + \dot{\mathbf{r}}_2), \tag{18}$$

where $d\mathbf{r}_1 = \delta(1,0,0)$ is the term with the micro-level physical uncertainty. For simplicity, we first only focus on the following three cases: $\delta = 0, \delta = +10^{-60}$ and $\delta = -10^{-60}$. *Mathematically*, the three initial positions have the tiny difference in the level of 10^{-60} (i.e. the physical uncertainty). However, the difference is even smaller than the Planck length so that all of these initial conditions can be regarded as the *same* in *physics*, since any lengths shorter than the Planck length do not make physical senses [26]. For the sake of simplicity, let us consider the case of equal masses, i.e. $\rho_j = 1$ $(j = 1,2,3)$. We are interested in the chaotic orbits of the three bodies in the time interval $0 \leqslant t \leqslant 1000$, which is long enough for our investigation interest, as shown below.

Note that the initial conditions satisfy

$$\sum_{j=1}^{3} \dot{\mathbf{r}}_j(0) = \sum_{j=1}^{3} \mathbf{r}_j(0) = 0.$$

Thus, due to the momentum conversation, we have

$$\sum_{j=1}^{3}\dot{\mathbf{r}}_j(t)=\sum_{j=1}^{3}\mathbf{r}_j(t)=0,\quad t\geqslant 0 \tag{19}$$

in general.

All data are expressed in 300-digit precision, i.e. $N=300$, using Mathematica. Thus, the round-off error is very small (compared to the physical uncertainty at the level 10^{-60}) and thus negligible. In addition, the higher the order M of Taylor expansion (4), the smaller the truncation error, i.e. the more accurate the results at $t=1000$. Assume that, at $t=1000$, we have the result $x_{1,1}=\mathbf{1.81510}12345$ by means of the M_1-order Taylor expansion and the result $x_{1,1}=\mathbf{1.81510}47535$ by means of the M_2-order Taylor expansion, respectively, where $M_2>M_1$. Then, the result $x_{1,1}$ by means of the lower-order (M_1) Taylor expansion is said to be in the accuracy of 5 significance digit, expressed by $n_s=5$. For more details about the CNS, please refer to Liao [17].

When $\delta=0$, the corresponding three-body problem has chaotic orbits with the Lyapunov exponent $\lambda=0.1681$, as pointed out by Sprott [29] (see Fig. 6.15 on page 137). It is well-known that a chaotic dynamic system has the sensitivity dependence on initial condition (SDIC). Thus, in order to gain reliable simulations of the chaotic orbits in such a long interval $0\leqslant t\leqslant 1000$, we employ the CNS approach using the high-enough order M of Taylor expansion with all data expressed in 300-digit precision.

It is found that, when $\delta=0$, the CNS simulations using $\Delta t=10^{-2}, N=300$ and $M=8,16,24,30,40$ and 50 agree each other in the *whole* interval [0,1000] at least in the accuracy of 11, 23, 38, 48, 64 and 81 significance digits, respectively. Approximately, n_s, the number of significance digits of the positions at $t=1000$, is linearly proportional to M (the order of Taylor expansion), say,

$$n_s\approx 1.6762M-2.7662,$$

as shown in Fig. 1. For example, the CNS simulation using the 50th-order Taylor expansion and data in 300-digit precision (with $\Delta t=10^{-2}$) provides us the position of Body 1 at $t=1000$ in the accuracy of 81 significance digit:

$$x_{1,1}=+1.81510475356295161721659400884544006456900303205574323759010328524043618135498683 4, \tag{20}$$

$$x_{2,1}=-1.44063514405828616113383505540674890012312900284853721976890817632270324748240993 8, \tag{21}$$

$$x_{3,1}=+1.98700788757862988109767764149857789791684629974639705175707 4.117730821734512844759. \tag{22}$$

Using the smaller time step $\Delta=10^{-3}$ and data in 300-digit precision (i.e. $N=300$), the CNS simulations given by the 8, 16, 24 and 30th-order Taylor expansion agree in the whole interval [0,1000] in the accuracy of 18, 38, 62 and 77 significance digits,

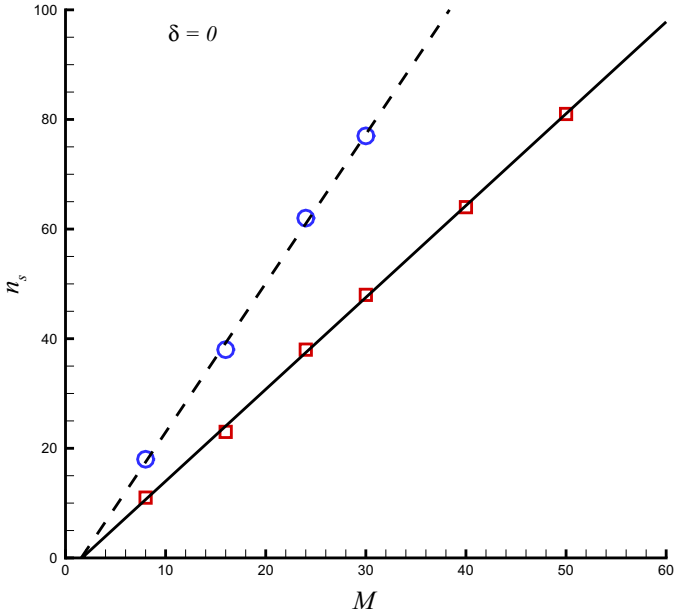

Fig. 1. Accuracy (expressed by n_s, number of significance digits) of the CNS results at $t=1000$ in the case of $\delta=0$ by means of $N=300$, the different time-step Δt and the different order (M) of Taylor expansion. Square: $\Delta t=10^{-2}$; Circle: $\Delta t=10^{-3}$. Solid line: $n_s\approx 1.6762M-2.7662$; dashed line: $n_s\approx 2.7182M-4.2546$.

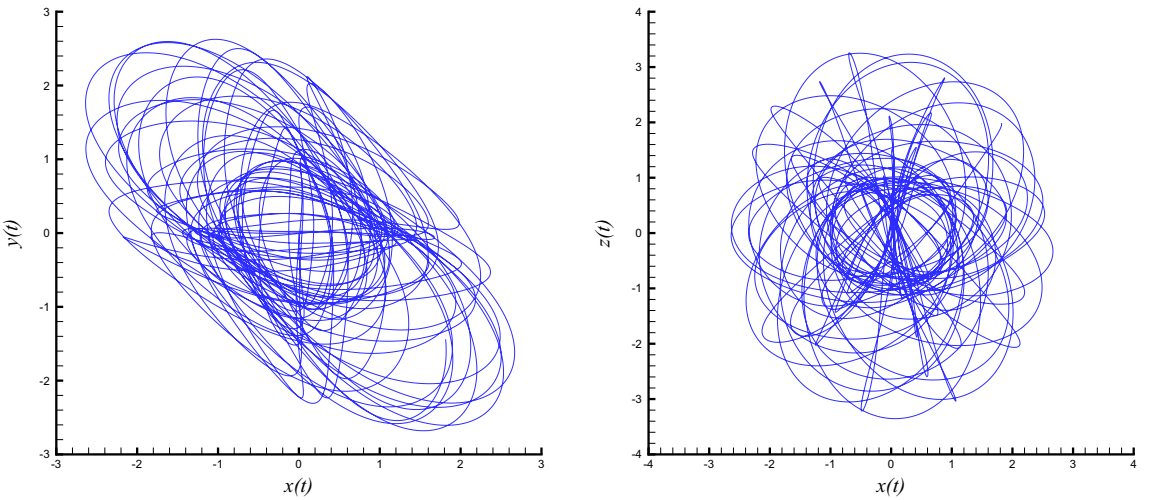

Fig. 2. $x-y$ and $x-z$ of Body 1 ($0 \leqslant t \leqslant 1000$) in the case of $\delta = 0$.

respectively. Approximately, n_s, the number of significance digits of the positions at $t = 1000$, is linearly proportional to M (the order of Taylor expansion), say,

$$n_s \approx 2.7182M - 4.2546,$$

as shown in Fig. 1. It should be emphasized that the CNS simulations given by $\Delta t = 10^{-3}$ and $M = 30$ agree (at least) in the 77 significance digits with those by $\Delta t = 10^{-2}$ and $M = 50$ in the *whole* interval $[0, 1000]$. In addition, the momentum conservation (19) is satisfied in the level of 10^{-295}. All of these confirm the mathematical correction and reliability of our CNS simulations[2]. Thus, although the considered three-body problem has chaotic orbits, we are quite sure that our numerical simulations given by the CNS using the 50th-order Taylor expansion and accurate data in 300-digit precision (with $\Delta t = 10^{-2}$) are *mathematically* reliable in the accuracy of 77 significance digits in the *whole* interval $0 \leqslant t \leqslant 1000$.

The orbits of the three bodies in the case of $\delta = 0$ are as shown in Figs. 2–4. The orbits of Body 1 and Body 3 are chaotic. This agrees well with Sprott's conclusion [29] (see Fig. 6.15 on page 137). However, it is interesting that Body 2 oscillates along a line on the plane $z = 0$. So, since $\sum_{j=1}^{3} \dot{\mathbf{r}}_j = \sum_{j=1}^{3} \mathbf{r}_j = 0$ due to the momentum conservation, the chaotic orbits of Body 1 and Body 3 must be symmetric about the regular orbit of Body 2. Thus, although the orbits of Body 1 and Body 3 are disorderly, the three bodies as a system have an elegant symmetry.

The initial conditions when $\delta = 10^{-60}$ have a tiny physical uncertainty

$$d\mathbf{r}_1 = 10^{-60}(1, 0, 0).$$

Since it is very small, it is reasonable to guess that the corresponding dynamic system is chaotic, too. Similarly, the corresponding orbits of the three bodies can be reliably simulated by means of the CNS. It is found that, when $\delta = 10^{-60}$, the CNS simulations by means of $\Delta t = 10^{-2}, N = 300$ and $M = 16, 24, 30, 40, 50, 60, 70, 80, 100$ agree each other in the whole interval $[0, 1000]$ in the accuracy of 1, 3, 5, 9, 11, 14, 17, 19 and 27 significance digits, respectively. Approximately, n_s (the number of significance digits) is linearly proportional to M (the order of Taylor expansion), say,

$$n_s \approx 0.2885M - 3.4684,$$

as shown in Fig. 5. According to this formula, in order to have the CNS simulations (in the interval $[0, 1000]$) in the precision of 81 significance digits by means of $\Delta t = 10^{-2}$, the 300th-order of Taylor expansion, i.e. $M = 300$, must be used. But, this needs much more CPU time. To confirm the correction of these CNS simulations, we further use the smaller time step $\Delta t = 10^{-3}$. It is found that, when $\delta = 10^{-60}$, the CNS simulations using $\Delta t = 10^{-3}, N = 300$ and $M = 8, 16, 24, 30, 40, 50$ agree well in the precision of 8, 21, 33, 43, 59, 72 significance digits in the whole interval $[0, 1000]$, respectively. Approximately, n_s, the number of significance digits of the corresponding CNS simulations in the interval $[0, 1000]$, is linearly proportional to M (the order of Taylor expansion), say,

$$n_s \approx 1.5386M - 3.7472$$

as shown in Fig. 5. For example, the position of Body 1 at $t = 1000$ given by the 50th-order Taylor expansion and data in 300-digit precision with $\Delta t = 10^{-3}$ reads

[2] Liao [17] proved a convergence-theorem and explained the validity and reliability of the CNS by using the mapping $x_{n+1} = \mathrm{mod}(2x_n, 1)$.

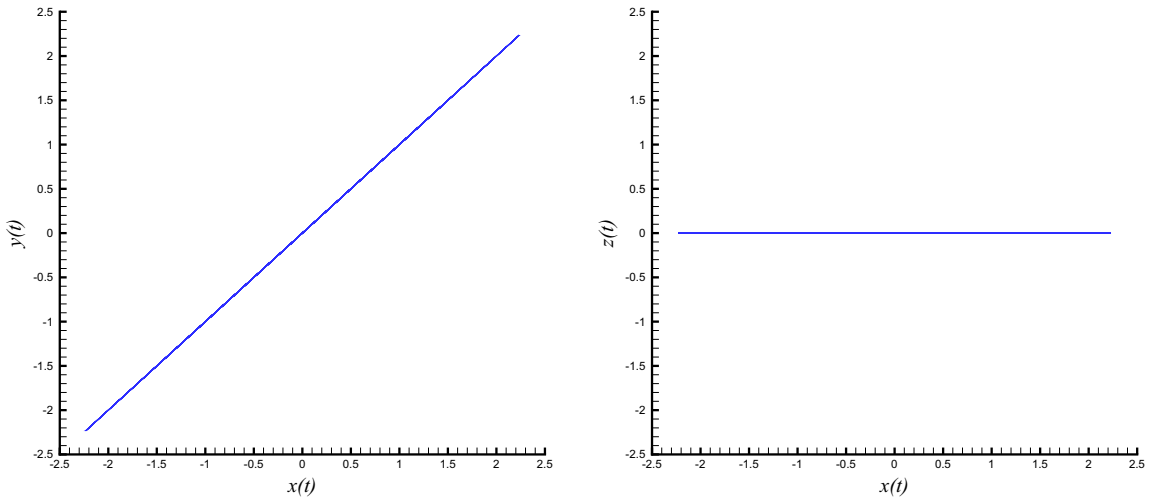

Fig. 3. $x-y$ and $x-z$ of Body 2 ($0 \leqslant t \leqslant 1000$) in the case of $\delta = 0$.

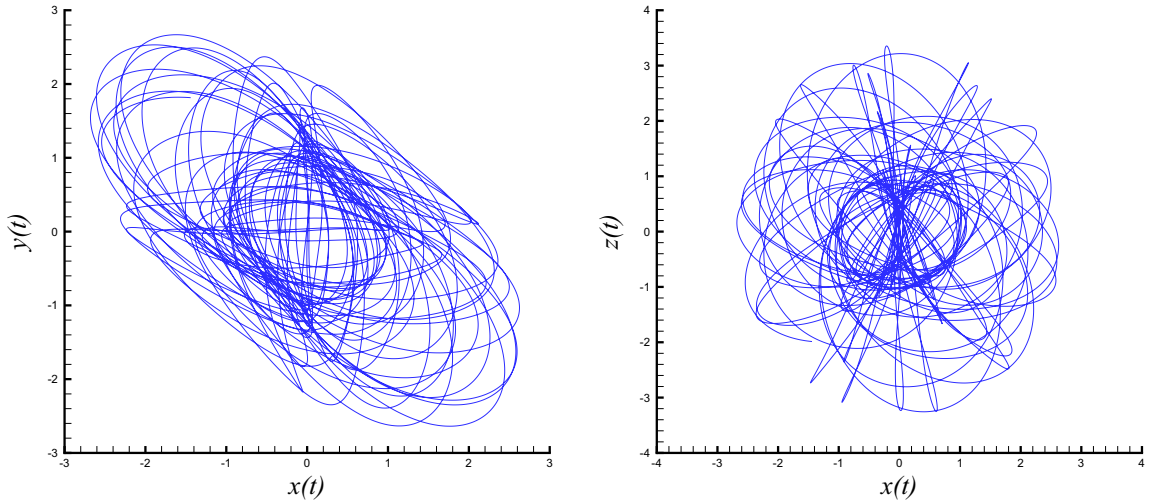

Fig. 4. $x-y$ and $x-z$ of Body 3 ($0 \leqslant t \leqslant 1000$) in the case of $\delta = 0$.

$$x_{1,1} = +10.571899177162684860539965118023338735518581619035776331820966524368101464, \quad (23)$$

$$x_{2,1} = -33.395686019658214703178151236176860240755968029199276186710038142873929480, \quad (24)$$

$$x_{3,1} = +20.284552739682192295213686921793884410440415394341528671055848805097262214, \quad (25)$$

which are in the precision of 72 significance digits. Note that the positions of the three bodies at $t = 1000$ given by $\Delta t = 10^{-2}$ and $M = 100$ agree well (in precision of 27 significance digits) with those by $\Delta t = 10^{-3}$ and $M = 50$. In addition, the momentum conservation (19) is satisfied in the level of 10^{-293}. Thus, our CNS simulations in the case of $\delta = 10^{-60}$ are *mathematically* reliable in the *whole* interval $0 \leqslant t \leqslant 1000$ as well.

The orbits of the three bodies in the case of $\delta = 10^{-60}$ are as shown in Figs. 6–8. It is found that, in the time interval $0 \leqslant t \leqslant 810$, the chaotic orbits of the three bodies are not obviously different from those in the case of $\delta = 0$, say, Body 2 oscillates along the same line on $z = 0$, Body 1 and Body 3 are chaotic with the same symmetry about the regular orbit of Body 2. However, the obvious difference of orbits appears when $t \geqslant 810$: Body 2 departs from the oscillations along the line on $z = 0$ and escapes (together with Body 3) along a complicated three-dimensional orbit. Besides, Body 1 and Body 3 escape in the opposite directions without any symmetry. As shown in Figs. 9–11, Body 2 and Body 3 go far and far away from Body 1 and thus become a binary-body system. It is very interesting that, when $t > 810$, the tiny physical uncertainty $d\mathbf{r}_1 = 10^{-60}(1,0,0)$ of the initial conditions disrupts not only the elegant symmetry of the orbits but also even the three-body system itself!

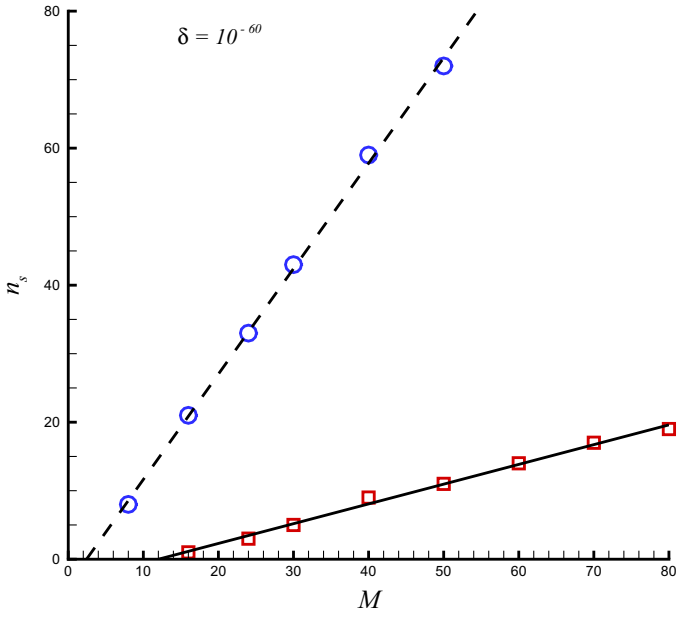

Fig. 5. Accuracy (expressed by n_s, number of significance digits) of the result at $t = 1000$ in the case of $\delta = 10^{-60}$ by means of $N = 300$, the different time-step Δt and the different order of Taylor expansion (M). Square: $\Delta t = 10^{-2}$; Circle: $\Delta t = 10^{-3}$. Solid line: $n_s \approx 0.2885M - 3.4684$; dashed line: $n_s \approx 1.5386M - 3.7472$.

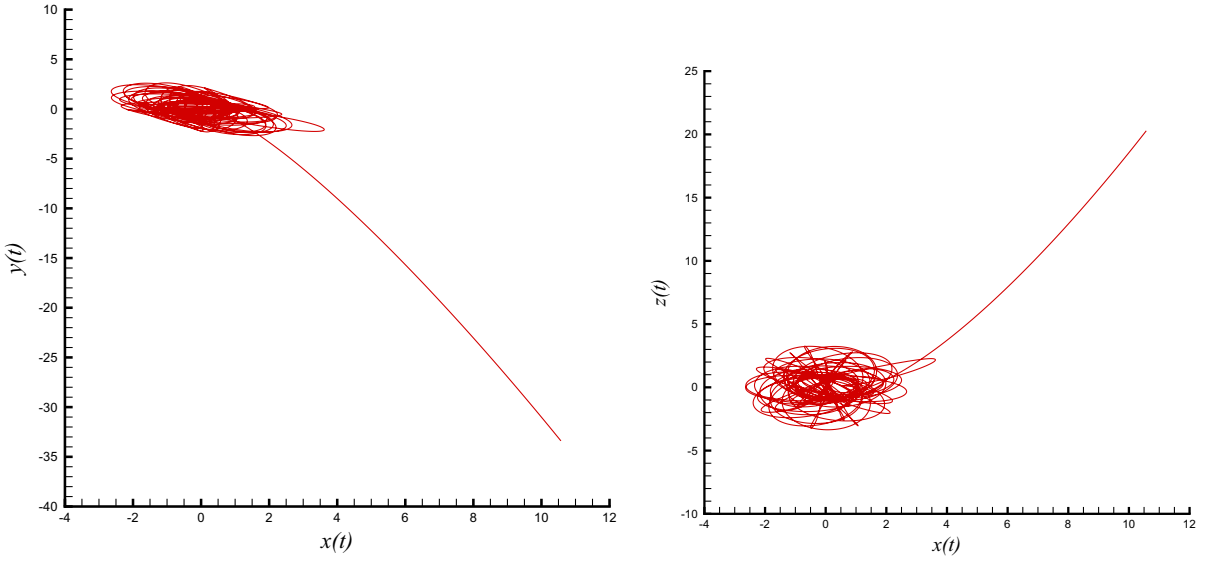

Fig. 6. $x - y$ and $x - z$ of Body 1 ($0 \leqslant t \leqslant 1000$) in the case of $\delta = 10^{-60}$.

Similarly, in the case of $\delta = -10^{-60}$, we gain the mathematically reliable orbits of the three bodies on the whole interval [0,1000] by means of the CNS with $\Delta t = 10^{-3}, N = 300$ and $M = 20$ (i.e. the 20th-order Taylor expansion). As shown in Figs. 12–14, the tiny physical uncertainty in the initial position disrupts not only the elegant symmetry of the orbits but also the three-body system itself as well: when $t > 810$, Body 2 departs from its oscillation along the line on $z = 0$ and escapes (but together with Body 1) in a complicated three-dimensional orbit, while Body 1 and Body 3 escape in the opposite direction without any symmetry. Note that, in the case of $\delta = -10^{-60}$, Body 1 and Body 2 go together far and far away from Body 3 to become a binary system. However, in the case of $\delta = +10^{-60}$, Body 2 and Body 3 escape together to become a binary system! This is very interesting. Thus, the chaotic orbits of the three-body system subject to the initial conditions with the micro-level physical uncertainty, corresponding to $\delta = 0, \delta = +10^{-60}$ and $\delta = -10^{-60}$, respectively, are completely *different*

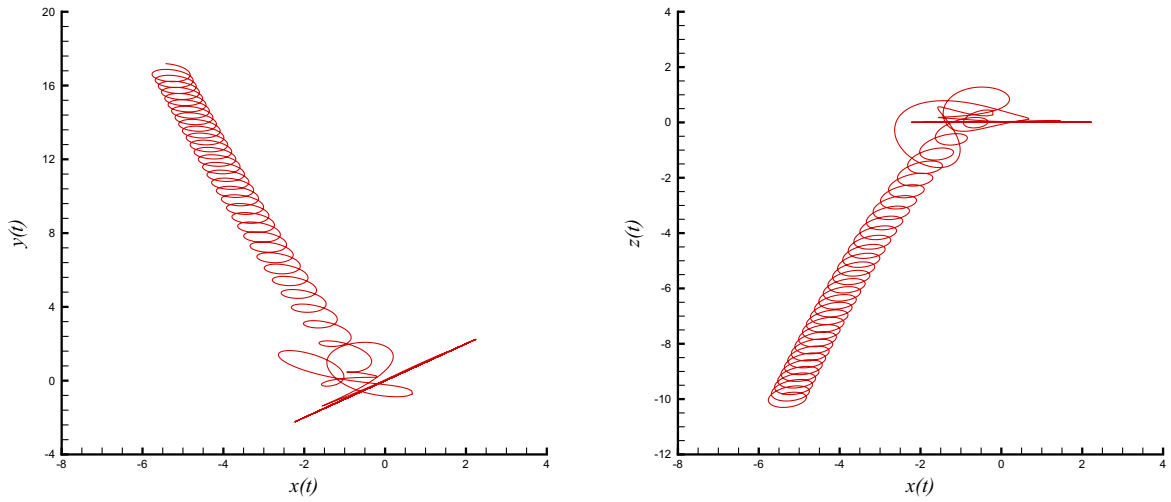

Fig. 7. $x-y$ and $x-z$ of Body 2 ($0 \leqslant t \leqslant 1000$) in the case of $\delta = 10^{-60}$.

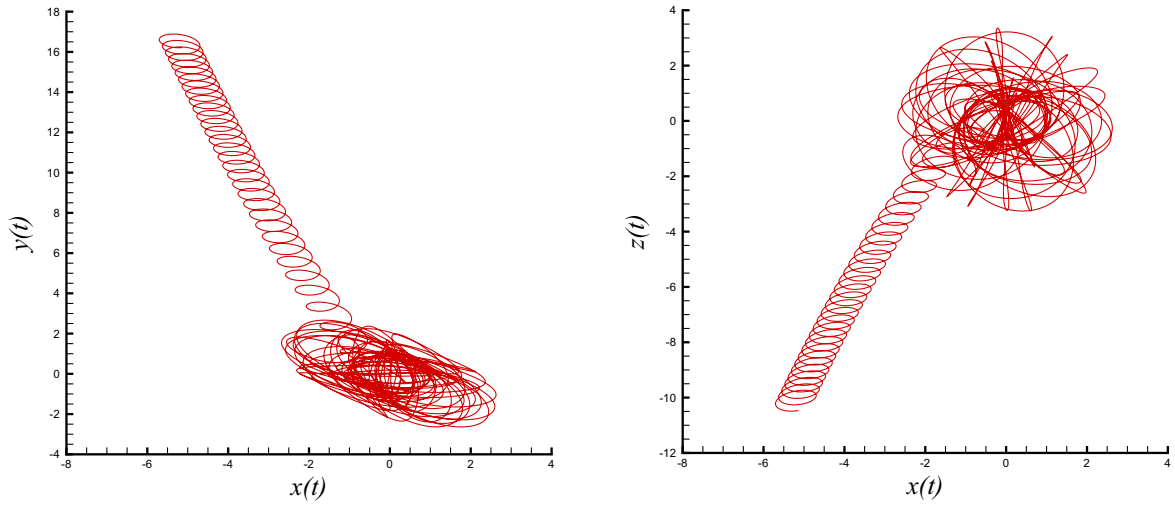

Fig. 8. $x-y$ and $x-z$ of Body 3 ($0 \leqslant t \leqslant 1000$) in the case of $\delta = 10^{-60}$.

when $t > 810$. It should be emphasized here that, these *mathematically* different initial conditions are *physically* the *same*, since any lengths shorter than the Planck length do not make physical senses [26].

Our mathematically reliable CNS simulations reveal that there exists such an interval $[0, T_p^{max}]$, where $T_{max}^p \approx 810$ for the special case of the three-body problem, that the chaotic trajectories of the three bodies have no obvious difference[3] in $[0, T_p^{max}]$, but become obviously different beyond it. It should be emphasized that, by means of the CNS with high enough order of Taylor series method and data in high enough precision, the numerical noises are negligible so that all chaotic trajectories are mathematically reliable in the whole interval $[0, 1000]$. On the other side, the micro-level physical uncertainty of positions of these initial conditions are shorter even than the Planck length so that they are the *same* in *physics*, according to the string theory [26]. In practice, each of these mathematically different (but physically the same) initial conditions can sample with equal probability, but we do not know which one will practically occur. Therefore, from physical viewpoint, the chaotic trajectories when $t > T_{max}^p$ are essentially uncertain, i.e. we can not predict the *physically* correct trajectories after $t > T_{max}^p$, even if they are mathematically reliable. So, micro-level physical uncertainty of initial condition put forwards a physical limit of prediction time, T_p^{max}, which is objective, i.e. independent of numerical methods and limited precision of measurement at all. The key point is that our ability of prediction is greatly restricted by physical limit of prediction time. It is a surprise that, for the chaotic motion

[3] In other words, they look like "deterministic".

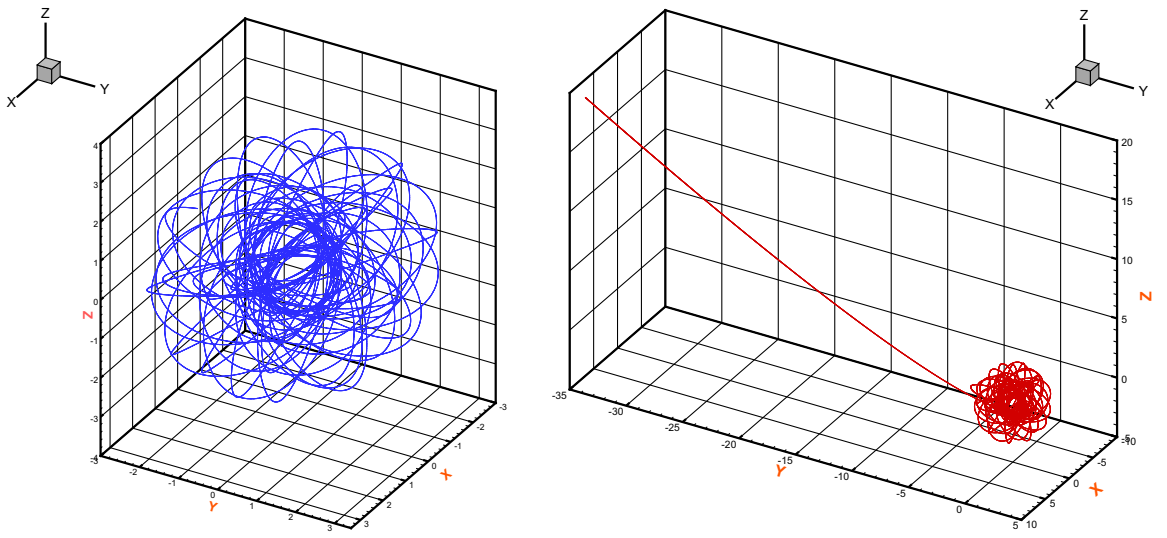

Fig. 9. Orbit of Body 1 ($0 \leqslant t \leqslant 1000$). Left: $\delta = 0$; Right: $\delta = 10^{-60}$.

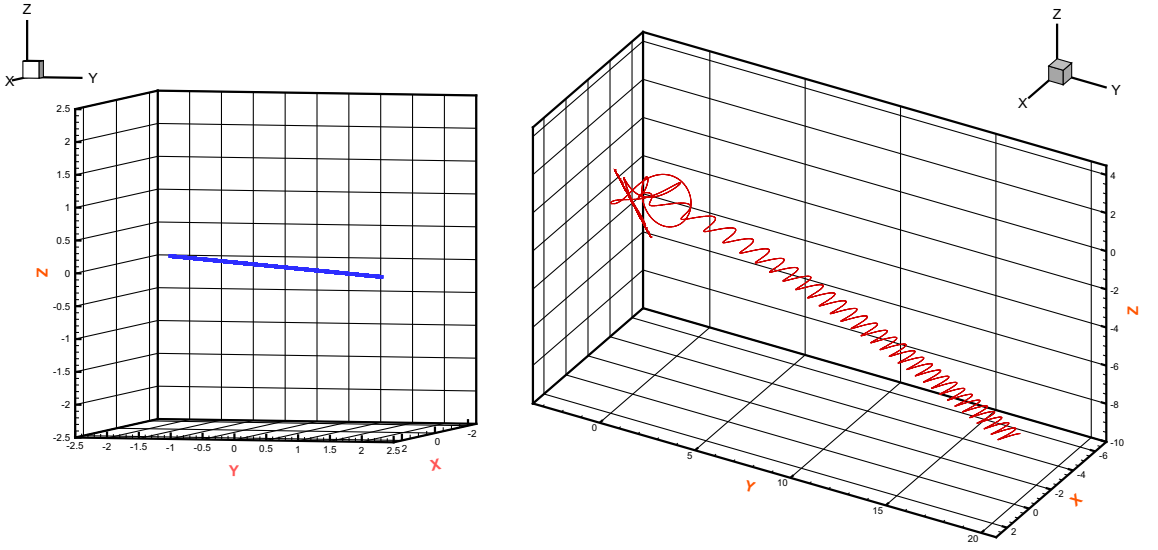

Fig. 10. Orbit of Body 2 ($0 \leqslant t \leqslant 1000$). Left: $\delta = 0$; Right: $\delta = 10^{-60}$.

of the special case of the three body problem considered in this article, the physical limit of prediction time is a little short, i.e. $T^p_{max} \approx 810$, although it is easy for us to apply the CNS to gain mathematically reliable chaotic trajectories in a much longer interval. The time of physical limit of prediction provides us a *time-scale* for at most how long a deterministic prediction of chaotic dynamic systems is *physically* correct.

In addition, the concept of the physical limit of prediction time suggests that the micro-level, objective, physical uncertainty of chaotic systems might transfer into macroscopic uncertainty for a long time, i.e. after $t > T^p_{max}$. At least, the considered special case of the three body problem provides us with a good example that the micro-level physical uncertainty might be an origin of some macroscopic uncertainty. This suggests that macroscopic uncertainty might have a close relationship with micro-level uncertainty and thus is essentially unavoidable. This conclusion is supported by the mathematical theory [1] and many physical experiments (such as those in [30,31]).

Finally, to confirm our above conclusions, we further consider such a special case with the micro-level uncertainty of the initial position $d\mathbf{r}_1 = 10^{-60}(1,1,1)$, i.e.

$$\mathbf{r}_1 = (0,0,-1) + 10^{-60}(1,1,1).$$

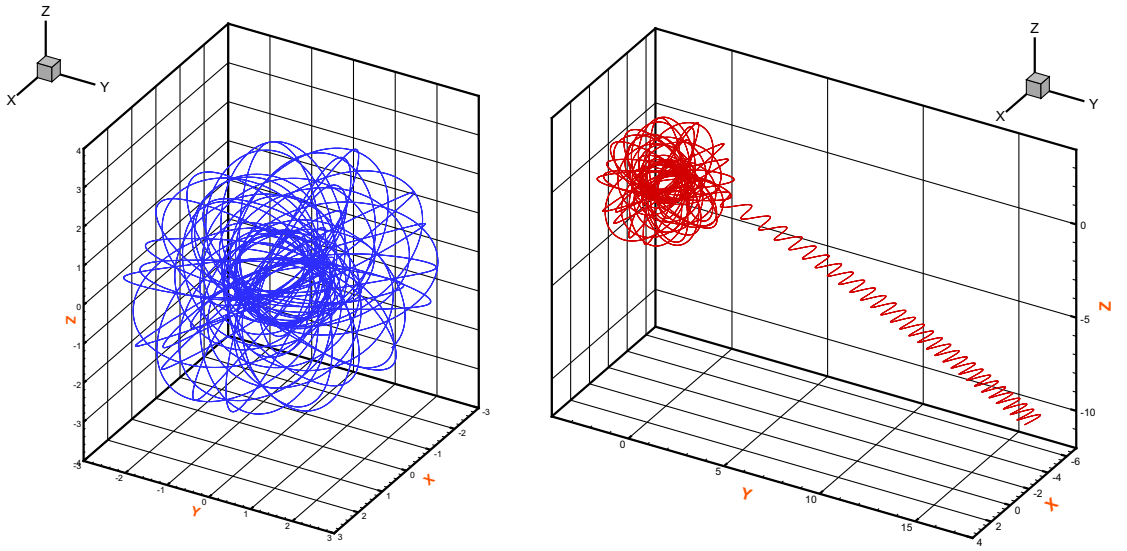

Fig. 11. Orbit of Body 3 ($0 \leqslant t \leqslant 1000$). Left: $\delta = 0$; Right: $\delta = 10^{-60}$.

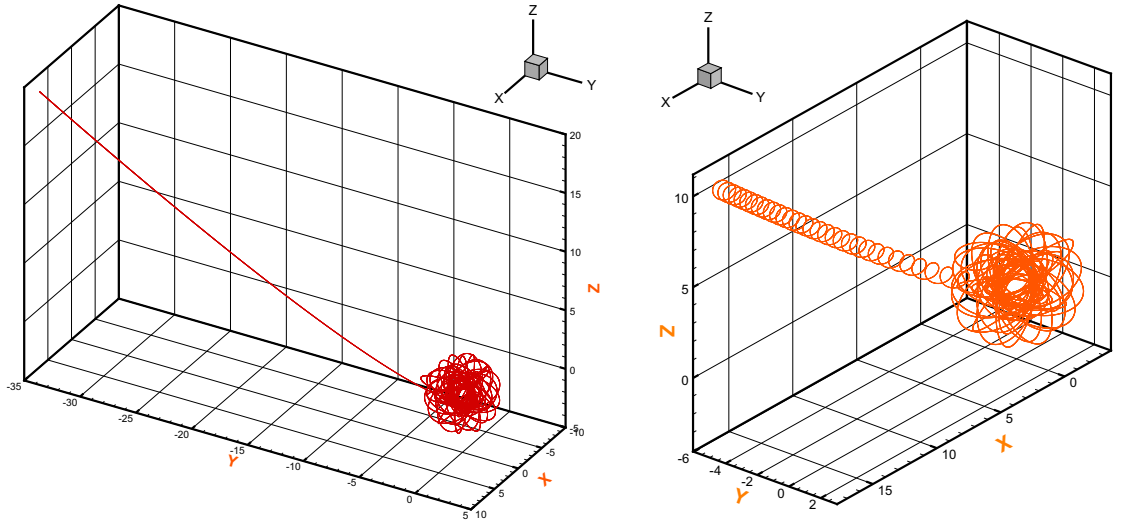

Fig. 12. Orbit of Body 1 ($0 \leqslant t \leqslant 1000$). Left: $\delta = +10^{-60}$; Right: $\delta = -10^{-60}$.

Since the physical uncertainty of position is even shorter than the Planck length, from the physical viewpoint mentioned above, the initial positions can be regarded as the *same* [26] as those of the above-mentioned three initial conditions. However, the corresponding mathematically reliable chaotic orbits of the three bodies (in the time interval $0 \leqslant t \leqslant 1200$) obtained by means of the CNS with $\Delta t = 10^{-3}, N = 300$ and the 30th-order Taylor expansion ($M = 30$) are almost the *same* in the interval $[0, T_p^{max}]$, where $T_p^{max} \approx 810$, but beyond it they become *quite* different from those of the above-mentioned three cases: Body 2 first oscillates along a line on $z = 0$ but departs from the regular orbit when $t > 810$ to move along a complicated three-dimensional orbits, while Body 1 and Body 3 first move with the symmetry but lose it when $t > 810$, as shown in Figs. 15–17. However, it is not clear whether any one of them might escape or not, i.e. the fate of the three-body system is uncertain. Since such kind of micro-level uncertainty of initial position is inherent and objective, the orbits of the three-body system beyond the time of the physical limit of prediction is essentially uncertain in physics. It should be emphasized that the uncertainty of chaotic trajectories beyond the time T_p^{max} of physical limit of prediction is *objective*, i.e. it has nothing to do with limited precision of measurement, numerical noises and Heisenberg uncertainty principle [27]. All of these confirm once again our conclusions mentioned above.

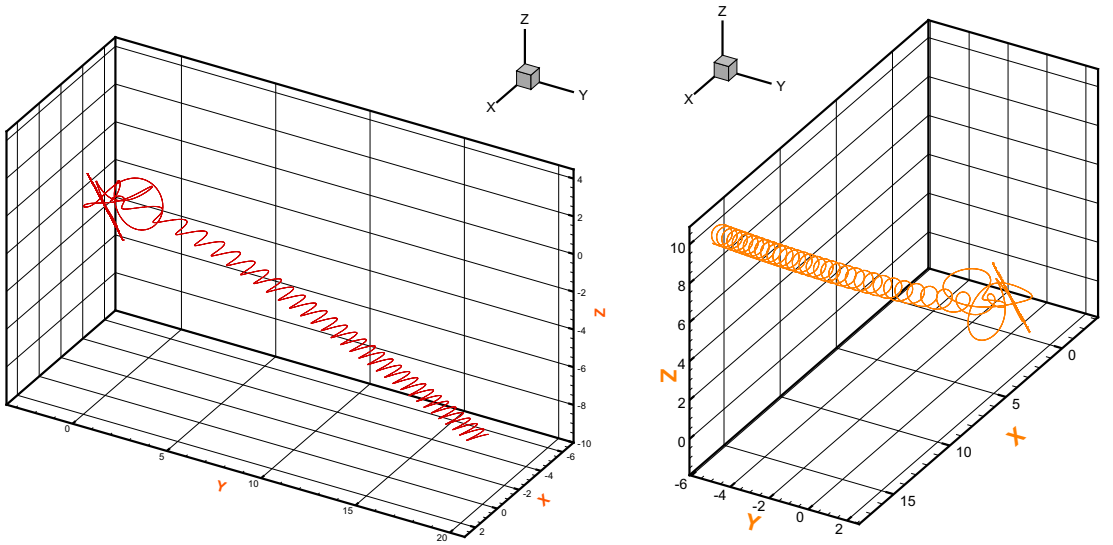

Fig. 13. Orbit of Body 2 ($0 \leqslant t \leqslant 1000$). Left: $\delta = +10^{-60}$; Right: $\delta = -10^{-60}$.

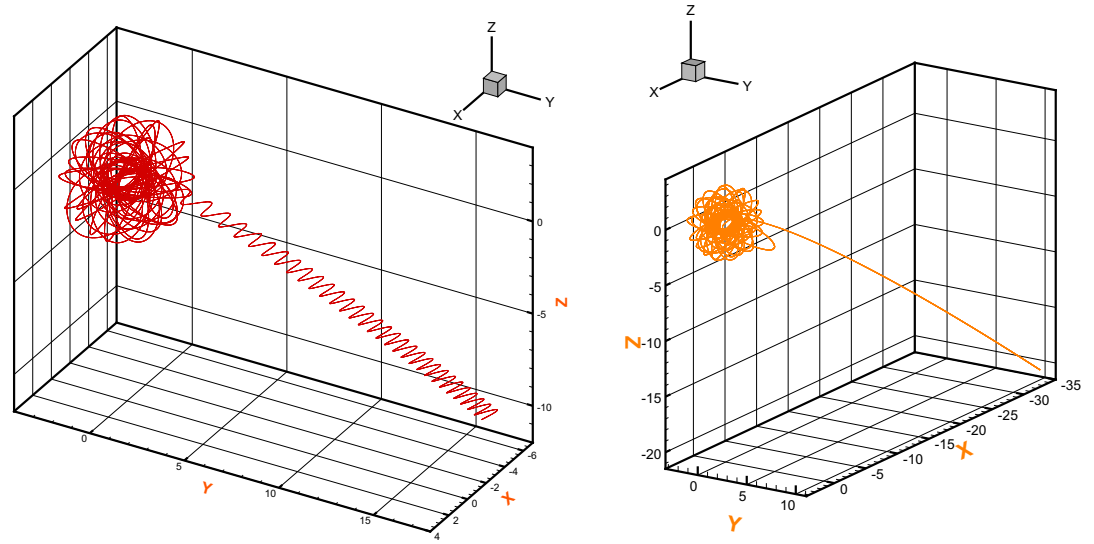

Fig. 14. Orbit of Body 3 ($0 \leqslant t \leqslant 1000$). Left: $\delta = +10^{-60}$; Right: $\delta = -10^{-60}$.

5. Concluding remarks and discussions

A half century ago, Lorenz [3] made a famous conclusion that "long-term prediction of chaos is impossible". However, what is the exact meaning of the "long-term" in the above famous claim? Is one day long enough? Or one million of years?.

In this article, using chaotic motion of the famous three body problem as an example, a new concept, namely the physical limit of prediction time, is put forwarded to provide us a *time-scale* for at most how long a *mathematically* reliable simulation of chaotic dynamic system is deterministic and *physically* correct. First of all, from mathematical viewpoint, given an exact initial condition, we can always gain reliable simulations of chaotic trajectories of the considered three body problem in a finite but long enough interval (such as $[0, 1000]$) by means of the so-called "Clean Numerical Simulation" (CNS) [9,17]. This is mainly because, based on the high enough order of Taylor series method and the data in high enough multiple precision, both of truncation and round-off errors of the CNS simulations are negligible in the finite but long enough interval. However, from physical viewpoint, the initial position of any a star/planet contains micro-level physical uncertainty, that is objective and unavoidable. Our mathematically reliable CNS simulations of chaotic trajectories of the considered three-body problem with the physical uncertainty at the micro-level 10^{-60} indicate that there exists such a time of physical limit of prediction, i.e.

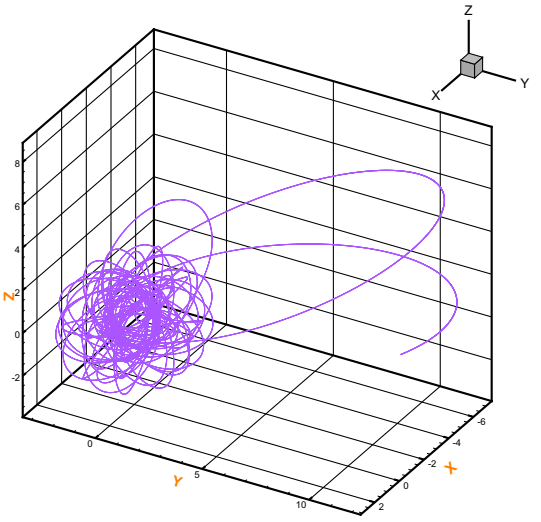

Fig. 15. Orbit of Body 1 ($0 \leqslant t \leqslant 1200$) when $d\mathbf{r}_1 = 10^{-60}\,(1,1,1)$.

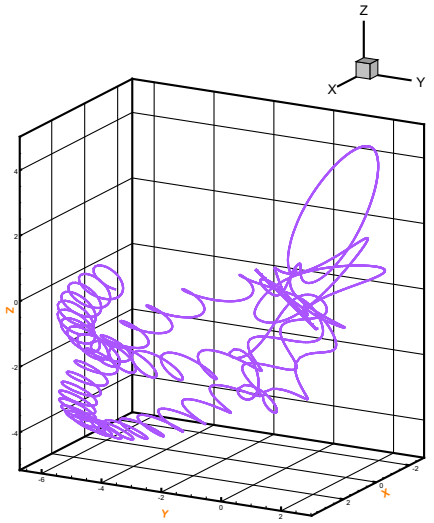

Fig. 16. Orbit of Body 2 ($0 \leqslant t \leqslant 1200$) when $d\mathbf{r}_1 = 10^{-60}\,(1,1,1)$.

T_p^{max}, that the chaotic trajectories of the three body problem are the same in the interval $[0, T_p^{max}]$, but beyond it, they become essentially uncertain in physics. Thus, the time T_p^{max} of the physical limit of prediction provides us a time-scale of *physically* correct prediction for chaos: it has no physical sense to talk about the "prediction" of chaotic trajectories beyond T_p^{max}, since they are essentially uncertain in physics. The concept of "the physical limit of prediction" also implies that the macroscopic model of the three body problem based on Newtonian gravitational law is essentially *not* deterministic, because the initial condition contains the micro-level physical uncertainty that is unavoidable and might become very important for the time beyond the physical limit of prediction, i.e. T_p^{max}.

On the other side, the concept of "physical limit of prediction" also suggests that the micro-level physical uncertainty of initial conditions might transfer into macroscopic uncertainty when $t > T_p^{max}$. In other word, our mathematically reliable CNS simulations of the chaotic motion of the three body problem under consideration suggest that the seemingly random distribution of stars/planets in the sky contains uncertainty, and that the micro-level physical uncertainty of position of a star/planet might be an origin of this kind of macroscopic uncertainty. So, it might provide us a theoretical explanation to many questions, such as why stars in the sky look random, why velocities of turbulent flows are so uncertain, and so on. All of these might enrich our knowledge and deepen our understandings about not only the three-body problem but also chaos. Besides, it also illustrates that the CNS might provide us a safe numerical method to understand the physical world better.

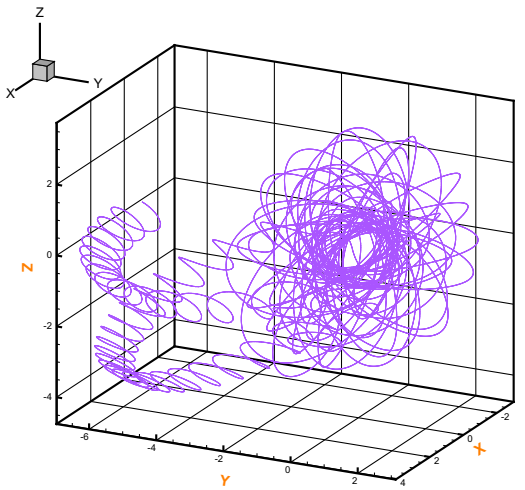

Fig. 17. Orbit of Body 3 ($0 \leqslant t \leqslant 1200$) when $d\mathbf{r}_1 = 10^{-60}\,(1,1,1)$.

A special three body problem, subject to the four mathematically different initial conditions with micro-level physical uncertainty, is considered in this paper. The four initial conditions contains a tiny difference due to the physical uncertainty of position in the micro-level of 10^{-60}: they are shorter even than the Planck length so that they are the *same* in *physics*, according to the string theory [26]. In practice, each of these *mathematically* different (but *physically* the same) initial conditions can sample with equal probability, but we do not know which one practically occurs. Using the CNS, we gain mathematically reliable chaotic trajectories in the interval [0,1000] for each of these four (mathematically different but physically same) initial conditions, and found that the corresponding chaotic trajectories agree well in the interval $[0, 810]$, but become quite different when $t > 810$. So, we have the time $T_p^{max} \approx 810$ of the physical limit of prediction, which provides us a time-scale for at most how long we can gain a *physically* correct, deterministic prediction of chaotic motion for the special three-body problem. Besides, the time T_p^{max} of the physical limit of prediction also provides us a better understanding about the "long-term" prediction in Lorenz's famous claim [3]. Finally, it should be emphasized that, given an *exact* initial condition, we can gain *mathematically* reliable simulations of chaotic dynamic system under consideration in a *larger* interval $[0, 1000]$ by means of the CNS. However, such kind of *exact* initial condition does *not* exist from the physical viewpoint. It is the interaction of this kind of uncertainty of the initial condition and the so-called butterfly effect of chaotic dynamic system that leads to the physical limit of prediction of chaos.

Note that the computation ability of human being plays an important role in the development of nonlinear dynamic systems. Without digit computer, it was impossible for Lorenz [3] to find out the famous butter-fly effect of chaos, although only 16-digits precision was able to use in his pioneering work [3]. Thanks to the great development of computer technology in the past half century, the CNS based on arbitrary order of Taylor series method and arbitrary precision of data provides us a *time-scale*, i.e. the time T_p^{max} of the physical limit of prediction, about "how long" we can give a *physically* correct, accurate, deterministic prediction of chaotic dynamic systems.

As reported by Sussman and Wisdom [32,33], the motion of Pluto and even the solar system is chaotic with a time scale in the range of 3 to 30 million years. Thus, our illustrative CNS simulations reported in this paper suggest that the solar system might be essentially uncertain beyond a time of physical limit of prediction which might be millions of years. Note that such kind of macroscopic uncertainty is closely related to the inherent physical micro-level uncertainty of position and thus is objective. In other words, it has nothing to do with limited precision of measurement: the history of human being is indeed too short, compared to the time scale of such kind of macroscopic uncertainty.

The proposed reliable numerical approach can be used to investigate the stability of periodic solutions of nonlinear dynamic systems. For example, it can be used to study the stability of the newly found three classes of Newtonian three-body planar periodic orbits [34].

Note that the time of physical limit of perdition might be quite different for different chaotic systems, different initial conditions, different physical parameters and so on. It would be interesting to investigate the relationship between the time of physical limit of prediction and some traditional characteristics of chaos, such as Lyapunov exponent, and so on.

Finally, it should be emphasized that the microscopic and macroscopic phenomena are described by two completely different systems of physical laws, i.e. the quantum mechanics [35] and classical (Newtonian) mechanics, respectively. It is widely believed that the microscopic phenomena are uncertain and described by the quantum mechanics, but most macroscopic phenomena are governed by deterministic physical laws of classical mechanics (such as the Newtonian second law and the Newtonian gravitational law for three body problem). Many scientists attempted to develop a unified theory valid for both of the microscopic and macroscopic phenomena. Unfortunately, such a unified theory dos not exist up to now. It is

not very clear whether or not and how the micro-level uncertainty has relationships with the macroscopic uncertainty. However, it is a fact that macroscopic phenomena with uncertainty exist widely, such as the distribution of stars in the universe, turbulent flows and so on. The origin of macroscopic uncertainty of these phenomena and especially their relationships to the micro-level uncertainty are not very clear. Our current work reported in this article strongly suggests that the micro-level physical uncertainty might be an origin of uncertainty of some macroscopic phenomena, although it might be not the only one. In other words, our work suggests that the macroscopic uncertainty might have a close relationship with the micro-level uncertainty, although we have not a unified theory now. Note that the nonlinearity is an inherent property of the nature and the uncertainty is a fundamental property of microscopic phenomena. Thus, our work suggests that the macroscopic uncertainty seems to be inescapable: it should be an inherent property of the nature, too. If this is indeed true, it could well explain the origin of uncertainty of many macroscopic phenomena, such as turbulent flows, the random distribution of stars in the universe and so on.

Acknowledgement

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