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A new branch of solutions of boundary-layer flows over a permeable stretching plate

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Abstract

The steady-state boundary-layer flows over a permeable stretching sheet are investigated by an analytic method for strongly non-linear problems, namely the homotopy analysis method (HAM). Two branches of solutions are obtained. One of them agrees well with the known numerical solutions. The other is new and has never been reported in general cases. The entrainment velocity of the new branch of solutions is always smaller than that of the known ones. For permeable stretching sheet with sufficiently large suction of mass flux, the difference between the shear stresses and velocity profiles of two branches of solutions is obvious: the shear stress of the new branch of solutions is considerably larger than that of the known ones. However, for impermeable sheet and permeable sheet with injection or small suction of mass flux, the shear stress and the velocity profile of two branches of solutions are rather close: in some cases the difference is so small that the new branch of solutions might be neglected even by numerical techniques. This reveals the reason why the new branch of solutions has not been reported. This work also illustrates that, for some non-linear problems having multiple solutions, analytic techniques are sometimes more effective than numerical methods.
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1. Introduction

The investigation of boundary-layer flows of incompressible viscous fluid over a stretching sheet has many important applications in engineering, such as aerodynamic extrusion of plastic sheets, boundary layers along liquid film condensation process, cooling process of metallic plate in a cooling bath, glass and polymer industries, and process of hot rolling, wire drawing and paper production. The investigations were made by a lots of researchers, such as Sakiadis [1], Crane [2], Banks [3], Banks and Zaturska [4], Grubka and Bobba [5], Ali [6] for the impermeable plate and Erickson et al. [7], Gupta and Gupta [8], Chen and Char [9], Chaudhary et al. [10], Elbashbeshy [11], Pop and Na [12], Magyari and Keller [13], and Magyari et al. [14] for the permeable plate. McLeod and Rajagopal investigated the uniqueness of flows of Navier–Stokes fluid due to a stretching boundary [15]. The non-uniqueness of flows of non-Newtonian fluids over a stretching sheet was investigated by Chang et al. [16] and Lawrence et al. [17].

Consider boundary-layer flows over a stretching permeable sheet, governed by

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2}$$

subject to the boundary conditions

$$u=U_w=a\ (x+b)^{\lambda},\quad v=V_w=A(x+b)^{(\lambda-1)/2}\quad \text{at } y=0, \tag{3}$$

$$u\to 0 \quad \text{as } y\to+\infty, \tag{4}$$

where $U_w=a(x+b)^{\lambda}$ denotes the stretching velocity of the sheet, $V_w=A(x+b)^{(\lambda-1)/2}$ the velocity of fluid through the permeable sheet, respectively. Note that $A>0$ corresponds to the injection, and $A<0$ the suction of mass flux, respectively.

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Following Crane [2] and Banks [3], we transfer the above non-linear partial differential equations into a set of ordinary differential equations. Let ψ denote the stream function. Under the similarity transformation:

$$\psi = a\sqrt{\frac{v}{a(1+\lambda)}}(x+b)^{(\lambda+1)/2}F(\xi),$$

$$\xi = \sqrt{\frac{a(1+\lambda)}{v}}(x+b)^{(\lambda-1)/2}y, \tag{5}$$

where $a \neq 0$ and $a(1+\lambda) > 0$, the original equations become

$$F'''(\xi) + \tfrac{1}{2}F(\xi)F''(\xi) - \beta F'^2(\xi) = 0, \tag{6}$$

subject to the boundary conditions

$$F(0) = \alpha, \quad F'(0) = 1, \quad F'(+\infty) = 0, \tag{7}$$

where

$$\alpha = -\frac{2A}{\sqrt{a(\lambda+1)v}}, \quad \beta = \frac{\lambda}{1+\lambda}. \tag{8}$$

Here, $\alpha=0$ denotes the impermeable sheet, $\alpha<0$ and $\alpha>0$ correspond to the lateral injection ($V_w>0$) and suction ($V_w<0$) of mass flux through the permeable sheet, respectively. Since $a(1+\lambda)>0$, it holds $a>0$ when $\lambda>-1$, corresponding to $-1<\beta\leqslant 1$; and $a<0$ when $\lambda<-1$, corresponding to $\beta>1$. When $a<0$, the stretching velocity of the sheet is negative, corresponding to the so-called backward boundary-layer flows that have physical meanings, as pointed out by Goldstein [18] and Kuiken [19].

From (5), we have the velocity field of the fluid

$$u(x,y) = a(x+b)^{\lambda}F'(\xi), \tag{9}$$

$$v(x,y) = -\tfrac{1}{2}\sqrt{a(1+\lambda)v}(x+b)^{(\lambda-1)/2} \times [F(\xi) + (2\beta-1)\xi F'(\xi)]. \tag{10}$$

The entrainment velocity of the fluid is given by

$$v(x,+\infty) = -\tfrac{1}{2}\sqrt{a(1+\lambda)v}(x+b)^{(\lambda-1)/2}F(+\infty), \tag{11}$$

and the shear stress on the stretching sheet is

$$\tau_w = -\rho v \left.\frac{\partial u}{\partial y}\right|_{y=0} = -a\rho\sqrt{a(1+\lambda)v}(x+b)^{(3\lambda-1)/2}F''(0). \tag{12}$$

Thus, $F''(0)$ and $F(+\infty)$ have clear physical meanings. For simplicity, we call $F(+\infty)$ the entrainment parameter.

The non-linear two-point boundary-value problem mentioned above is not easy to solve even by means of numerical techniques. Using the shooting method, Banks [3] gave a branch of numerical solutions when $-1<\beta<+\infty$ for impermeable plate ($\alpha=0$). Banks' solutions exist for $-1<\beta<+\infty$ and have the property $f'(\xi)>0$ in the whole region $0\leqslant\xi<+\infty$. Banks [3] showed that the boundary-layer flow problem does not admit similarity solutions for $\beta<-1$. As mentioned by Liao and Pop [20], when $\lambda>-1$, corresponding to a restricted region $-1<\beta\leqslant 1$, the boundary-layer flow over an impermeable plate ($\alpha=0$) is mathematically equivalent to (but physically different from) the steady free convection flow over a vertical semi-infinite plate embedded in a fluid-saturated porous medium of ambient temperature T_∞, governed by the so-called Cheng–Minkowycz's equation [21]. But, different from Banks [3], Ingham and Brown [21] found two branches of solutions in case of $\frac{1}{2}\leqslant\beta\leqslant 1$. Therefore, at least in case of $\frac{1}{2}<\beta\leqslant 1$, the boundary-layer flows over a stretching impermeable plate ($\alpha=0$) should have two branches of solutions. Currently, exact solutions for boundary-layer flows over a permeable stretching sheet at some special values of β, such as $\beta=-\frac{1}{2}$ and $\beta=\frac{1}{2}$, are given by Magyari and Keller [13]. When $\beta=\frac{1}{2}$ and $\beta=-\frac{1}{2}$, one has the close-form solution

$$F(\xi) = \frac{\alpha+\sqrt{\alpha^2+8}}{2}\left\{1 - \frac{(\alpha-\sqrt{\alpha^2+8})^2}{8} \times \exp\left[\left(-\frac{\alpha+\sqrt{\alpha^2+8}}{4}\right)\xi\right]\right\} \tag{13}$$

and

$$F(\xi) = \sqrt{\alpha^2+4}\tanh\left[\left(\frac{\sqrt{\alpha^2+4}}{4}\right)\xi + \mathrm{sign}(F_w)\mathrm{ArcSech}\left(\frac{2}{\sqrt{\alpha^2+4}}\right)\right], \tag{14}$$

respectively, where $\tanh(z)$ is the hyperbolic tangent of z, $\mathrm{ArcSech}(z)$ is the inverse hyperbolic secant of z, and $\mathrm{sign}(x)$ is defined by

$$\mathrm{sign}(x) = \begin{cases} 1, & x\geqslant 0, \\ -1, & x<0. \end{cases}$$

For details, refer to Magyari and Keller [13]. In a physically different but mathematically identical context, Chaudhary et al. [10] showed that the similarity solutions also exist for $\beta<-1$ if suction ($\alpha>0$) is applied at the surface. Currently, Magyari et al. [14] showed that, in the special case $\lambda=-1$, there exist multiple solutions for the permeable plate with sufficiently large suction. Liao and Magyari [22] found that there exist an infinite number of algebraically decaying solutions of the boundary-layer flows over a stretching impermeable sheet in case of $-\frac{1}{2}<\lambda<0$. All of these support the conclusion that there exist multiple solutions for the considered problem at hand. However, to the best of our knowledge, multiple solutions of the boundary-layer flows over a stretching permeable sheet ($\alpha\neq 0$) have never been reported in general cases of $\lambda\neq -1$.

Generally speaking, it is not easy to solve a non-linear differential equation. Based on the homotopy [23], a basic concept in topology [24], some powerful numerical techniques for non-linear problems, such as the continuation method [25] and the homotopy continuation method [26–30], were developed. There is a suite of FORTRAN subroutines in Netlib for numerically solving non-linear systems of equations by homotopy methods, called HOMPACK. Recently, a kind of analytic method, namely the homotopy analysis method (HAM)

[31–33], is developed to solve highly non-linear problems. Different from perturbation techniques [34], the HAM does not depend upon any small/large physical parameters. Besides, as proved by Liao in his book [32], it logically contains other non-perturbation techniques such as Lyapunov's small parameter method [35], the δ-expansion method [36], and Adomian's decomposition method [37]. Currently, Sajid et al. [38] and Abbasbandy [39,40] pointed out that the so-called "homotopy perturbation method" [41] proposed in 1999 is only a special case of the HAM [31,32] propounded in 1992. Thus, the HAM is rather general and valid for most of non-linear problems in science and engineering. Especially, different from all other analytic techniques, the HAM provides us with a simple way to ensure the convergence of solution series. The HAM has been successfully applied to many non-linear problems, such as similarity boundary-layer flows [42–45], non-linear heat transfer [39], non-linear oscillations [32,33], non-linear waves [40], Thomas–Fermi atom model [32], Volterra's population model [32], high-dimensional Gelfand equation [33] and so on. It has been applied in many fields of science and engineering. For example, Zhu [46,47] applied the HAM to obtain, for the first time, an explicit series solution of the famous Black–Scholes type equation in finance for American put option, which is a system of non-linear PDEs with an unknown moving boundary. Besides, the HAM has been successfully applied to solve some PDEs, such as the unsteady boundary-layer viscous flows [48], the unsteady non-linear heat transfer problem [49], and so on. Currently, by means of the HAM, Liao and Magyari [22] found an infinite number of algebraically decaying solutions for boundary-layer flows over a stretching impermeable sheet in case of $-\frac{1}{2} < \lambda < 0$, and this kind of solutions have been not reported. All of these verify the validity of the HAM. It seems that the HAM is suitable for most types of non-linear problems in different fields of science and engineering, except non-linear problems whose solutions are chaotic, as pointed out by Liao [32].

Liao [50] successfully applied the HAM to the boundary-layer flows over an *impermeable* sheet, and found not only all of Ingham and Brown's solutions for $1/2 \leqslant \beta \leqslant 1$ [21] but also a new branch of solutions for $\beta > 1$. It is interesting that the new branch of solutions for $\beta > 1$ found by Liao [50] has never been reported even by numerical techniques. In this paper, the HAM is further employed to the boundary-layer flows over a *permeable* plate, and a new branch of solutions is found in a similar way.

2. Series solutions given by the HAM

By means of the HAM, Liao [50] proposed two analytic approaches for the boundary-layer flows over a stretching *impermeable* plate. One is for given value of β, the other is for given value of the entrainment parameter $F(+\infty)$. Here, for the boundary-layer flows over a stretching permeable plate, we also propose two analytic approaches by means of the HAM in a similar way: one is for given values of α and β, the other is for given values of α and the entrainment parameter $F(+\infty)$, respectively. The difference is that the mass flux through the permeable sheet must be considered. Because the two approaches used here are rather similar to those described in [50], we neglect all mathematical formulas in this paper. For details, refer to Liao [50].

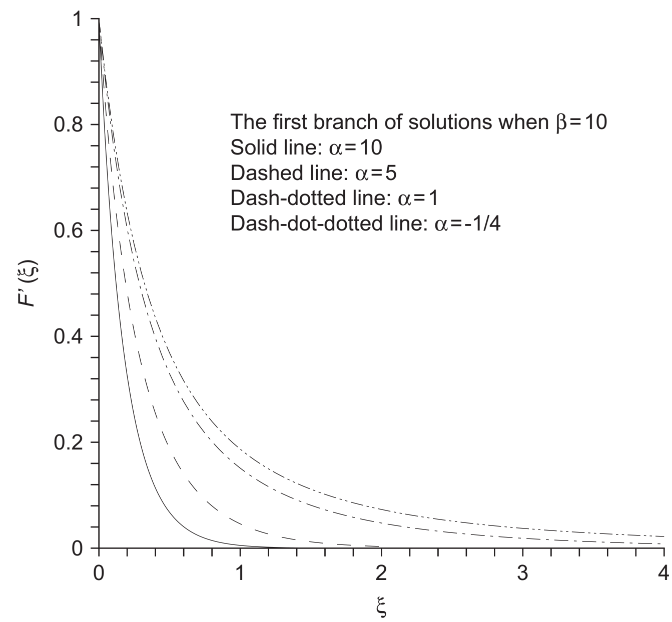

Fig. 1. $F'(+\infty)$ of the 1st branch of solutions when $\beta = 10$.

Obviously, the shear stress on the sheet, the velocity profile and the entrainment velocity of the fluid depend on both the stretching velocity of the sheet and the suction/injection of mass flux through the sheet, denoted by β and α, respectively, where $\alpha < 0$ corresponds to lateral injection and $\alpha > 0$ the suction of mass flux.

2.1. Series solutions for given β

The 1st HAM approach is for given values of α and β, which is rather similar to the 1st approach for impermeable stretching sheet ($\alpha = 0$) described in [50]. Without loss of generality, let us consider the two cases, $\beta = 1$ and 10, with several given values of α. In these cases, our HAM approach gives two branches of series solutions for given α and β, as shown in Figs. 1–6. The convergent series solution of $\delta = F(+\infty)$ and $F''(0)$ are listed in Tables 1–4.

The velocity profiles of the 1st branch of solutions are shown in Fig. 1. This kind of branch of solutions was reported by Magyari and Keller [13] for some special values of β. It has no reversed velocity in all cases. Like the 1st branch of solutions, the 2nd branch of solutions has no reversed velocity in case of injection or small suction of mass flux. However, different from the 1st branch of solutions, the 2nd branch of solutions has the reversed velocity in case of large suction of mass flux, as shown in Fig. 2. Besides, the maximum reversed velocity enlarges as the suction ($\alpha > 0$) of mass flux increases. It should be emphasized that, to the best of our knowledge, the 2nd branch of solutions for given values of α has not been reported in general, except some special values such as $\alpha = 0$ considered

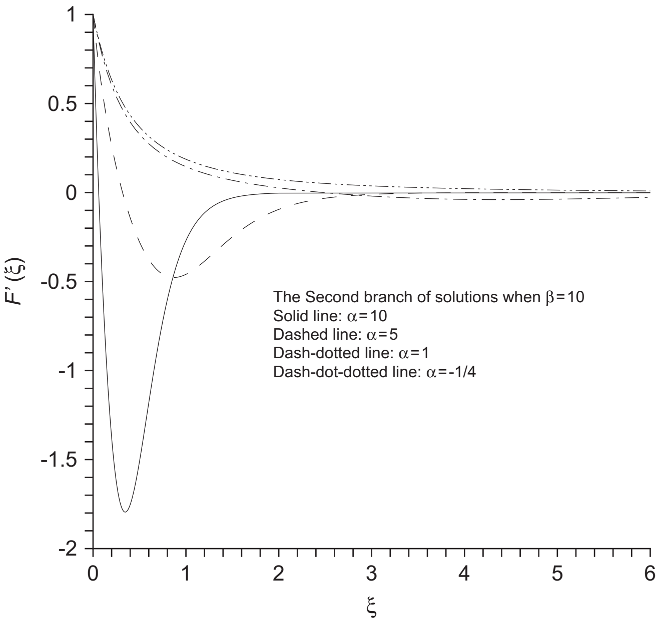

Fig. 2. $F'(+\infty)$ of the 2nd branch of solutions when $\beta = 10$.

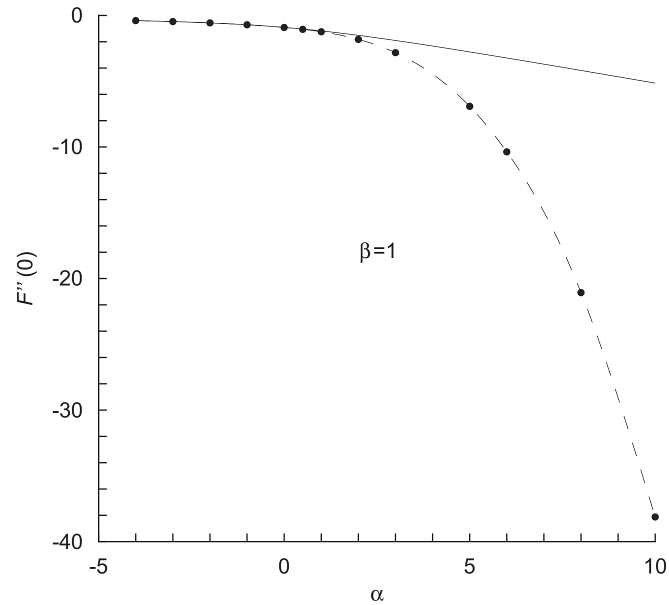

Fig. 4. $F''(0)$ of the two branches of solutions when $\beta = 1$. Solid line: the 1st branch of solution; dashed line with symbols: the 2nd branch of solution.

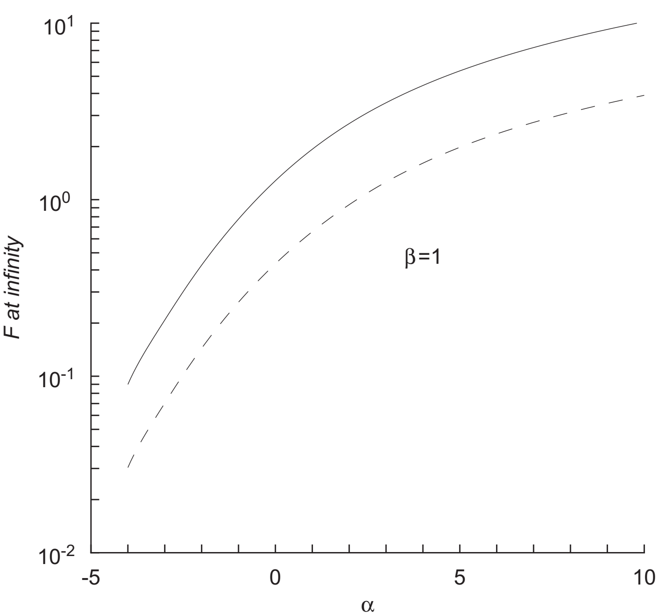

Fig. 3. $F(+\infty)$ of the two branches of solutions when $\beta = 1$. Solid line: the 1st branch of solution; dashed line: the 2nd branch of solution.

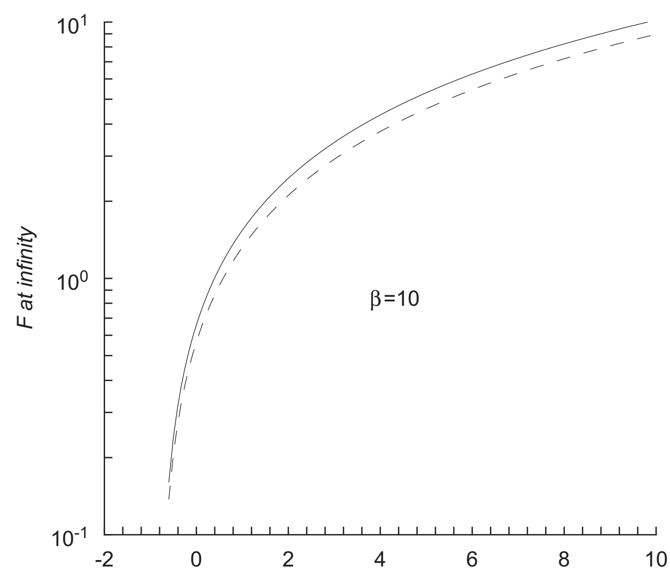

Fig. 5. $F(+\infty)$ of the two branches of solutions when $\beta = 10$. Solid line: the 1st branch of solution; dashed line: the 2nd branch of solution.

by Ingham and Brown [21] for impermeable sheet, or $\lambda = -\frac{1}{3}$, $\frac{1}{2}$ and 1 investigated by Magyari and Keller [13] for permeable sheet.

For the given values of β, the entrainment parameter $\delta = F(+\infty)$ of two branches of solutions increases as α enlarges from negative to positive values. Thus, the suction of mass flux enlarges the entrainment velocity $v(x, +\infty)$ for the two branches of solutions. Physically, this is easy to understand. For given stretching velocity of sheet, i.e. a given value of β, the entrainment velocity $v(x, +\infty)$ of the new branch of solutions is always smaller than that of the known branch of solutions, as shown in Tables 1–4, and Figs. 3 and 5. Besides, the larger the suction of mass flux, the smaller the entrainment velocity of the new branch of solutions than that of the known ones. Therefore, the suction of mass flux enlarges the difference of the entrainment velocity of the two branches of solutions.

According to (12), $F''(0)$ is related with the shear stress (skin friction). For given β, the shear stress of two branches of solutions increases as α enlarges from negative (corresponding

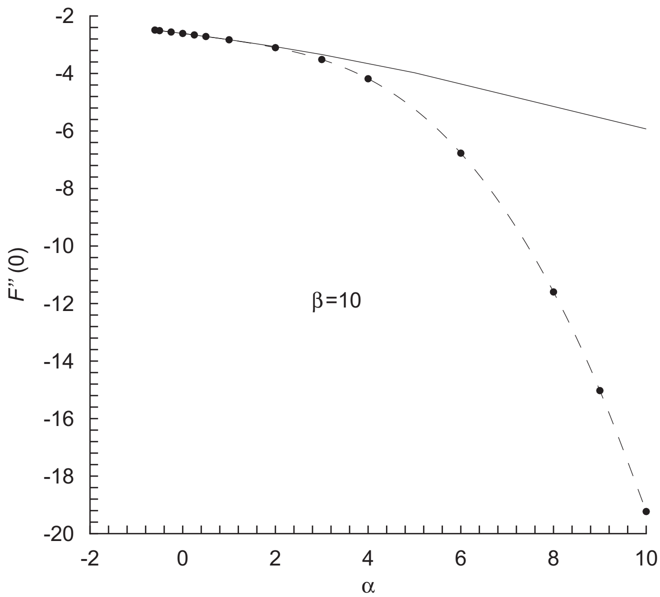

Fig. 6. $F''(0)$ of the two branches of solutions when $\beta=10$. Solid line: the 1st branch of solution; dashed line with symbols: the 2nd branch of solution.

Table 1
$F(+\infty)$ and $F''(0)$ of the 1st branch of solutions when $\beta=1$

α	$\hbar$	$F(+\infty)$	$F''(0)$
-4	$-\frac{1}{3}$	0.089829775350	-0.391815781402
-3	$-\frac{1}{3}$	0.208116758086	-0.466406144776
-2	$-\frac{1}{2}$	0.42570039	-0.56773509
-1	$-\frac{1}{2}$	0.77693585	-0.70852950
0	$-\frac{3}{4}$	1.28077378	-0.90637551
1	$-\frac{3}{4}$	1.93111056	-1.17561409
2	$-\frac{3}{4}$	2.69794357	-1.51344932
5	$-\frac{3}{4}$	5.36649491	-2.77335832
10	$-\frac{3}{4}$	10.19523827	-5.14619724

to injection) to positive values (corresponding to suction), as shown in Figs. 4 and 6. Physically, this is easy to understand, because the suction of mass flux enlarges the gradient of the velocity of the fluid on the sheet. Note that, for given β, the shear stresses of the two branches of solutions are nearly the same for impermeable plate ($\alpha=0$) and permeable plate with injection ($\alpha<0$). However, for permeable sheet with large suction of mass flux, the shear stress of the new branch of solutions is much larger than those of the known ones, as shown in Table 1–4 and Figs. 4 and 6. For example, in case of $\beta=1$ and $\alpha=20$, the shear stress of the new branch of solution is about 6 times larger than that of the known ones, as shown in Tables 1 and 2.

In summary, by means of the 1st HAM approach (refer to [50] for details), we find two branches of solutions of boundary-layer flows over permeable plate for given β. To the best of our knowledge, one of them has never been reported in general

Table 2
$F(+\infty)$ and $F''(0)$ of the 2nd branch of solutions when $\beta=1$

α	$\hbar$	$F(+\infty)$	$F''(0)$
-4	$-\frac{1}{4}$	0.030369809556	-0.391815781392
-3	$-\frac{1}{2}$	0.070360482202	-0.466406145105
-2	$-\frac{1}{2}$	0.143922	-0.567736
-1	$-\frac{1}{2}$	0.262687	-0.708761
0	$-\frac{1}{2}$	0.433654	-0.913339
$\frac{1}{2}$	$-\frac{1}{2}$	0.539488	-1.055678
1	$-\frac{1}{2}$	0.658921	-1.241337
2	$-\frac{1}{2}$	0.936984	-1.819762
3	$-\frac{1}{2}$	1.259310	-2.829113
5	$-\frac{1}{2}$	1.980378	-6.912612
6	$-\frac{1}{2}$	2.358324	-10.36825
8	$-\frac{1}{2}$	3.126363	-21.06924
10	$-\frac{1}{2}$	3.900930	-38.12400

Table 3
$F(+\infty)$ and $F''(0)$ of the 1st branch of solutions when $\beta=10$

α	$\hbar$	$F(+\infty)$	$F''(0)$
$-\frac{3}{5}$	$-\frac{1}{8}$	0.160088	-2.4901117
$-\frac{1}{2}$	$-\frac{1}{6}$	0.239927	-2.509225
$-\frac{1}{4}$	$-\frac{1}{5}$	0.446081	-2.557970
0	$-\frac{1}{5}$	0.658388	-2.608148
1	$-\frac{1}{5}$	1.541900	-2.824345
2	$-\frac{1}{5}$	2.457505	-3.067872
3	$-\frac{1}{5}$	3.392704	-3.340911
5	$-\frac{1}{5}$	5.300584	-3.973654
10	$-\frac{1}{4}$	10.181382	-5.928155

Table 4
$F(+\infty)$ and $F''(0)$ of the 2nd branch of solutions when $\beta=10$

α	$\hbar$	$F(+\infty)$	$F''(0)$
$-\frac{3}{5}$	$-\frac{1}{5}$	0.137297	-2.49012
$-\frac{1}{2}$	$-\frac{1}{4}$	0.205770	-2.509226
$-\frac{1}{4}$	$-\frac{1}{3}$	0.382575	-2.557972
0	$-\frac{1}{3}$	0.564658	-2.608169
$\frac{1}{2}$	$-\frac{1}{2}$	1.065042	-2.713439
1	$-\frac{3}{4}$	1.322510	-2.827165
2	$-\frac{3}{4}$	2.109078	-3.102492
3	$-\frac{3}{4}$	2.916481	-3.513958
4	$-\frac{3}{4}$	3.743084	-4.183302
6	$-\frac{3}{4}$	5.446575	-6.771344
8	$-\frac{3}{4}$	7.193990	-11.596886
10	$-\frac{3}{4}$	8.962150	-19.231412

Table 5
$F(+\infty)$ and $F''(0)$ of the 1st branch of solutions when $\alpha = 1$

β	$\hbar$	$F(+\infty)$	$F''(0)$
−1.058585	$-\frac{3}{4}$	20	451.784
−1.122963	$-\frac{3}{4}$	10	56.690207
−1.156731	$-\frac{3}{4}$	8	28.880524
−1.213242	$-\frac{3}{4}$	6	11.914562
−1.255601	$-\frac{3}{4}$	5	6.659478
−1.280142	$-\frac{3}{4}$	9/2	4.689289
−1.303342	$-\frac{3}{4}$	4	3.096833
−1.313026	$-\frac{3}{4}$	7/2	1.830559
−1.264897	$-\frac{3}{4}$	3	0.823920
−0.977969	$-\frac{3}{4}$	5/2	−0.0334866
−0.842905	$-\frac{3}{4}$	12/5	−0.204443
−0.656192	$-\frac{3}{4}$	23/10	−0.381592
−0.395769	$-\frac{3}{4}$	11/5	−0.569180
0	$-\frac{1}{3}$	2.093662	−0.786400
1	$-\frac{3}{4}$	1.931111	−1.175614
3	$-\frac{1}{2}$	1.764465	−1.699431
5	$-\frac{1}{3}$	1.670900	−2.088843
10	$-\frac{1}{5}$	1.541900	−2.824345
20	$-\frac{1}{10}$	1.422621	−3.880318

cases. For impermeable sheet ($\alpha = 0$) and permeable sheet with injection ($\alpha < 0$) or small suction of mass flux, the shear stress, the entrainment velocity $v(x, +\infty)$ and the velocity profiles of two branches of solutions are nearly the same. However, for permeable sheet with large suction of mass flux ($\alpha > 0$), the new branch of solutions has the reversed velocity profile, considerably larger shear stress on the sheet but smaller entrainment velocity at infinity.

2.2. Series solutions for given entrainment parameter $F(+\infty)$

As mentioned in Section 2.1, the difference of the shear stress of the two branches of solutions is small for impermeable plate (α=0) and permeable plate with injection ($\alpha < 0$). So, it is interesting to investigate the boundary-layer flows over a stretching permeable plate with suction velocity ($\alpha > 0$). Thus, we propose here the 2nd analytic approach by means of the HAM for given α and the entrainment parameter $F(+\infty)$, which is rather similar to the 2nd approach described in [50].

Without loss of generality, let us consider here two cases, $\alpha = 1$ and 10, with several given values of β. For each given α, the series solutions for $\beta \geqslant 0$ are obtained by means of the 1st HAM approach, and the series solutions for $\beta < 0$ are given by the 2nd HAM approach, respectively. It is found that there exist two branches of solutions for the given values of α, as shown in Tables 5–8, and Figs. 7–13. The 1st branch of solutions are the same as published numerical ones. However, to the best of our knowledge, the 2nd branch of solutions, which exist for $\beta > \frac{1}{2}$, has never been reported in general.

Table 6
$F(+\infty)$ and $F''(0)$ of the 2nd branch of solutions when $\alpha = 1$

β	$\hbar$	$F(+\infty)$	$F''(0)$
$\frac{51}{100}$	$-\frac{2}{25}$	0.19274	−1.05429
$\frac{11}{20}$	$-\frac{1}{3}$	0.281051	−1.078487
$\frac{3}{5}$	$-\frac{1}{2}$	0.350757	−1.101572
$\frac{4}{5}$	$-\frac{1}{2}$	0.534353	−1.176962
$\frac{9}{10}$	$-\frac{1}{2}$	0.601365	−1.210051
1	$-\frac{1}{2}$	0.658921	−1.241337
2	$-\frac{3}{4}$	0.986856	−1.505238
3	$-\frac{3}{4}$	1.130957	−1.725947
5	$-\frac{3}{4}$	1.253584	−2.100642
10	$-\frac{3}{4}$	1.322510	−2.827164
20	$-\frac{1}{3}$	1.313713	−3.880830

Table 7
$F(+\infty)$ and $F''(0)$ of the 1st branch of solutions when $\alpha = 10$

β	$\hbar$	$F(+\infty)$	$F''(0)$
−2.428948	$-\frac{1}{2}$	18	265.53889
−3.873406	$-\frac{1}{2}$	14	89.283353
−4.833631	$-\frac{1}{2}$	13	58.896300
−6.736765	$-\frac{1}{2}$	12	33.334878
−8.603705	$-\frac{1}{2}$	11.5	22.268703
−12.166495	$-\frac{1}{2}$	11	12.286156
−14.625826	$-\frac{1}{2}$	10.8	8.566258
−18.160463	$-\frac{1}{2}$	10.6	4.934705
−19.232754	$-\frac{1}{2}$	10.55	4.020142
−20.346912	$-\frac{1}{2}$	10.5	3.089445
−21.430725	$-\frac{1}{2}$	10.45	2.128818
−22.317707	$-\frac{1}{2}$	10.4	1.115722
−22.627422	$-\frac{1}{2}$	10.35	0.011961
−21.462442	$-\frac{1}{2}$	10.3	−1.250274
−16.556065	$-\frac{3}{4}$	10.25	−2.799140
−14.702033	$-\frac{3}{4}$	10.24	−3.162613
−12.393548	$-\frac{3}{4}$	10.23	−3.550309
−9.52221	$-\frac{3}{4}$	10.22	−3.965951
−5.949297	$-\frac{3}{4}$	10.21	−4.413988
−1.495596	$-\frac{3}{4}$	10.2	−4.899776
0	$-\frac{3}{4}$	10.197086	−5.049351
1	$-\frac{3}{4}$	10.195238	−5.146197
3	$-\frac{3}{4}$	10.191759	−5.333035
5	$-\frac{1}{2}$	10.188534	−5.511670
10	$-\frac{1}{4}$	10.181382	−5.928155
20	$-\frac{1}{4}$	10.169894	−6.663634
30	$-\frac{1}{5}$	10.160896	−7.307361
40	$-\frac{1}{5}$	10.153542	−7.886163

Table 8
$F(+\infty)$ and $F''(0)$ of the 2nd branch of solutions when $\alpha = 10$

β	$\hbar$	$F(+\infty)$	$F''(0)$
0.5000048	$-\frac{1}{3}$	$\frac{1}{2}$	-15.72888
0.50080929	$-\frac{1}{5}$	$\frac{3}{4}$	-19.571466
0.50670877	$-\frac{1}{2}$	1	-22.487530
0.59926472	$-\frac{1}{2}$	2	-30.176567
0.77336669	$-\frac{1}{2}$	3	-35.222229
1	$-\frac{1}{2}$	3.900930	-38.124001
3	$-\frac{1}{2}$	7.055465	-34.392807
5	$-\frac{3}{4}$	8.056920	-27.769041
10	$-\frac{3}{4}$	8.962150	-19.231412
20	$-\frac{3}{4}$	9.485879	-13.466843
30	$-\frac{3}{4}$	9.677017	-11.541606
40	$-\frac{3}{4}$	9.777200	-10.781134

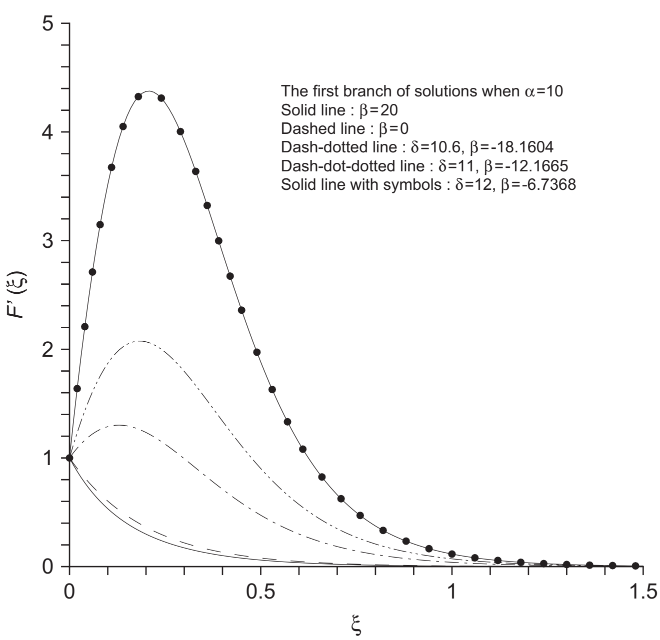

Fig. 7. $F'(\xi)$ of the 1st branch of solution when $\alpha = 10$. Solid line: when $\beta = 20$; dashed line: when $\beta = 0$; dash-dotted line: when $F(+\infty) = 10.6$ and $\beta = -18.16046344$; dash-dot-dotted line: when $F(+\infty) = 11$ and $\beta = -12.16649517$; solid line with symbols: when $F(+\infty) = 12$ and $\beta = -6.73676515$.

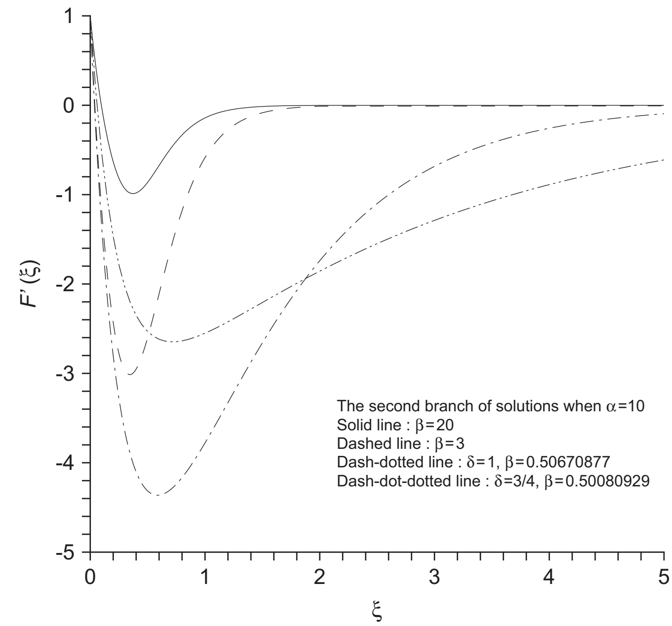

Fig. 8. $F'(\xi)$ of the 2nd branch of solution when $\alpha = 10$. Solid line: when $\beta = 20$; dashed line: when $\beta = 5$; dash-dotted line: when $F(+\infty) = 2$ and $\beta = 0.59926472$; dash-dot-dotted line: when $F(+\infty) = \frac{3}{4}$ and $\beta = 0.50080929$.

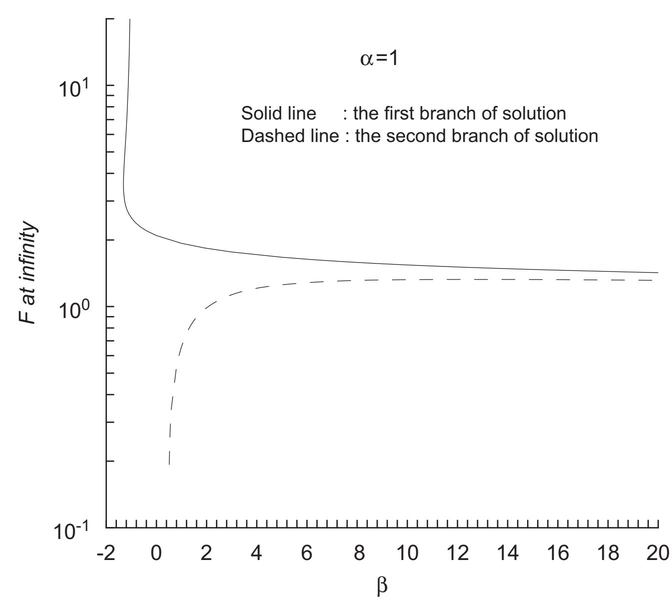

Fig. 9. $F(+\infty)$ of the two branches of solutions when $\alpha = 1$. Solid line: the 1st branch of solution; dashed line: the 2nd branch of solution.

In case of permeable stretching sheet with suction velocity, the 1st branch of solutions has the property $F'(\xi) > 0$ in the whole region $0 \leqslant \xi < +\infty$. The velocity of the 1st branch of solution decreases monotonously for $\beta \geqslant 0$, but there exists the velocity overshooting when $\beta < 0$, as shown in Fig. 7. This character was reported by some previous researchers [13]. The 2nd branch of solutions shows the reversed velocity and is therefore completely different from the 1st ones, as shown in Fig. 8. It should be emphasized that such a kind of solutions has never been reported. Note that the new branch of solution exist only for $\beta \geqslant \frac{1}{2}$. However, the 1st branch of solutions exist in a larger region of β, as shown in Figs. 9 and 12. So, there exist multiple solutions for permeable stretching sheet in case of $\beta \geqslant \frac{1}{2}$, corresponding to $1 \leqslant \lambda < +\infty$ or $-\infty < \lambda < -1$.

For impermeable plate ($\alpha = 0$), Banks' numerical results [3] indicate that the entrainment velocity decreases monotonously in the region $\beta \in (-1, +\infty)$ and tends to infinity as $\beta \to -1$. However, for permeable plate with suction ($\alpha > 0$), the 1st branch of solutions exist in a region of β larger than

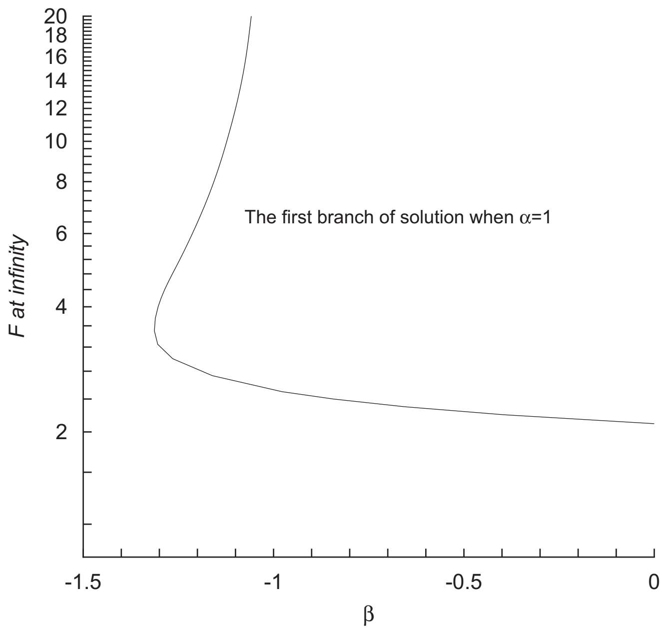

Fig. 10. $F(+\infty)$ of the 1st branch of solution near $\beta=-1$ when $\alpha=1$.

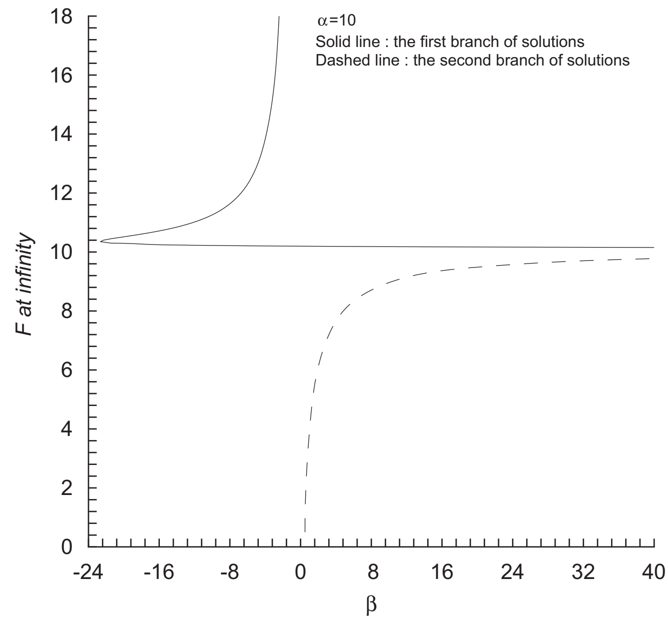

Fig. 12. $F(+\infty)$ of the two branches of solutions when $\alpha=10$. Solid line: the 1st branch of solution; dashed line: the 2nd branch of solution.

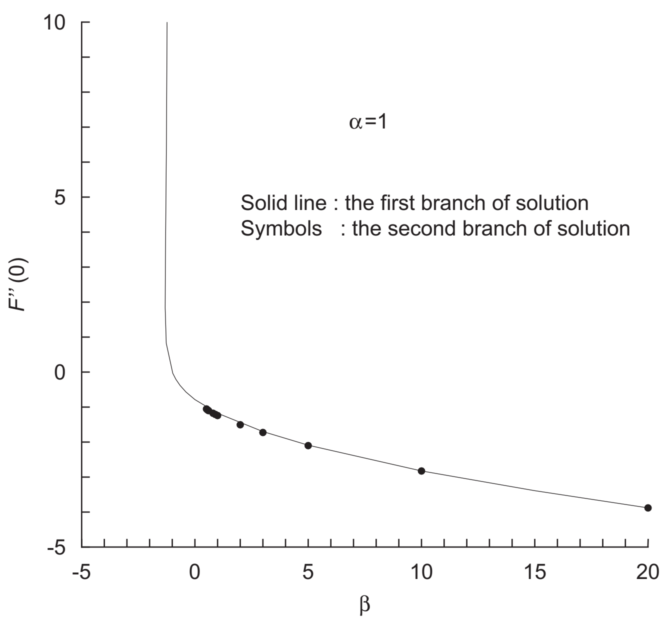

Fig. 11. $F''(0)$ of the two branches of solutions when $\alpha=1$. Solid line: the 1st branch of solution; symbols: the 2nd branch of solution.

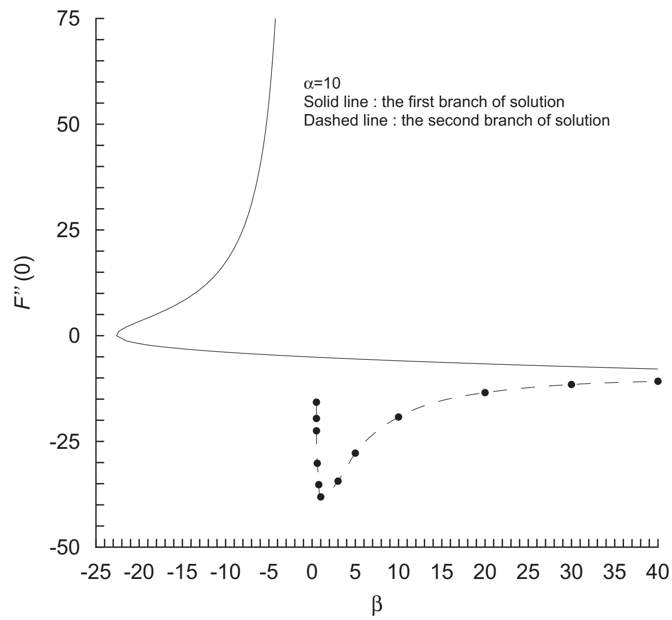

Fig. 13. $F''(0)$ of the two branches of solutions when $\alpha=10$. Solid line: the 1st branch of solution; dashed line with symbols: the 2nd branch of solution.

$(-1,+\infty)$, and besides there exists a turning point in the region $\beta<-1$, as shown in Figs. 9, 10 and 12. It is interesting that the entrainment velocity of the new branch of solutions monotonously increases in the region $\beta\in(\frac{1}{2},+\infty)$, and is always smaller than that of the known ones, as shown in Figs. 9 and 12.

As mentioned in Section 2.1, the difference between the shear stress of the two branches of solutions is rather small for the impermeable sheet ($\alpha=0$) and the permeable sheet with injection ($\alpha<0$). For the permeable sheet with small suction of mass flux, such as $\alpha=1$, the difference of shear stress of the two branches of solutions is still not obvious, as shown in Fig. 11. However, as the suction of mass flux increases, the new branch of solution has much larger shear stress than the known ones, as shown in Fig. 13 for $\alpha=10$.

2.3. *Sensitivity of the solutions to $F''(0)$*

It was a little surprise that, to the best of our knowledge, the new branch of solutions, which exists for $\beta>\frac{1}{2}$, has not been reported in general cases even by numerical techniques. Why?

Generally speaking, it is much more difficult to solve non-linear boundary-value problems (BVPs) than initial-value problems (IVPs). The so-called shooting method is often used to get numerical results of BVPs. However, as pointed out by Shampine and his co-workers [51,52], "unstable IVPs can cause a shooting code to fail because the integral blows up before reaching the end of the interval", thus, "when the IVPs are very unstable, shooting is just not a natural approach to solving BVPs". To overcome the disadvantages of shooting methods, Shampine and his co-workers [51,52] provided a numerical approach for non-linear BVPs. Different from shooting method, Shampine's method is based on solving a set of non-linear algebraic equations by iterations over the whole interval, while the boundary conditions are taken into account at all time. They developed a MATHLAB program **bvp4c**, which can be used to give accurate numerical solutions of some non-linear BVPs. Shampine and his co-workers [51,52] successfully applied the MATHLAB program **bvp4c** to solve many non-linear BVPs, some of them even have multiple solutions.

In this paper, by means of the analytic approaches based on the HAM method (for details, refer to [50]), we obtain two branches of convergent series solutions. Thus, we can get very accurate analytic value of $F''(0)$. So, using the analytic result of $F''(0)$, it is easy for us to directly apply the Runge–Kutta's method to get the numerical solutions. However, our series solutions indicate that, for impermeable sheet ($\alpha = 0$) and permeable sheet with injection ($\alpha < 0$), the difference between the values of $F''(0)$ of the two branches of solutions (when $\beta > \frac{1}{2}$) is rather small. For example, when $\beta = 1$ and $\alpha = -3$, our analytic result (by means of homotopy-Padé acceleration) gives

$$F''(0) = -0.46640, 61447, 76$$

for the 1st branch of solution, and

$$F''(0) = -0.46640, 61451, 05$$

for the 2nd one, respectively: the difference between them is less than 4×10^{-10}. Besides, the corresponding IVP with the given $F''(0)$ seems rather *unstable* for each solution: if the value of $F''(0)$ is not accurate enough, for example, $F''(0) \approx -0.46640614$, whose first 8 decimals are exactly the *same* as those given by our analytic approaches, one cannot obtain a satisfied numerical solution by means of Runge–Kutta's method, as shown in Figs. 14 and 15. Therefore, by means of shooting methods, one cannot solve this kind of BVPs. Even by means of the MATHLAB code **bvp4c** [51,52], one must use a very good guess and the domain must be large enough (i.e. at least more than 100), but it is very difficult to known all of these if one has no ideas about the exact solutions. Note that the profiles of $F(\xi)$ of the two branches of solutions are nearly the same in a considerably large region near the sheet, but are obviously different out of that region, as shown in Fig. 16. This clearly indicates that there indeed exist two different solutions for impermeable sheet ($\alpha = 0$) and permeable sheet with injection ($\alpha < 0$) or small suction of mass flux. For impermeable sheet and permeable sheet with injection (for example $\alpha = -3$) of mass flux, the velocity profiles of $u(x, y)$ of the two branches of solutions are too close to distinguish, as shown in Fig. 17. So, for impermeable sheet and permeable sheet with injection of mass flux, the difference of $F''(0)$ of the two branches of solutions is so small and the corresponding IVPs are so unstable that the 2nd branch of solutions are neglected even by numerical methods. This might be the reason why the new branch of solutions were not reported before. This example also reveals the restrictions of numerical techniques for non-linear problems with multiple solutions. It should be emphasized that, different from numerical techniques, even our 10th-order analytic

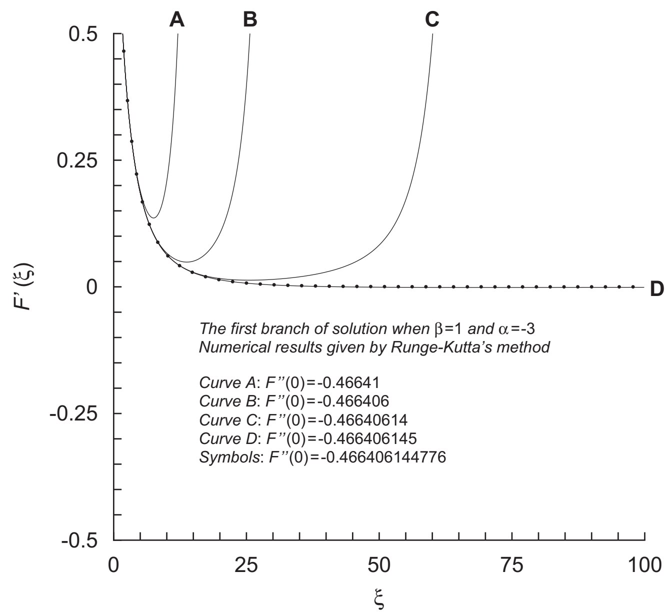

Fig. 14. Numerical results of the 1st-branch of solution given by Runge–Kutta method when $\beta = 1$ and $\alpha = -3$.

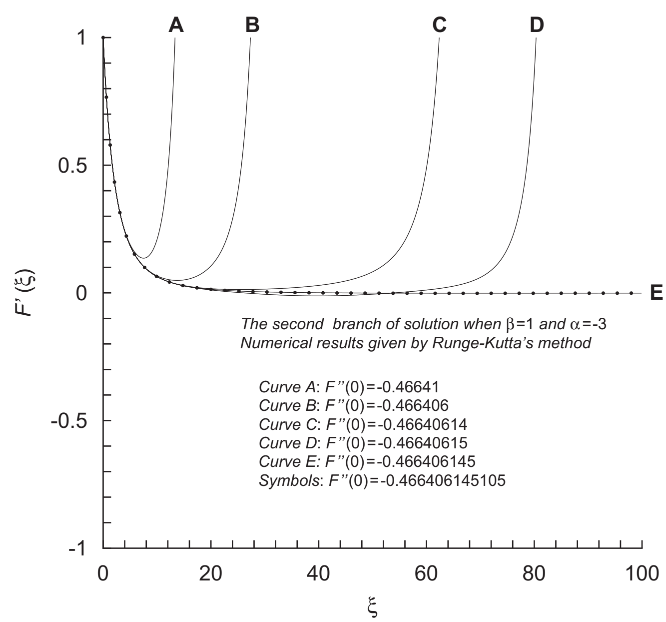

Fig. 15. Numerical results of the 2nd-branch of solution given by Runge–Kutta method when $\beta = 1$ and $\alpha = -3$.

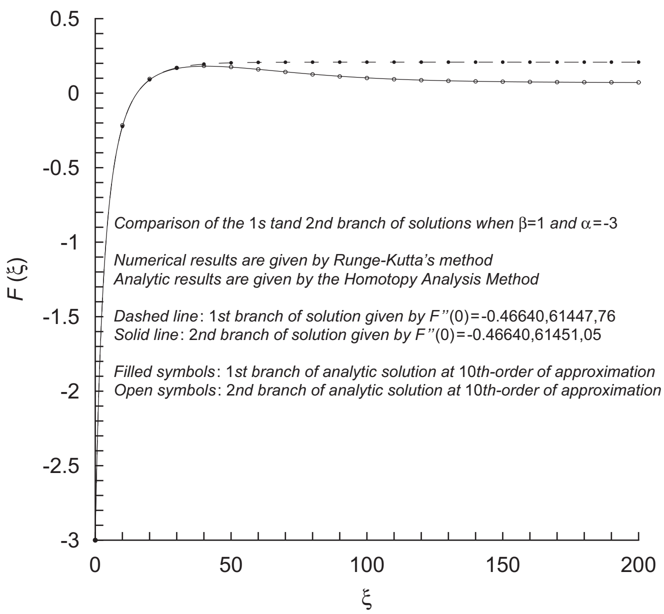

Fig. 16. Comparison of $F(\xi)$ of the 1st and 2nd branch of solutions when $\beta=1$ and $\alpha=-3$. Dashed line: 1st branch of solutions given by Runge–Kutta method; solid line: 2nd branch of solutions given by Runge–Kutta method; filled circle: 1st branch of analytic solution at the 10th-order of approximation; open circle: 2nd branch of analytic solution at 10th-order of approximation.

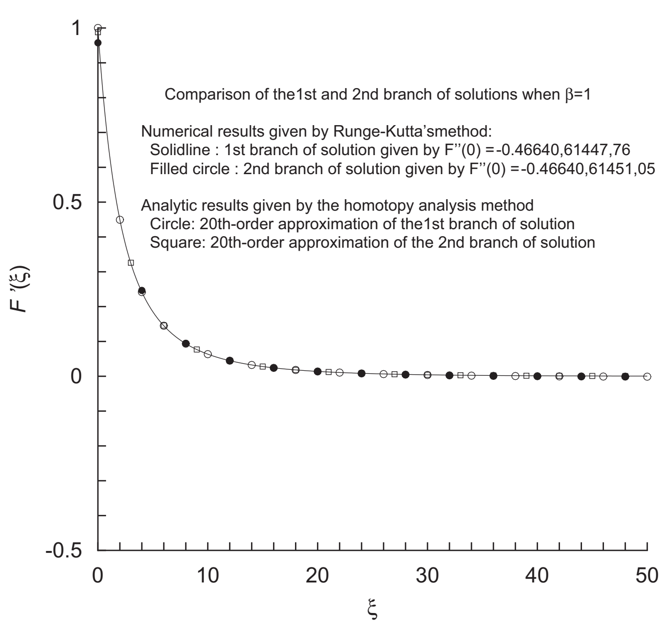

Fig. 17. Comparison of $F'(\xi)$ of the 1st and 2nd branch of solutions when $\beta=1$ and $\alpha=-3$. Solid line: 1st branch of solution given by Runge–Kutta method; filled circle: 2nd branch of solution given by Runge–Kutta method; open circles: 1st branch of analytic solutions at 20th-order of approximation; open square: 2nd branch of analytic solutions at 20th-order of approximation.

approximation is accurate enough in the whole region, as shown in Fig. 16, although the corresponding $F''(0)$ is far less accurate than $F''(0) \approx -0.46640614$. Thus, it is *unnecessary* for our HAM approaches to have a very accurate approximation of $F''(0)$, i.e. our HAM approaches is *insensitive* to the accuracy of $F''(0)$. This is mainly because we solve the non-linear BVP directly, and besides the accuracy of the HAM series solution does not depend up the accuracy of $F''(0)$. This reveals the advantage of our HAM approaches over numerical techniques.

The example given above is not a special case: when $\beta = 1$ and $\alpha = -4$, our analytic approach (by means of homotopy-Padé acceleration) gives

$$F''(0) = -0.39181, 57814, 02$$

for the 1st branch of solution, and

$$F''(0) = -0.39181, 57813, 92$$

for the 2nd one, respectively. The corresponding difference of $F''(0)$ is even smaller, i.e. less than 10^{-11}. So, when $\beta=1$ and $\alpha=-10, -100, -1000$ and so on, it becomes harder and harder to find multiple solutions by means of numerical techniques.

It seems that many non-linear BVPs have the similar property mentioned above. For example, Liao [53] investigated a simple non-linear BVP, which has an infinite number of closed-form analytic solutions. The corresponding IVPs of this non-linear BVP are very unstable: these solutions are rather "sensitive" to the second-order derivative at the boundary, and besides the difference of the second derivatives of two obviously different solutions might be less than 10^{-10}, 10^{-100}, 10^{-1000} or even smaller. Therefore, it seems impossible to find out all of these solutions by means of current numerical methods, as pointed out by Liao [53], mainly because most of our today's computers have no such a high accuracy as 10^{-1000}.

So, by means of the analytic approaches based on the HAM [31,32], we can find multiple solutions of non-linear problems, even if they are very close. This indicates that the HAM [31,32] is a useful tool to solve non-linear BVPs with multiple solutions. It also illustrates that, for some non-linear problems with multiple solutions, analytic techniques might be more powerful than numerical ones.

3. Conclusions and discussions

In this paper, the steady-state boundary-layer flows over a permeable stretching sheet are investigated by means of an analytic method for non-linear problems, namely the homotopy analysis method (HAM) [31,32]. Two analytic approaches are proposed, which are rather similar to those described in [50]. To shorten the length of this paper, we neglect all formulas here. For details, refer to Liao [50].

Two branches of convergent series solutions are obtained. One of them agrees well with the known numerical solutions. The other is new and has never been reported in general cases. The new branch of solutions exists in case of $\beta > \frac{1}{2}$ for both impermeable and permeable stretching sheet. The entrainment velocity $v(x, +\infty)$ of the new branch of solutions is always less than that of the known ones. For permeable sheet with sufficiently large suction ($\alpha > 0$) of mass flux, the shear stress of the new branch of solutions is considerably larger than that of the known one. However, for impermeable sheet ($\alpha = 0$) and permeable sheet with injection ($\alpha < 0$) or small suction, the shear stress on the sheet and the velocity profiles of two branches of solutions are rather close. It is found that, for impermeable sheet and permeable sheet with injection or small suction, the difference of $F''(0)$ of two branches of solutions is so small and the corresponding BVPs are so unstable that the new branch of solutions were neglected even by means of the numerical techniques. This paper illustrates that the HAM is more powerful than numerical methods for some non-linear problems with multiple solutions.

For impermeable sheet ($\alpha = 0$) and permeable plate with injection ($\alpha < 0$) of mass flux, it is rather hard to distinguish the profiles of $F'(\xi)$ of the two branches of solutions, as shown in Fig. 17 in case of $\alpha = -3$ and $\beta = 1$. Besides, the difference of $F(\xi)$ of two branches of solutions is also hard to distinguish in a considerably large region near the sheet but becomes obvious out of that region, as shown in Fig. 16. Traditionally, the boundary-layer thickness is defined by 1% of the mainstream velocity. According to this definition, the boundary-layer thickness in case of $\alpha = -3$ and $\beta = 1$ is about 22.5. In this region, it is hard to distinguish $F(\xi)$ of two branches of solutions, as shown in Fig. 16. Thus, according to (9) and (10), the horizontal velocity $u(x, y)$ of two branches of solutions are almost the same in the *whole* spatial region, and the vertical velocity $v(x, y)$ of two branches of solutions are nearly the same in the *whole* boundary-layer thickness. However, the vertical velocities $v(x, +\infty)$ of two branches of solutions at infinity are obviously different indeed, as shown in Fig. 16. Note that, physically, the vertical velocity $v(x, y)$ is much smaller than the horizontal velocity $u(x, y)$, because of $v(x, y)/u(x, y) \sim \sqrt{v}$. Thus, for impermeable sheet ($\alpha = 0$) and permeable sheet with injection of mass flux ($\alpha < 0$), the difference of the velocity fields of the two branches of solutions is small from the physical point of view, although there exist indeed two branches of solution from the mathematical points of view. As mentioned before, for impermeable sheet ($\alpha = 0$) and permeable sheet with injection ($\alpha < 0$) of mass flux, the corresponding IVPs are rather unstable. But, it is not clear whether the injection of mass flux increases the instability of the flows or not.

On the other side, for the stretching permeable plate with the large suction of mass flux, both of the horizontal velocity $u(x, y)$ and vertical velocity $v(x, y)$ of two branches of solutions are obviously different even near the plate. For example, when $\alpha = 10$, the velocity $u(x, y)$ of the 1st branch of solutions either decreases monotonously or has overshooting, as shown in Figs. 1 and 7. However, the new branch of solutions has reversed velocity $u(x, y)$, as shown in Figs. 2 and 8. Therefore, for the stretching permeable plate with large suction of mass flux, the two branches of solutions are obviously different, not only mathematically but also physically.

It should be emphasized that, using the very accurate analytic results of $F''(0)$ obtained by our HAM approaches and by means of the Runge–Kutta's method, we obtain the corresponding numerical results, which agree very well with our convergent series solutions. Thus, mathematically, there is no doubt that the new branch of solutions indeed exist. However, some experiments are necessary to prove its physical existence.

Many non-linear problems have multiple solutions [16,17]. For example, all related heat transfer problems over a stretching permeable sheet have multiple solutions. Besides, there exist many physically different but mathematically identical non-linear problems. All of these multiple solutions could be found in a similar way. Thus, the HAM [31,32] provides us a new approach to get convergent series of multiple solutions of non-linear problems.

It is a pity that many things are not very clear now and thus need be further investigated in details. For example, it is unknown whether the new branch of solutions is stable or not. Can we separate multiple solutions of the unsteady boundary-layer flows over a permeable sheet, which are very close to each other? Does the injection of mass flux through the stretching permeable sheet increases the instability of the flows? Does the new branch of solutions still exist if the boundary-layer equations (1) and (2) are replaced by the exact Navier–Stokes equations, especially for the impermeable sheet ($\alpha = 0$) and permeable sheet with injection ($\alpha < 0$) or small suction of mass flux?

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