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HIGHER-ORDER STREAMFUNCTION–VORTICITY FORMULATION OF 2D STEADY-STATE NAVIER–STOKES EQUATIONS

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SUMMARY

Based on regular boundary element method and a kind of linearity invariance under homotopy, a kind of numerical scheme of 2D steady-state Navier–Stokes equation in streamfunction–vorticity formulation is described. The flow inside a square cavity is used to illustrate this numerical scheme.

KEY WORDS 2D Navier–Stokes equation Streamfunction–Vorticity formulation Regular boundary element method Linearity invariance under homotopy

1. INTRODUCTION

Boundary element techniques in solving Navier–Stokes equations[1–4] have developed very quickly in recent years. The fundamental solutions have been found for steady and unsteady incompressible viscous flow via integral formulations with primitive variables, as mentioned in References 8 and 9. The boundary element methods of solving both steady and unsteady Navier–Stokes equations in streamfunction–vorticity formulation were presented in References 6 and 7. The latter is based on a set of fundamental solutions providing a complete coupling between the streamfunction and vorticity equations, so that iteration is not needed in the case $Re=0$. The non-linear terms are considered as inhomogeneities and treated by simple direct iteration, but this numerical scheme seems unstable for Reynolds number greater than 300, as mentioned in Reference 7.

Perhaps, the divergence in the case of $Re>300$ comes from the simple direct iteration. We know that nearly all iterative models are more or less sensitive to the initial solutions. In the case of strong non-linearity, the initial solutions obtained generally from the corresponding simplified linear problem are often far from the exact solution, so that more attempts must be done in order to make the iteration convergent.

The author has tried to avoid iteration in solving non-linear problems in order to make numerical schemes insensitive to initial solutions. In Reference 10, the process of the continuous change of homotopy is investigated and a kind of linear property of homotopy is described. A numerical method, namely the finite process method, is obtained by discretizing this continuous-change process of homotopy in the imbedding region. By means of the finite process method, we can indeed avoid the use of iterative techniques in solving non-linear problems, although

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much CPU time is needed. A more thorough study reported in Reference 11 shows that the finite process method can give a family of iterative formulas, which are more insensitive to the initial solutions than other simple iterative models and, therefore, can be seen as iterative models at higher orders. In fact, the more complex the formula in this family, the more insensitive it is to the initial solutions. In the limiting state, no iteration is necessary. So, iteration seems to be a special case of the finite process method.

In this paper, the linear property of homotopy[10–12] is applied to give a numerical model of 2D steady-state Navier–Stokes equation in streamfunction–vorticity formulation. Instead of discretizing a homotopy in embedding domain, $p\in[0, 1]$, as mentioned in Reference 10, we analyse it by Taylor's series. A higher-order streamfunction–vorticity formula of the boundary element method of 2D steady-state Navier–Stokes equations is given. As an example, the flow inside a square cavity is used to examine the numerical scheme. The results are obtained for the case of $Re \gg 300$.

2. MATHEMATICAL FORMULATION

Two-dimensional steady flow of an incompressible Newtonian fluid is formulated in terms of the streamfunction (ψ) and the vorticity (ω) as follows:

$$\nabla^2\omega = Re\left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right), \tag{1}$$

$$\nabla^2\psi + \omega = 0, \tag{2}$$

with boundary conditions

$$\psi = \psi_b \qquad \text{on } \Gamma, \tag{3}$$

$$\frac{\partial\psi}{\partial n} = \left(\frac{\partial\psi}{\partial n}\right)_b \qquad \text{on } \Gamma. \tag{4}$$

The above equations are expressed in dimensionless form. $Re = \rho v_0 L/\mu$ is the Reynolds number expressed in terms of a characteristic length L and a characteristic velocity v_0. The two independent co-ordinates x and y have also been made dimensionless.

For simplicity, let

$$h(x, y) = Re\left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right). \tag{5}$$

From equations (1)–(4), we can construct a new system of equations as follows:

$$\nabla^2\omega(x, y; p) = ph(x, y; p) + (1-p)\nabla^2\omega_0(x, y), \tag{6}$$

$$\nabla^2\psi(x, y; p) + \omega(x, y; p) = (1-p)[\nabla^2\psi_0(x, y) + \omega_0(x, y)], \tag{7}$$

with boundary conditions

$$\psi(x, y; p) = p\psi_b + (1-p)\psi_0 \qquad \text{on } \Gamma, \tag{8}$$

$$\frac{\partial\psi(x, y; p)}{\partial n} = p\left(\frac{\partial\psi}{\partial n}\right)_b + (1-p)\frac{\partial\psi_0}{\partial n} \qquad \text{on } \Gamma, \tag{9}$$

where

$$h(x, y; p) = Re\left[\frac{\partial\psi(x, y; p)}{\partial y}\frac{\partial\omega(x, y; p)}{\partial x} - \frac{\partial\psi(x, y; p)}{\partial x}\frac{\partial\omega(x, y; p)}{\partial y}\right], \tag{10}$$

and $\omega_0(x, y)$ and $\psi_0(x, y)$ are either the known approximate solutions or the solutions in the case $Re=0$. Note that the streamfunction ψ and vorticity ω in equations (6)–(9) are dependent not only on the original variables x, y but also on the embedding variable $p\in[0, 1]$.

When $p=0$, we obtain from equations (6)–(9) the *initial equations* as follows:

$$\nabla^2\omega(x, y; 0)=\nabla^2\omega_0(x, y), \tag{11}$$

$$\nabla^2\psi(x, y; 0)+\omega(x, y; 0)=\nabla^2\psi_0(x, y)+\omega_0(x, y), \tag{12}$$

with the corresponding boundary conditions

$$\psi(x, y; 0)=\psi_0 \quad \text{on } \Gamma, \tag{13}$$

$$\frac{\partial\psi(x, y; 0)}{\partial n}=\frac{\partial\psi_0}{\partial n} \quad \text{on } \Gamma. \tag{14}$$

So, $\psi_0(x, y)$ and $\omega_0(x, y)$ are just the solutions of the initial equations (11)–14), called the *initial solutions.*

When $p=1$, we have the *final equations* as follows:

$$\nabla^2\omega(x, y; 1)=h(x, y; 1), \tag{15}$$

$$\nabla^2\psi(x, y; 1)+\omega(x, y; 1)=0 \tag{16}$$

with the corresponding boundary conditions

$$\psi(x, y; 1)=\psi_b \quad \text{on } \Gamma, \tag{17}$$

$$\frac{\partial\psi(x, y; 1)}{\partial n}=\left(\frac{\partial\psi}{\partial n}\right)_b \text{ on } \Gamma. \tag{18}$$

For convenience, let $\psi_f(x, y)$ and $\omega_f(x, y)$ denote the solutions of the *non-linear* final equations (15)–(18), called the *final solution.* Clearly, the final equations (15)–(18) are the same as the original equations (1)–(4). So, the final solutions $\psi_f(x, y)$ and $\omega_f(x, y)$ are just what we want to know.

Let $\mathscr{E}_0$ denote the initial equations (11)–(14), $\mathscr{E}_f$ denote the final equations (15)–(18). Clearly, equations (6)–(9) connect the initial equations $\mathscr{E}_0$ with the final equation $\mathscr{E}_f$. The process of the continuous change of p from zero to unity is just that of the continuous *deformation* from initial equations $\mathscr{E}_0$ to final equations $\mathscr{E}_f$. So, equations (6)–(9) construct, in fact, a homotopy $\mathscr{E}(p)$:$\mathscr{E}_0\simeq\mathscr{E}_f$; $\mathscr{E}_0$ and $\mathscr{E}_f$ are called homotopic in topology.

Suppose that equations (6)–(9) have solutions at any $p\in[0. 1]$. Then, equations (6)–(9) connect also the initial solutions $\psi_0(x, y)$ and $\omega_0(x, y)$ with the final solutions $\psi_f(x, y)$ and $\omega_f(x, y)$ by the embedding variable p. The process of the continuous change of p from zero to unity is also just the process of $\omega(x, y; p)$ and $\psi(x, y; p)$ from the initial solutions $\omega_0(x, y)$, $\psi_0(x, y)$ to the final solutions $\omega_f(x, y)$, $\psi_f(x, y)$. Thus, $\psi_0(x, y)$ and $\psi_f(x, y)$ are also homotopic, and so are $\omega_0(x, y)$ and $\omega_f(x, y)$, denoted, respectively, as $\psi_0(x, y)\simeq\psi_f(x, y)$ and $\omega_0(x, y)\simeq\omega_f(x, y)$.

So, equations (6)–(9) construct in fact, two kinds of homotopy. One is for equations, $\mathscr{E}(p)$:$\mathscr{E}_0\simeq\mathscr{E}_f$, and the other is for the functions (solutions), $\psi(x, y; p):\psi_0(x, y)\simeq\psi_f(x, y)$ and $\omega(x, y; p):\omega_0(x, y)\simeq\omega_f(x, y)$. For convenience, we call equations (6)–(9) the *zero-order process equations.*

Suppose $\psi(x, y; p)$ and $\omega(x, y; p)$ have Mth-order partial derivatives with respect to p at $p=0$, denoted as follows:

$$\omega_0^{[k]}(x, y)=\left.\frac{\partial^k\omega(x, y; p)}{\partial p^k}\right|_{p=0}, \tag{19}$$

$$\psi_0^{[k]}(x, y) = \left.\frac{\partial^k \psi(x, y; p)}{\partial p^k}\right|_{p=0}. \tag{20}$$

For simplicity, call $\omega_0^{[k]}(x, y)$ and $\psi_0^{[k]}(x, y)$ the *kth-order process derivatives* of vorticity and streamfunction at $p=0$, respectively.

Suppose $\omega(x, y; p)$, $\psi(x, y; p)$, $\omega^{[k]}(x, y; p)$ and $\psi^{[k]}(x, y; p)$ $(k>0)$ exist and are continuous. Then, according to Taylor's theory, we have

$$\begin{aligned}\omega(x, y; p) &= \omega(x, y, 0) + \sum_{k=1}^{\infty} \frac{\omega_0^{[k]}(x, y)}{k!} p^k \\ &= \omega_0(x, y) + \sum_{k=1}^{\infty} \frac{\omega_0^{[k]}(x, y)}{k!} p^k,\end{aligned} \tag{21}$$

$$\begin{aligned}\psi(x, y; p) &= \psi(x, y, 0) + \sum_{k=1}^{\infty} \frac{\psi_0^{[k]}(x, y)}{k!} p^k \\ &= \psi_0(x, y) + \sum_{k=1}^{\infty} \frac{\psi_0^{[k]}(x, y)}{k!} p^k.\end{aligned} \tag{22}$$

For convenience, call the above expressions *Taylor's homotopy series*. Clearly, Taylor's series have generally a finite radius ρ of convergence. If the radius ρ of convergence is equal to or greater than unity, then a kind of relation between $\omega_0(x, y)$, $\psi_0(x, y)$ and $\omega_f(x, y)$, $\psi_f(x, y)$ can be given in the form of process derivatives as follows:

$$\omega_f(x, y) = \omega_0(x, y) + \sum_{k=1}^{\infty} \frac{\omega_0^{[k]}(x, y)}{k!}, \tag{23}$$

$$\psi_f(x, y) = \psi_0(x, y) + \sum_{k=1}^{\infty} \frac{\psi_0^{[k]}(x, y)}{k!}. \tag{24}$$

The above two expressions give a kind of relation between the initial solutions $\psi_0(x, y)$, $\omega_0(x, y)$ and the final solutions $\psi_f(x, y)$, $\omega_f(x, y)$ by means of the kth-order process derivatives $\psi_0^{[k]}(x, y)$, $\omega_0^{[k]}(x, y)$ $(k=1, 2, \ldots, \infty)$. We call this kind of relation *Taylor's series relation under homotopy.*

The equations of the kth-order process derivatives $\omega_0^{[k]}(x, y)$ and $\psi_0^{[k]}(x, y)$ can be obtained in the following way.

Differentiating equation (6) k times with respect to p, we have

$$\begin{aligned}\nabla^2 \frac{\partial^k \omega}{\partial p^k} &= \frac{\partial^k}{\partial p^k}[p h(x, y; p) + (1-p)\nabla^2 \omega_0(x, y)] \\ &= \sum_{i=0}^{k} \binom{i}{k} \frac{\partial^i}{\partial p^i}(p) \frac{\partial^{k-i} h(x, y; p)}{\partial p^{k-i}} - D_k \nabla^2 \omega_0(x, y) \\ &= p \frac{\partial^k}{\partial p^k}[h(x, y; p)] + k \frac{\partial^{k-1} h(x, y; p)}{\partial p^{k-1}} - D_k \nabla^2 \omega_0(x, y),\end{aligned} \tag{25}$$

where

$$D_k = \begin{cases} 1{\cdot}0 & \text{when } k=1, \\ 0{\cdot}0 & \text{when } k>1. \end{cases} \tag{26}$$

At $p=0$, we obtain from equation (25) that

$$\nabla^2\omega_0^{[k]}=k\frac{\partial^{k-1}}{\partial p^{k-1}}[h(x,y;p)]|_{p=0}-D_k\nabla^2\omega_0(x,y)$$
$$=h_k(x,y)-D_k\nabla^2\omega_0(x,y)\quad (k\geqslant 1), \tag{27}$$

where

$$h_1(x,y)=Re\left(\frac{\partial\psi_0}{\partial y}\frac{\partial\omega_0}{\partial x}-\frac{\partial\psi_0}{\partial x}\frac{\partial\omega_0}{\partial y}\right) \tag{28}$$

$$h_k(x,y)=kRe\sum_{i=0}^{k-1}\binom{i}{k-1}\left(\frac{\partial\psi_0^{[i]}}{\partial y}\frac{\partial\omega_0^{[k-1-i]}}{\partial x}-\frac{\partial\psi_0^{[i]}}{\partial x}\frac{\partial\omega_0^{[k-1-i]}}{\partial y}\right)\quad (k\geqslant 2). \tag{29}$$

Differentiating equation (7) k times with respect to p and then setting $p=0$, we have

$$\nabla^2\psi_0^{[k]}(x,y)+\omega_0^{[k]}(x,y)=-D_k(\nabla^2\psi_0+\omega_0). \tag{30}$$

Similarly, from boundary conditions (8) and (9), we can easily obtain

$$\psi_0^{[k]}(x,y)=D_k(\psi_b-\psi_0)\qquad \text{on }\Gamma,\quad k\geqslant 1, \tag{31}$$

$$\frac{\partial\psi_0^{[k]}(x,y)}{\partial n}=D_k\left[\left(\frac{\partial\psi}{\partial n}\right)_b-\frac{\partial\psi_0}{\partial n}\right]\qquad \text{on }\Gamma,\quad k\geqslant 1. \tag{32}$$

The system of equations (27) and (30)–(32) is called the *kth-order process equation* ($k\geqslant 1$).
Define

$$\bar{\omega}_0^{[k]}(x,y)=\omega_0^{[k]}(x,y)+D_k\omega_0(x,y)\qquad (k\geqslant 1), \tag{33}$$

$$\bar{\psi}_0^{[k]}(x,y)=\psi_0^{[k]}(x,y)+D_k\psi_0(x,y)\qquad (k\geqslant 1). \tag{34}$$

Then, according to equations (27) and (30)–(32), $\bar{\omega}_0^{[k]}(x,y)$ and $\bar{\psi}_0^{[k]}(x,y)$ can be described in the same form as follows:

$$\nabla^2\bar{\omega}_0^{[k]}(x,y)=h_k(x,y),\qquad k\geqslant 1, \tag{35}$$

$$\nabla^2\bar{\psi}_0^{[k]}(x,y)+\bar{\omega}_0^{[k]}(x,y)=0,\quad k\geqslant 1, \tag{36}$$

with boundary conditions

$$\bar{\psi}_0^{[k]}=D_k\psi_b\qquad \text{on }\Gamma,\ k\geqslant 1, \tag{37}$$

$$\frac{\partial\bar{\psi}_0^{[k]}}{\partial n}=D_k\left(\frac{\partial\psi}{\partial n}\right)_b\qquad \text{on }\Gamma,\ k\geqslant 1. \tag{38}$$

Perhaps, it should be emphasized that $h_k(x,y)$ ($k=1,2,\ldots,\infty$) are just functions of $\psi_0^{[i]}(x,y)$ and $\omega_0^{[i]}(x,y)$ for $i=0,1,2,\ldots,k-1$. Therefore $h_k(x,y)$ is known in solving $\bar{\omega}_0^{[k]}(x,y)$ and $\bar{\psi}_0^{[k]}(x,y)$. Note that the above equations are *linear* in $\bar{\omega}_0^{[k]}(x,y)$ and $\bar{\psi}_0^{[k]}(x,y)$. According to Reference 7, equations (35)–(38) can be written in the form of a boundary integral as follows:

$$\oint_\Gamma\left(G_\omega\frac{\partial\bar{\omega}_0^{[k]}}{\partial n}-\bar{\omega}_0^{[k]}\frac{\partial G_\omega}{\partial n}\right)\mathrm{d}\Gamma=\iint_\Omega h_kG_\omega\,\mathrm{d}\Omega+a(\xi)\bar{\omega}_0^{[k]}(\xi), \tag{39}$$

and

$$\oint_{\Gamma}\left(G_{\omega}\frac{\partial\bar{\psi}_0^{[k]}}{\partial n}-\bar{\psi}_0^{[k]}\frac{\partial G_{\omega}}{\partial n}\right)\mathrm{d}\Gamma+\oint_{\Gamma}\left(G_{\psi}\frac{\partial\bar{\omega}_0^{[k]}}{\partial n}-\bar{\omega}_0^{[k]}\frac{\partial G_{\psi}}{\partial n}\right)\mathrm{d}\Gamma$$

$$=\iint_{\Omega}h_kG_{\psi}\,\mathrm{d}\Omega+a(\xi)[\bar{\omega}_0^{[k]}(\xi)+\bar{\psi}_0^{[k]}(\xi)],\tag{40}$$

where Ω is the domain of fluid flow, Γ is the boundary of Ω and the integrations are performed with respect to $\mathbf{r}$. The parameter $a(\xi)$ is a geometric factor with the following values, depending on the location of ξ:

$$a(\xi)=\begin{cases}1 & \text{if } \xi\in\Omega,\\ 0 & \text{if } \xi\in\Omega^c,\\ \theta/2\pi & \text{if } \xi\in\Gamma,\end{cases}\tag{41}$$

where Ω^c denotes the exterior of the domain Ω, excluding its boundary Γ. θ is the angle formed between the tangents to the boundary at point ξ, approaching it from each side. For points at which the boundary is differentiable, $\theta=\pi$.

G_{ω} and G_{ψ} are fundamental solutions which satisfy, respectively, the following equations:

$$\nabla^2G_{\omega}=-\delta(\mathbf{r}-\xi),\tag{42}$$

$$\nabla^2G_{\psi}+G_{\omega}=-\delta(\mathbf{r}-\xi).\tag{43}$$

According to Reference 7,

$$G_{\omega}=-\frac{1}{2\pi}\ln r\tag{44}$$

$$G_{\psi}=-\frac{1}{2\pi}\ln r+\frac{r^2}{8\pi}(\ln r-1).\tag{45}$$

Subtracting equation (39) from equation (40), we have

$$\oint_{\Gamma}\left(F_{\psi}\frac{\partial\bar{\omega}_0^{[k]}}{\partial n}-\bar{\omega}_0^{[k]}\frac{\partial F_{\psi}}{\partial n}\right)\mathrm{d}\Gamma+\oint_{\Gamma}\left(G_{\omega}\frac{\partial\bar{\psi}_0^{[k]}}{\partial n}-\bar{\psi}_0^{[k]}\frac{\partial G_{\omega}}{\partial n}\right)\mathrm{d}\Gamma=\iint_{\Omega}h_kF_{\psi}\,\mathrm{d}\Omega+a(\xi)\bar{\psi}_0^{[k]}(\xi),\tag{46}$$

where

$$F_{\psi}=G_{\psi}-G_{\omega}$$

$$=\frac{r^2}{8\pi}(\ln r-1).\tag{47}$$

According to equations (39) and (46), we obtain

$$-a(\xi)\bar{\omega}_0^{[k]}(\xi)+\oint_{\Gamma}\left(G_{\omega}\frac{\partial\bar{\omega}_0^{[k]}}{\partial n}-\bar{\omega}_0^{[k]}\frac{\partial G_{\omega}}{\partial n}\right)\mathrm{d}\Gamma=\iint_{\Omega}h_kG_{\omega}\,\mathrm{d}\Omega,\tag{48}$$

$$-a(\xi)\bar{\psi}_0^{[k]}(\xi)+\oint_{\Gamma}\left(F_{\psi}\frac{\partial\bar{\omega}_0^{[k]}}{\partial n}-\bar{\omega}_0^{[k]}\frac{\partial F_{\psi}}{\partial n}\right)\mathrm{d}\Gamma=\iint_{\Omega}h_kF_{\psi}\,\mathrm{d}\Omega-\oint_{\Gamma}\left(G_{\omega}\frac{\partial\bar{\psi}_0^{[k]}}{\partial n}-\bar{\psi}_0^{[k]}\frac{\partial G_{\omega}}{\partial n}\right)\mathrm{d}\Gamma.\tag{49}$$

Substituting boundary conditions (37) and (38) into the above equations, we obtain

$$-a(\xi)\bar{\omega}_0^{[k]}(\xi)+\oint_{\Gamma}\left(G_{\omega}\frac{\partial\bar{\omega}_0^{[k]}}{\partial n}-\bar{\omega}_0^{[k]}\frac{\partial G_{\omega}}{\partial n}\right)\mathrm{d}\Gamma=\iint_{\Omega}h_kG_{\omega}\,\mathrm{d}\Omega,\tag{50}$$

$$-a(\xi)\bar{\psi}_0^{[k]}(\xi)+\oint_\Gamma\left(F_\psi\frac{\partial\bar{\omega}_0^{[k]}}{\partial n}-\bar{\omega}_0^{[k]}\frac{\partial F_\psi}{\partial n}\right)\mathrm{d}\Gamma=\iint_\Omega h_kF_\psi\,\mathrm{d}\Omega-D_k\oint_\Gamma\left[G_\omega\left(\frac{\partial\psi}{\partial n}\right)_\mathrm{b}-\psi_\mathrm{b}\frac{\partial G_\omega}{\partial n}\right]\mathrm{d}\Gamma. \tag{51}$$

According to equation (41), if $\xi\in\Omega^c$ then $a(\xi)=0$. Thus, for points in the exterior of the domain Ω, excluding the boundary Γ, we have

$$\oint_\Gamma\left(G_\omega\frac{\partial\bar{\omega}_0^{[k]}}{\partial n}-\bar{\omega}_0^{[k]}\frac{\partial G_\omega}{\partial n}\right)\mathrm{d}\Gamma=\iint_\Omega h_kG_\omega\,\mathrm{d}\Omega, \tag{52}$$

$$\oint_\Gamma\left(F_\psi\frac{\partial\bar{\omega}_0^{[k]}}{\partial n}-\bar{\omega}_0^{[k]}\frac{\partial F_\psi}{\partial n}\right)\mathrm{d}\Gamma=\iint_\Omega h_kF_\psi\,\mathrm{d}\Omega-D_k\oint_\Gamma\left[G_\omega\left(\frac{\partial\psi}{\partial n}\right)_\mathrm{b}-\psi_\mathrm{b}\frac{\partial G_\omega}{\partial n}\right]\mathrm{d}\Gamma, \tag{53}$$

These are the equations for the determination of both $\bar{\omega}^{[k]}$ and $\partial\bar{\omega}^{[k]}/\partial n$ ($k=1,2,3,\ldots,\infty$) on boundary Γ. Note that the points ξ are now in the exterior of the flow domain Ω, so that the integrals on boundary Γ appearing in the above two equations are regular. This is the basic idea of the regular boundary element method described in detail in Reference 5. Note that both $\bar{\omega}^{[k]}$ and $\partial\bar{\omega}^{[k]}/\partial n$ on boundary Γ are now as unknowns, so that no boundary conditions about vorticity are necessary. This is one of the advantages of this method.

Note that, after discretizing the above two equations by the regular boundary element method, the matrix of the set of the corresponding *linear* algebraic equations is the same in solving $\bar{\omega}^{[k]}$ and $\partial\bar{\omega}^{[k]}/\partial n$ for any $k\geq 1$. Thus, once its inverse matrix is obtained, it can be stored and then used again and again. Then a little more CPU time of computer is needed to solve the corresponding sets of algebraic equations. This is especially important in case many similar linear problems must be solved.

Once $\bar{\omega}_0^{[k]}$ and $\partial\bar{\omega}_0^{[k]}/\partial n$ on the boundary Γ are known, then according to equations (50), (51) and (41), we can obtain $\bar{\omega}_0^{[k]}$, $\bar{\psi}_0^{[k]}$ in the domain Ω:

$$\bar{\omega}_0^{[k]}(x,y)=\oint_\Gamma\left(G_\omega\frac{\partial\bar{\omega}_0^{[k]}}{\partial n}-\bar{\omega}_0^{[k]}\frac{\partial G_\omega}{\partial n}\right)\mathrm{d}\Gamma-\iint_\Omega h_kG_\omega\,\mathrm{d}\Omega, \tag{54}$$

$$\bar{\psi}_0^{[k]}(x,y)=\oint_\Gamma\left(F_\psi\frac{\partial\bar{\omega}_0^{[k]}}{\partial n}-\bar{\omega}_0^{[k]}\frac{\partial F_\psi}{\partial n}\right)\mathrm{d}\Gamma-\iint_\Omega h_kF_\psi\,\mathrm{d}\Omega+D_k\oint_\Gamma\left[G_\omega\left(\frac{\partial\psi}{\partial n}\right)_\mathrm{b}-\psi_\mathrm{b}\frac{\partial G_\omega}{\partial n}\right]\mathrm{d}\Gamma. \tag{55}$$

If the Taylor's homotopy series of $\psi(x,y;p)$, $\omega(x,y;p)$ have the radius of convergence equal to or greater than unity, then according to equations (23) and (34), the final solutions at Mth-order of approximation are as follows:

$$\begin{aligned}\omega_\mathrm{f}(x,y)&\approx\omega_0(x,y)+\sum_{k=1}^{M}\frac{\omega_0^{[k]}(x,y)}{k!}\\&\approx\sum_{k=1}^{M}\frac{\bar{\omega}_0^{[k]}(x,y)}{k!},\end{aligned} \tag{56}$$

$$\begin{aligned}\psi_\mathrm{f}(x,y)&\approx\psi_0(x,y)+\sum_{k=1}^{M}\frac{\psi_0^{[k]}(x,y)}{k!}\\&\approx\sum_{k=1}^{M}\frac{\bar{\psi}_0^{[k]}(x,y)}{k!}.\end{aligned} \tag{57}$$

But unfortunately, the radius ρ of convergence is generally less than unity. In this case, we can obtain the results only at $p=\lambda\leqslant\rho<1$, as follows:

$$\omega(x, y; \lambda)\approx\omega_0(x, y)+\sum_{k=1}^{M}\frac{\omega_0^{[k]}(x, y)}{k!}\lambda^k$$
$$\approx(1-\lambda)\omega_0(x, y)+\sum_{k=1}^{M}\frac{\bar{\omega}_0^{[k]}(x, y)}{k!}\lambda^k, \tag{58}$$

$$\psi(x, y; \lambda)\approx\psi_0(x, y)+\sum_{k=1}^{M}\frac{\psi_0^{[k]}(x, y)}{k!}\lambda^k$$
$$\approx(1-\lambda)\psi_0(x, y)+\sum_{k=1}^{M}\frac{\bar{\psi}_0^{[k]}(x, y)}{k!}\lambda^k. \tag{59}$$

Clearly, the above approximate solutions are better than the initial solutions $\omega_0(x, y)$ and $\psi_0(x, y)$. So, they can be used as the new initial solutions to make a new similar computation. This is the basic idea of iteration. Formulas (58) and (59) give a family of iterative models. Clearly, the larger the M, the more complex is the corresponding iterative model. When $M=1$, it is just the same as SOR model mentioned in Reference 7. This means that the above formulas can be seen as a kind of higher-order iterative model for 2D Navier–Stokes equations. Note that if the initial solutions $\omega_0(x, y)$ and $\psi_0(x, y)$ are just the exact solutions of the original equations, then $\omega_0^{[k]}(x, y)=0$ and $\psi_0^{[k]}=0$ for any $k\geqslant 1$, so that the radii ρ of convergence of the corresponding Taylor's homotopy series are infinite. It is obvious that a couple of better initial solutions would be corresponding to a greater radius ρ of convergence. So, the radius ρ of convergence would be greater and greater in the process of iteration.

Perhaps, it should be emphasized again that the kth-order process equations ($k\geqslant 1$), consisting of equations (27) and (30)–(32) are *linear* in $\omega_0^{[k]}(x, y)$ and $\psi_0^{[k]}(x, y)$ ($k\geqslant 1$), although the original equations (1)–(4) and the zero-order process equations (6)–(9) are non-linear. *Generally, any kth-order process equations* ($k\geqslant 1$) *are linear in their corresponding kth-order process derivatives.* This is an important invariance under homotopy. An abstract mathematical proof about it has been given in Reference 11. In Reference 10, we discretize the first-order process equation in process domain $p\in[0, 1]$; but in this paper, we *analyse* the continuous process of homotopy at $p=0$ by Taylor's series. So, we call this method the *process analysis method*. Other applications of it have been given in References 11 and 12.

3. SIMPLE APPLICATIONS

In order to examine the numerical scheme described above, consider the flow inside a square cavity whose upper lid is moving at a constant velocity. The geometry and the corresponding boundary conditions are shown in Figure 1.

Let the number of boundary elements on each side be the same, denoted as N. For convenience, let N be an even number. The co-ordinates of node points on the upper and the lower side are as follows:

$$x_k=\begin{cases}\left(\dfrac{k}{N}\right)^q+\varepsilon C_k & (k=0, 1, 2, \ldots, N/2-1),\\ 1/2 & (k=N/2),\\ 1-\left(1-\dfrac{k}{N}\right)^q-\varepsilon C_k & (k=N/2+1, N/2+2, \ldots, N),\end{cases} \tag{60}$$

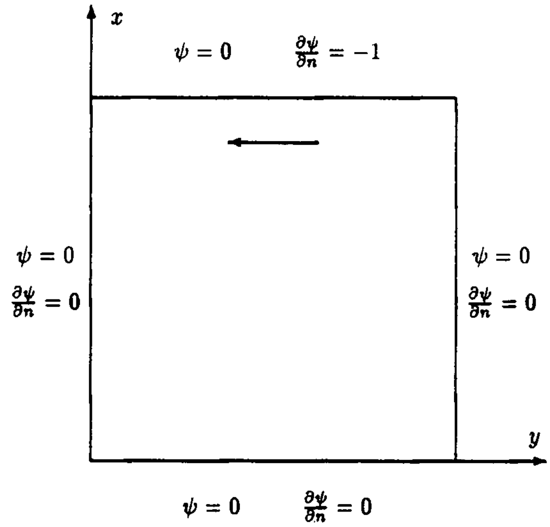

Figure 1. Geometry and boundary conditions for viscous flow inside a square cavity

where

$$C_k=\begin{cases}1 & \text{when } k=0 \text{ or } k=N,\\ 0 & \text{otherwise,}\end{cases} \tag{61}$$

with $y_k=0$ for lower side and $y_k=1{\cdot}0$ for upper side and where $\varepsilon=0{\cdot}0001$ is a small quantity.

In a similar way, the co-ordinates of node points on the left and the right side are as follows:

$$y_k=\begin{cases}\left(\dfrac{k}{N}\right)^q+\varepsilon C_k & (k=0,1,2,\ldots,N/2-1),\\ 1/2 & (k=N/2),\\ 1-\left(1-\dfrac{k}{N}\right)^q-\varepsilon C_k & (k=N/2+1,N/2+2,\ldots,N),\end{cases} \tag{62}$$

with $x_k=0{\cdot}0$ for the left side and $x_k=1{\cdot}0$ for the right side.

Here we use two nodes close to each other at the corners, one belonging to each side. This technique, called the double-node approach in Reference 5, is applied in order to treat the singularity at each corner.

We use linear boundary elements for both $\bar{\omega}^{[k]}$ and $\partial\bar{\omega}^{[k]}/\partial n$ on boundary Γ. Similar to Reference 5, we locate the singular points outside the domain of the square cavity so that the integrals on boundary are regular. The domain of square cavity is discretized into $N\times N$ small regions, where the volume integrals are computed by four-point Gauss numerical integral method.

Note that the solutions in case $Re=0$ can be obtained *without* iteration in the same way as in Reference 7. In all cases of $Re>0$, we use the solutions for the case $Re=0$ as the initial solutions.

All results given in this paper are obtained for the case $N=26$. So, we have $(26+1)\times 4=108$ nodes and 216 unknowns. We use formulas at second-order of approximation, i.e. we let $M=2$ in equations (58) and (59). If the relative error $(f^{m+1}-f^m)/f^m<5\%$, then iteration is stopped. Double precision variables are used in our computer program.

The convergent results are given in Table I. The contours of streamfunction ψ and vorticity ω and the velocity distribution of U at $x=0.5$ and V at $y=0.5$ are shown, respectively, in Figures 2–13. Comparing the above results with those given by other researchers, they seem reasonable.

Note that the formulas in case $M=1$ are the same as those given in Reference 7, which are unstable for Reynolds number greater than 300, as mentioned in Reference 7. In case $Re=400$, $q=1.050$, $\lambda=0.20$ but $M=1$, we find that the corresponding iteration is really divergent. But, as shown in Table I, we obtain convergent results for the case of $0\leqslant Re\leqslant 2000$ by using formulas at second order of approximation. This is so mainly because the formulas at second order are more accurate than those at first order. This can be easily understood from the viewpoint of Taylor's expansion. So, expressions (58) and (59) can be seen as higher-order iterative formulas. In fact, expressions (58) and (59) can give a family of iterative formulas at different orders of approximation M.

It seems that the radius ρ of convergence of the Taylor's homotopy series [equations (21) and (22)] would decrease with the increase in Reynolds number. For a higher Re, we must use smaller

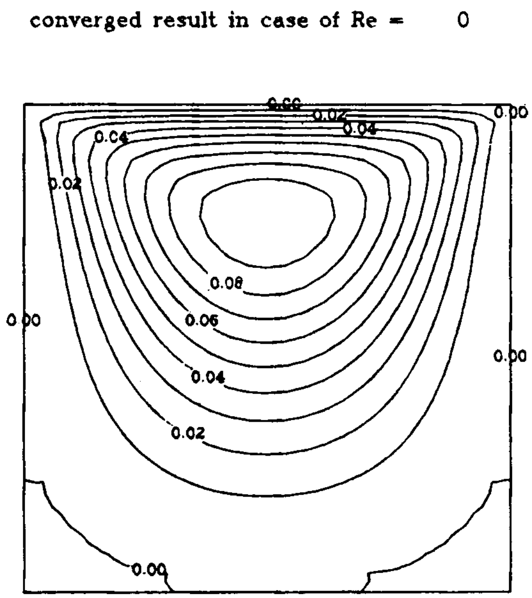

Figure 2. Contour of streamfunction ψ in the case $Re=0$

Table I. Convergent results of steady viscous flow in a square cavity ($M=2$)

Re	q	λ	Iteration times	ω_c	ψ_c	x_c	y_c
0	1.000	1.000	0	3.245	0.0991	0.500	0.769
100	1.050	0.850	15	3.336	0.1054	0.367	0.748
200	1.050	0.350	18	2.686	0.1057	0.405	0.672
400	1.050	0.200	45	2.311	0.1089	0.444	0.633
1000	1.050	0.050	129	2.026	0.1092	0.444	0.595
2000	1.050	0.025	373	1.898	0.1068	0.500	0.556

converged result in case of Re = 0

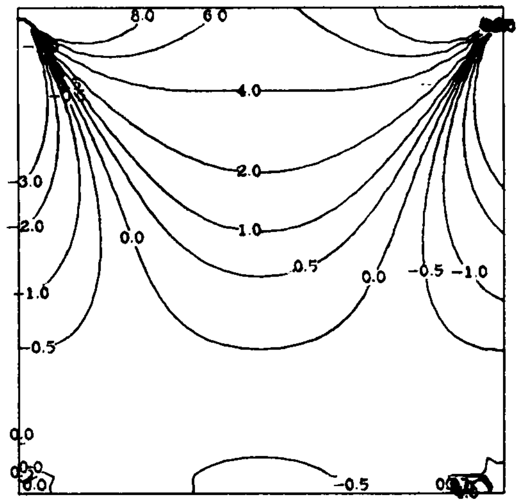

Figure 3. Contour of vorticity ω in the case $Re = 0$

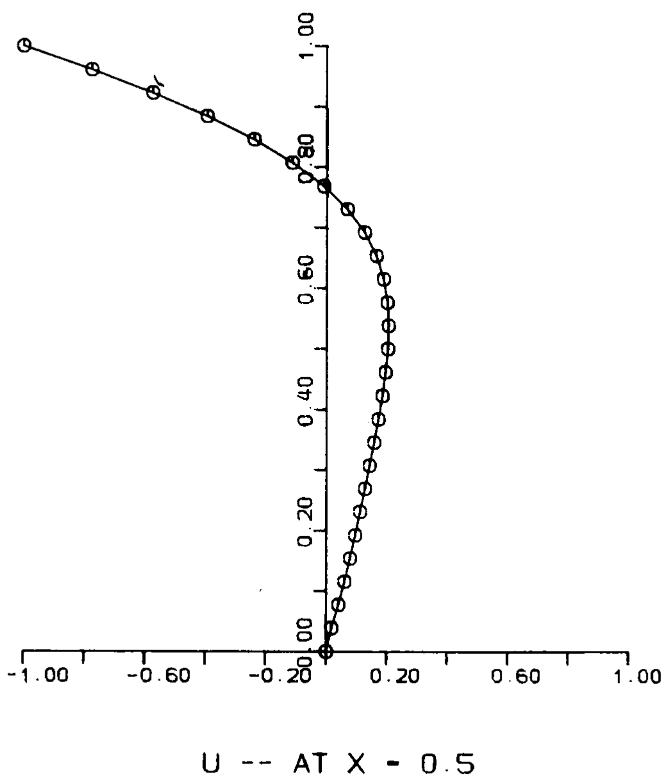

Figure 4. Velocity distribution U at $x = 0{\cdot}5$ in the case $Re = 0$

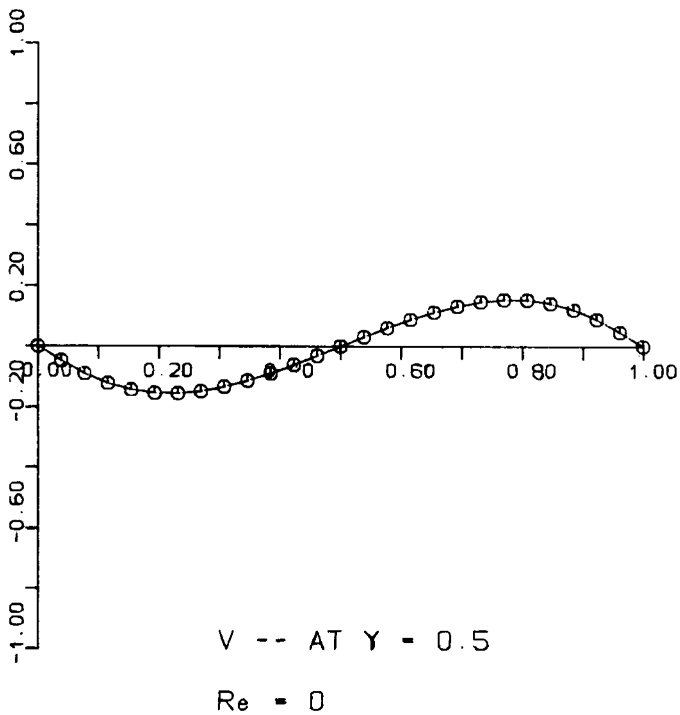

Figure 5. Velocity distribution V at $y=0{\cdot}5$ in the case $Re=0$

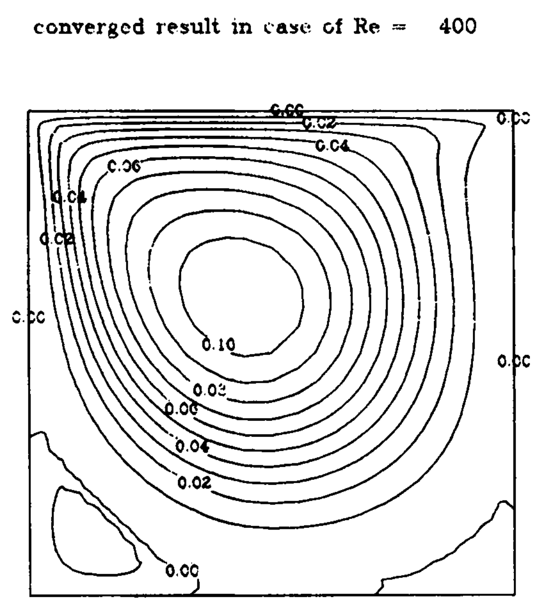

Figure 6. Contour of streamfunction ψ in the case $Re=400$

converged result in case of Re = 400

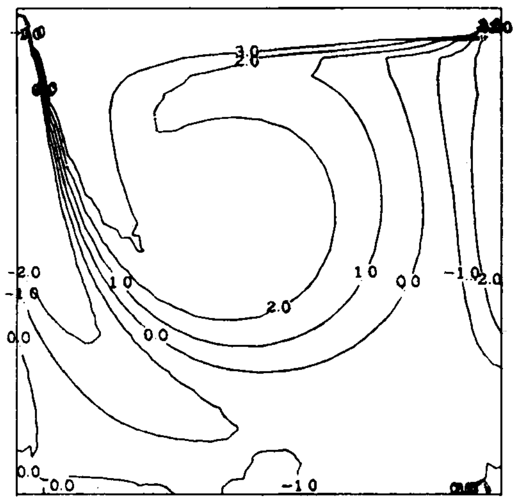

Figure 7. Contour of vorticity ω in the case $Re=400$

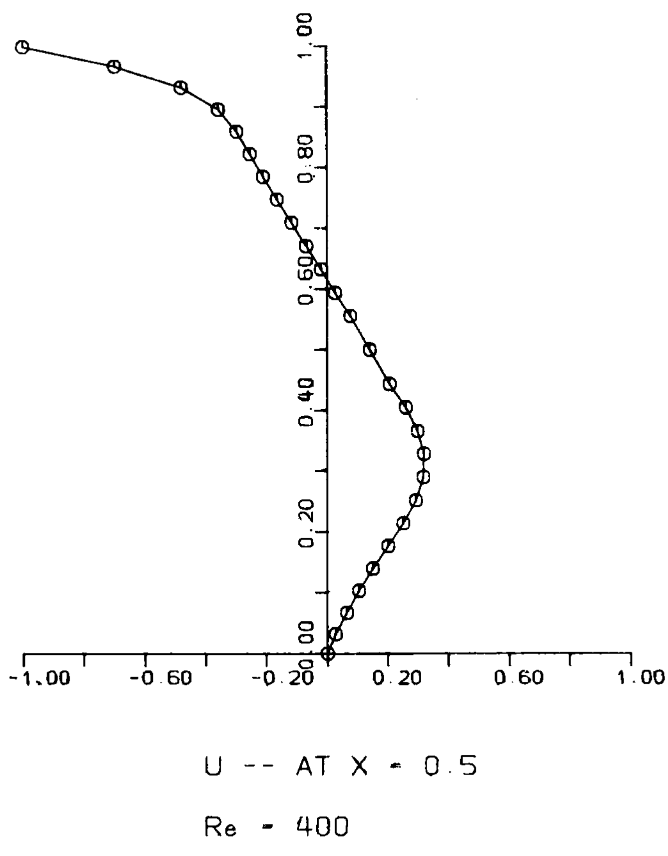

Figure 8. Velocity distribution U at $x=0{\cdot}5$ in the case $Re=400$

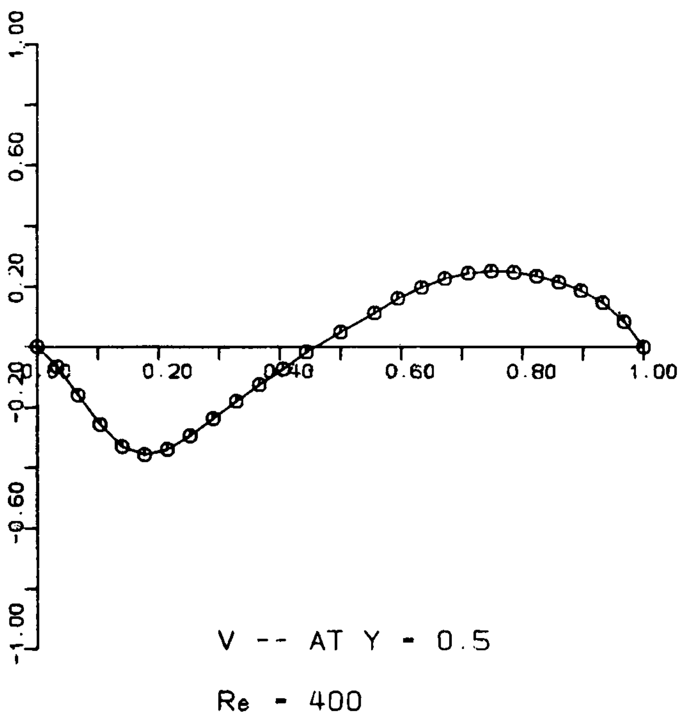

Figure 9. Velocity distribution V at $y=0{\cdot}5$ in the case $Re=400$

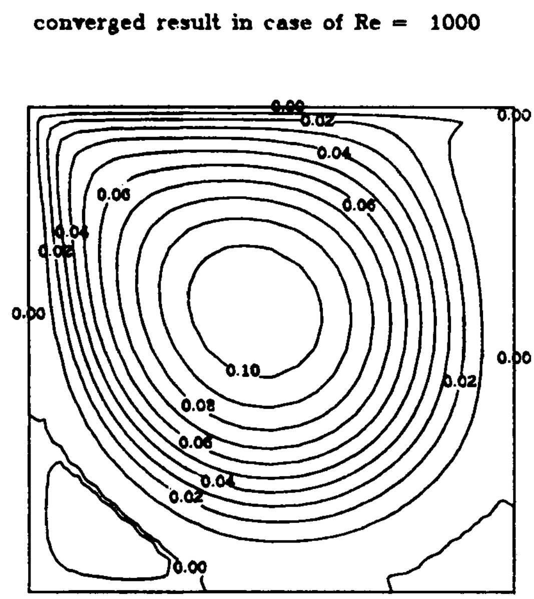

Figure 10. Contour of streamfunction ψ in the case $Re=1000$

converged result in case of Re = 1000

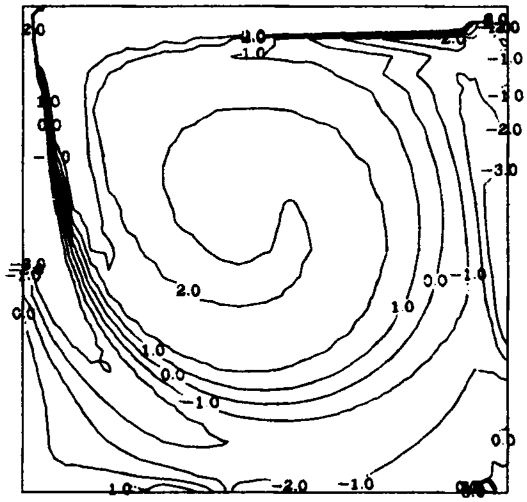

Figure 11. Contour of vorticity ω in the case $Re = 1000$

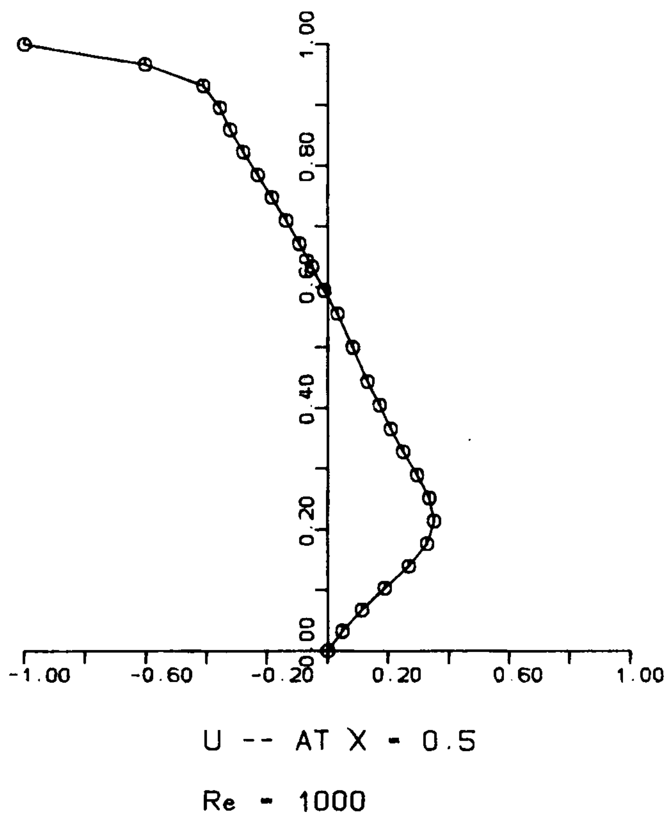

Figure 12. Velocity distribution U at $x = 0{\cdot}5$ in the case $Re = 1000$

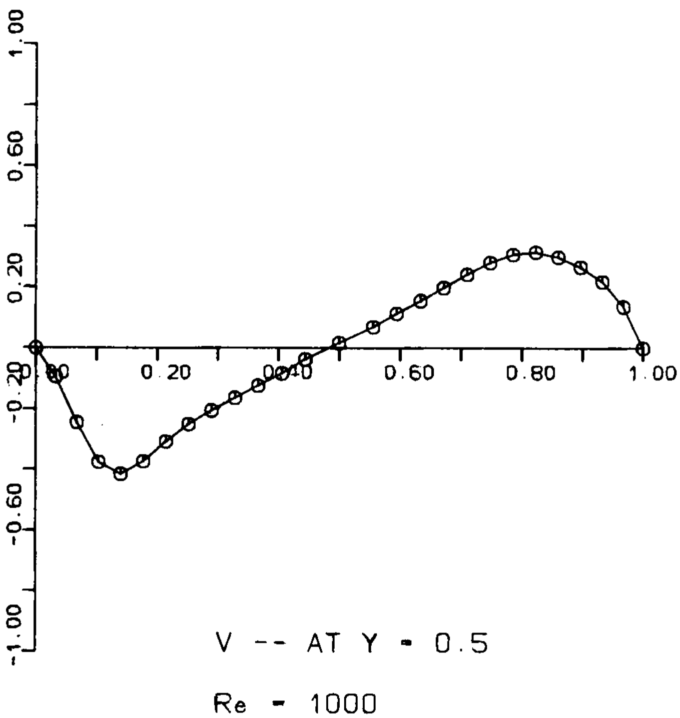

Figure 13. Velocity distribution V at $y=0{\cdot}5$ in the case $Re=1000$

$\lambda \leqslant \rho < 1$ in order to let iteration convergent. We know many researchers use a relaxation parameter smaller than one in their iterative procedures of Navier–Stokes equation to obtain convergent results. This can be easily explained from the viewpoint of the radius of convergence of the corresponding Taylor's homotopy series described in this paper.

Many researchers treat the non-linear terms of Navier–Stokes equations as inhomogeneities. But from the viewpoint of homotopy, the treatment of the non-linear terms described in this paper seems more natural.

For each value of Reynolds number, we use only a constant value of λ in iteration. Clearly, the radius ρ of convergence of the corresponding Taylor's homotopy series [equations (21) and (22)] would be greater and greater in the process of iteration by modifying successively the initial solutions so that the value of λ would be increased progressively. So, if we increase progressively the value of λ in iteration, less iterations are needed.

The matrix and its inverse matrix of the corresponding set of linear algebraic equations should be computed only once so that a little more CPU time is needed to obtain solutions of the corresponding set of linear algebraic equations. In fact, nearly 90% CPU time is used to compute the integrals in domain Ω. We know that integral is suitable for parallel computation. So, if we could develop a kind of sufficiently effective parallel computer, this boundary element method should be powerful in solving Navier–Stokes equations

In this paper, we obtain convergent results in cases $Re = 0$–2000. We believe that the results for the case of much higher Re could be obtained if we apply iterative formulas at higher orders of approximation and use finer numerical nets.

4. CONCLUSION

In this paper, we derive a kind of higher-order iterative model of a regular boundary element method in solving 2D steady-state Navier–Stokes equations in streamfunction–vorticity formulation by applying a kind of linearity invariance under homotopy. The formulas at lowest order of approximation are the same as those given in Reference 7, which are really divergent for $Re > 300$. However, using the formulas at second order of approximation, we obtain convergent results for $0 \leqslant Re \leqslant 2000$. We believe that convergent results for the case $Re > 2000$ could be obtained if we apply iterative formulas at higher orders of approximation and use finer numerical nets.

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APPENDIX: NOMENCLATURE

G_ω, G_ψ	fundamental solutions of 2D Navier–Stokes equations
p	embedding variable of homotopy
q	parameter for distribution of node points on boundary
M	order of approximation
Re	Reynolds number
U, V	velocity in x and y direction, respectively

Greek letters

Γ	boundary of Ω
ρ	radius of convergence
ψ	streamfunction
$\psi_b, \left(\dfrac{\partial\psi}{\partial n}\right)_b$	boundary conditions of streamfunction
ω	vorticity
Ω	domain of fluid flow
Ω^c	exterior of domain Ω

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