Available online at www.sciencedirect.com
ScienceDirect
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
Journal of Computational and Applied Mathematics 212 (2008) 320–330
www.elsevier.com/locate/cam

An explicit solution of the large deformation of a cantilever beam under point load at the free tip

Ji Wang[a], Jian-Kang Chen[a], Shijun Liao[b,*]

[a]Mechanics and Materials Science Research Center, School of Engineering, Ningbo University, 818 Fenghua Road, Ningbo, Zhejiang 315211, China

[b]School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiaotong University, Shanghai 200030, China

Received 30 July 2006; received in revised form 20 November 2006

Abstract

The large deformation of a cantilever beam under point load at the free tip is investigated by an analytic method, namely the homotopy analysis method (HAM). The explicit analytic formulas for the rotation angle at the free tip are given, which provide a convenient and straightforward approach to calculate the vertical and horizontal displacements of a cantilever beam with large deformation. These explicit formulas are valid for most practical problems, thus providing a useful reference for engineering applications. The corresponding Mathematica code is given in the Appendix.

MSC: 74K10; 34B15; 35C10; 74G10

Keywords: Large deformation; Beam; Nonlinear; Deflection; Homotopy analysis method

1. Introduction

We consider the large deformation of a cantilever beam under point load at the free tip as shown in Fig. 1. The bending equation of a uniform cross-section beam with large deformation is [7]

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{P}{\mathrm{EI}}(l_1 - x), \quad \theta(0) = 0, \quad \theta'(l) = 0, \tag{1}$$

where s is the arc-coordinate of the neutral axis of the beam, x is the horizontal coordinate from the fixed end, l is the length of the beam, P the point load at the free tip, EI is the bending stiffness of the beam, θ is the rotation of cross-section of the beam, and l_1 is the horizontal distance of two ends and is unknown. The axial elongation of the beam is much smaller than the deflection at the free tip, and therefore is neglected.

Differentiating the equation with respect to s and then using the dimensionless variables $\xi = s/l$, the original equation becomes

$$\theta'' + \alpha \cos\theta = 0, \quad \theta(0) = 0, \quad \theta'(1) = 0, \tag{2}$$

* Corresponding author.
E-mail addresses: wangji@nbu.edu.cn (J. Wang), chenjiankang@nbu.edu.cn (J.-K. Chen), sjliao@sjtu.edu.cn (S. Liao).

0377-0427/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2006.12.009

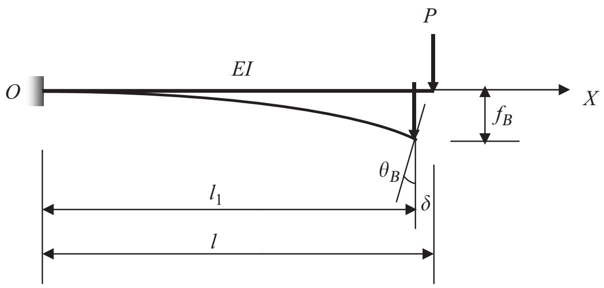

Fig. 1. Large deformation of a beam under point load at free tip.

where the prime denotes the differentiation with respect to ξ, and

$$\alpha = \frac{Pl^2}{\mathrm{EI}}.$$

The rotation angle of cross-section plane at tip is denoted by $\theta_B = \theta(1)$. From [7,20], the exact vertical displacement of the free tip is given by

$$f_B = l - 2[E(\mu) - E(\phi, \mu)]\sqrt{\frac{EI}{P}},$$

where $E(\mu)$ is the complete elliptic integral of the second kind, $E(\phi, \mu)$ the elliptic integral of the second kind, and

$$\mu = \sqrt{\frac{1 + \sin\,\theta_B}{2}}, \quad \phi = \arcsin\left(\frac{1}{\sqrt{2}\mu}\right). \tag{3}$$

Consequently, the dimensionless vertical displacement at free tip is given by

$$\frac{f_B}{l} = 1 - \frac{2}{\sqrt{\alpha}}[E(\mu) - E(\phi, \mu)]. \tag{4}$$

Besides, we have

$$\frac{l_1}{l} = \sqrt{\left(\frac{2EI}{Pl^2}\right)\sin\theta_B} = \sqrt{\frac{2\sin\theta_B}{\alpha}}. \tag{5}$$

Thus, the dimensionless horizontal displacement of the free tip is given by

$$\frac{\delta_B}{l} = \frac{l - l_1}{l} = 1 - \sqrt{\frac{2\sin\theta_B}{\alpha}}. \tag{6}$$

Clearly, the vertical and horizontal displacements f_B and δ_B can be easily calculated as long as θ_B is known.

For infinitesimal deformation, it is enough to use the linear equation

$$\theta'' + \alpha = 0, \quad \theta(0) = 0, \quad \theta'(1) = 0. \tag{7}$$

The corresponding solution is

$$\theta(\xi) = \frac{\alpha}{2}(2 - \xi)\xi, \tag{8}$$

which gives the linear result

$$\theta_B = \frac{\alpha}{2}. \tag{9}$$

If the large deformation is considered, one has to solve a nonlinear algebraic equation

$$\sqrt{\alpha} = K(\mu) - F(\phi, \mu), \tag{10}$$

where μ and ϕ are defined by (3), and $K(\mu)$ is the complete elliptic integral of the first kind, and $F(\phi, \mu)$ is the elliptic integral of the first kind, respectively. In general, one has to use numerical methods to solve this transcendental equation. Therefore, the above expressions of θ_B, l_1 and δ are *not* explicit, and are difficult to evaluate. Clearly, it would be interesting and convenient if an explicit and accurate expression for θ_B can be given. The traditional analytical methods for solving nonlinear equations such as perturbation [16], Adomian decomposition method [3], and δ-expansion method [10], in addition to the elliptical integrals mentioned above, can not provide solutions which are accurate, convergent, and simple for this widely known problem. On the other hand, problems similar to this one are widely encountered in many engineering and scientific fields with important applications. A recent development on solving strongly nonlinear problems arising from many disciplines, the homotopy analysis method (HAM) [11–15,4–6,8,17–19,1,2], which is a powerful technique in obtaining accurate solutions that cannot be given otherwise by perturbation and other methods, is applied to demonstrate its simplicity, effectiveness, and accuracy for this widely-known problem in structural analysis.

2. Solutions by the HAM

We now solve this nonlinear boundary value equation (2) by the HAM, and give an explicit formula for θ_B.

First of all, it is obvious that $\theta(\xi)$ can be expressed by the power series of ξ, i.e.

$$\theta(\xi) = \sum_{k=1}^{+\infty} a_k \xi^k, \tag{11}$$

where a_k is a constant coefficient. Obviously, (8) given by the linear equation is a good initial guess. For this reason, we choose

$$\theta_0(\xi) = \frac{\alpha}{2}(2-\xi)\xi, \tag{12}$$

as the initial guess of $\theta(\xi)$. Based on (2), we define the nonlinear operator

$$\mathscr{N}[\psi(\xi;q), q] = \frac{\partial^2 \psi(\xi;q)}{\partial \xi^2} + \alpha \cos[q\psi(\xi;q)]. \tag{13}$$

Let

$$\mathscr{L}\theta = \frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} \tag{14}$$

be an auxiliary linear operator, $q \in [0,1]$ an embedding parameter, and $\hbar$ a non-zero auxiliary parameter. Next, we construct such a homotopy

$$\mathscr{H}[\psi(\xi;q), q] := (1-q)\mathscr{L}[\psi(\xi;q) - \theta_0(\xi)] - q\hbar\mathscr{N}[\psi(\xi;q), q]. \tag{15}$$

Clearly, when $q = 0$, one has

$$\mathscr{H}[\psi(\xi;0), 0] := \mathscr{L}[\psi(\xi;0) - \theta_0(\xi)].$$

If $q = 1$, it becomes

$$\mathscr{H}[\psi(\xi;1), 1] := -\hbar\mathscr{N}[\psi(\xi;1), 1] = -\hbar\left(\frac{\partial^2\psi(\xi;1)}{\partial\xi^2} + \alpha\cos[\psi(\xi;1)]\right).$$

Thus, by enforcing

$$\mathscr{H}[\psi(\xi;q), q] = 0,$$

we have a family of equations

$$(1-q)\mathscr{L}[\psi(\xi;q)-\theta_0(\xi)] = q\hbar\mathscr{N}[\psi(\xi;q),q], \tag{16}$$

subject to the related boundary conditions

$$\psi(0;q)=0, \quad \psi'(1;q)=0, \tag{17}$$

where the prime denotes the differentiation with respect to ξ.

Obviously, when $q=0$, it holds

$$\psi(\xi;0)=\theta_0(\xi). \tag{18}$$

When $q=1$, Eqs. (16) and (17) are equivalent to the original (2), provided

$$\psi(\xi;1)=\theta(\xi). \tag{19}$$

Thus, as q increases from 0 to 1, $\psi(\xi;q)$ varies smoothly from the known initial guess $\theta_0(\xi)$ to the solution $\theta(\xi)$ of (2).

By expanding $\psi(\xi;q)$ into Taylor series of the embedding parameter q and using (18), we have

$$\psi(\xi;q)=\theta_0(\xi)+\sum_{n=1}^{+\infty}\theta_n(\xi)q^n, \tag{20}$$

where

$$\theta_n = \frac{1}{n!}\left.\frac{\partial^n\psi(\xi;q)}{\partial q^n}\right|_{q=0}.$$

Assume that $\hbar$ is so properly chosen that the series (20) is convergent at $q=1$. Thus, using (19), we have the series

$$\theta(\xi)=\theta_0(\xi)+\sum_{n=1}^{+\infty}\theta_n(\xi). \tag{21}$$

The governing equations of θ_n can be deduced from the zeroth-order deformation equations (16) and (17). Now substituting (20) into (16), and differentiating (16) n times with respect to the embedding parameter q, then dividing by $n!$, and finally setting $q=0$, we have the nth-order deformation equation

$$\mathscr{L}[\theta_n(\xi)-\chi_n\theta_{n-1}(\xi)]=\hbar R_n(\theta_0,\theta_1,\ldots,\theta_{n-1}), \tag{22}$$

subject to boundary conditions

$$\theta_n(0)=0, \quad \theta_n'(1)=0, \tag{23}$$

where

$$R_n=\frac{1}{(n-1)!}\left.\frac{\partial^{n-1}\mathscr{N}[\psi(\xi;q),q]}{\partial q^{n-1}}\right|_{q=0}, \tag{24}$$

and

$$\chi_k=\begin{cases}0, & k\leqslant 1,\\ 1, & k>1.\end{cases} \tag{25}$$

It should be emphasized that, as pointed by Sajid et al. [18], one can obtain exactly the *same* equations as (22)–(24) by directly substituting the series (20) into the zero-order deformation equations (16)–(17) and then expanding them in Taylor series about q and finally equating the coefficients of the like-power of q, no matter one regards q as a small parameter or not. Using (13) and (24), we have

$$R_1 = \theta_0''(\xi) + \alpha, \tag{26}$$

$$R_2 = \theta_1''(\xi), \tag{27}$$

$$R_3 = \theta_2''(\xi) - \frac{\alpha}{2}\theta_0^2(\xi), \tag{28}$$

$$R_4 = \theta_3''(\xi) - \alpha\theta_0(\xi)\theta_1(\xi), \tag{29}$$

$$R_5 = \theta_4''(\xi) - \alpha\left[\theta_0(\xi)\theta_2(\xi) + \frac{\theta_1^2(\xi)}{2} - \frac{\theta_0^4(\xi)}{24}\right], \tag{30}$$

$$\vdots$$

In this way, we can easily obtain the right-hand side term R_n by means of the symbolic calculation software such as Mathematica, Matlab and so on. Thereafter, it is straightforward to solve the linear high-order deformation equations (22) and (23). The corresponding Mathematica code is given Appendix,[1] which gives the 10th-order HAM approximation in a few seconds! Noting that, different from *all* other analytical techniques such as the perturbation method, Adomian decomposition method, δ-expansion method, and so on, the solutions given by the HAM contains an auxiliary parameter $\hbar$, which can be used to control and adjust the convergence region and rate of the HAM solution series. For example, the 10th-order approximation of θ_B with $\hbar = -1$,

$$\theta_B = 0.5\alpha - 0.0458333\alpha^3 + 0.00893105\alpha^5 - 0.00223058\alpha^7 + 9.092700^{-6}\alpha^9, \tag{31}$$

is convergent in the region $0 \leqslant \alpha \leqslant 1.5$, and the 10th-order approximation of θ_B with $\hbar = -\frac{1}{2}$,

$$\begin{aligned}\theta_B = {} & 0.5\alpha - 0.0456543\alpha^3 + 8.02901 \times 10^{-3}\alpha^5 \\ & - 8.59983 \times 10^{-4}\alpha^7 + 2.28499 \times 10^{-6}\alpha^9,\end{aligned} \tag{32}$$

is convergent in the range $0 \leqslant \alpha \leqslant 2$, as shown in Figs. 2 and 3. The 30th-order approximation of θ_B with $\hbar = -1/10$, i.e.

$$\begin{aligned}\theta_B = {} & 0.5\alpha - 4.34347 \times 10^{-2}\alpha^3 + 6.83591 \times 10^{-3}\alpha^5 - 1.09091 \times 10^{-3}\alpha^7 \\ & + 1.49396 \times 10^{-4}\alpha^9 - 1.62863 \times 10^{-5}\alpha^{11} + 1.34028 \times 10^{-6}\alpha^{13} \\ & - 7.90696 \times 10^{-8}\alpha^{15} + 3.12877 \times 10^{-9}\alpha^{17} - 7.51666 \times 10^{-11}\alpha^{19} \\ & + 9.27641 \times 10^{-13}\alpha^{21} - 4.2676 \times 10^{-15}\alpha^{23} + 3.43512 \times 10^{-18}\alpha^{25} \\ & - 4.92252 \times 10^{-23}\alpha^{27} + 1.75157 \times 10^{-32}\alpha^{29},\end{aligned} \tag{33}$$

is convergent in the range $0 \leqslant \alpha \leqslant 4$, as shown in Figs. 2 and 3. Obviously, $\hbar = -1$ gives the smallest convergence region, and the convergence region increases as $\hbar$ changes from -1 to 0. Note that, as pointed by Sajid et al. [18] and Abbasbandy [1], results given by the so-called "homotopy perturbation method" [9] are exactly the *same* as those given by the HAM when one sets $\hbar = -1$, because the "homotopy perturbation method" (proposed in 1999) is only a special case of the HAM (propounded in 1992) when $\hbar = -1$. Note that, the results given by the so-called "homotopy perturbation method" corresponds to the *worst* one among all of our solutions given by the HAM.

[1] To get free electronic version of this code, please email a message to the corresponding author.

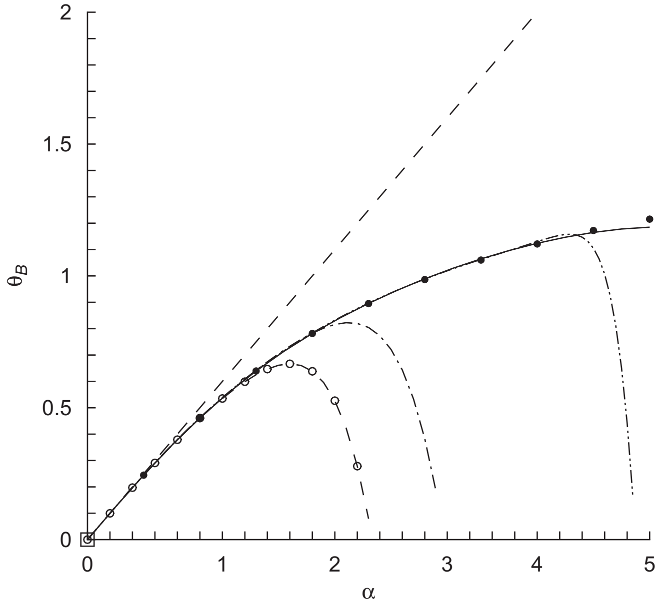

Fig. 2. The rotation angle at the free end θ_B vs. α. Dashed line: linear result $\theta_B = \alpha/2$; dashed line with open circles: HAM solution (31); dash-dotted line: HAM result (32); dash-dot-dotted line: HAM result (33); solid line: HAM result (34); and filled circle: exact solution.

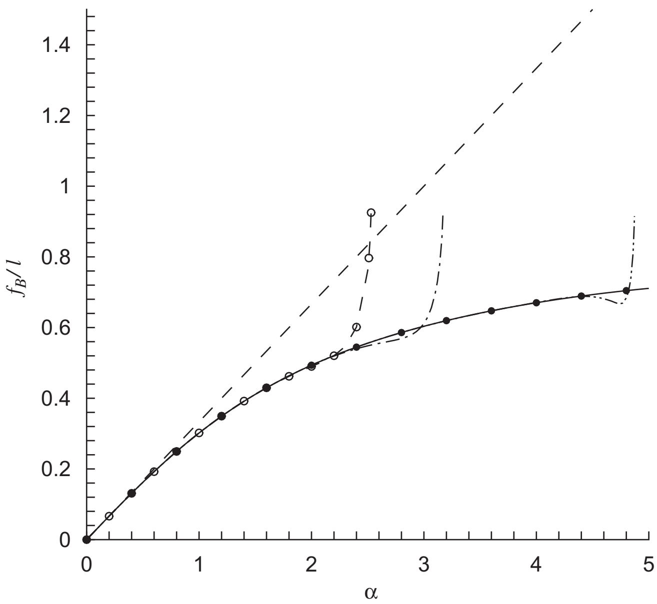

Fig. 3. The dimensionless vertical displacement f_B/l at the free tip. Dashed line: linear result $f_B/l = \alpha/3$; dashed line with open circles: HAM solution based on (31); dash-dotted line: HAM solution based on (32); dash-dot-dotted line: HAM result based on (33); solid line: HAM result based on (34); and filled circle: exact solution.

Using the so-called homotopy-Pàde acceleration technique [13], the convergence region can be greatly enlarged. For example, when $\hbar = -\frac{1}{10}$, the [8,8] homotopy-Páde approximation of θ_B, i.e.

$$\theta_B = \frac{\alpha}{2}\frac{f(\alpha)}{g(\alpha)}, \tag{34}$$

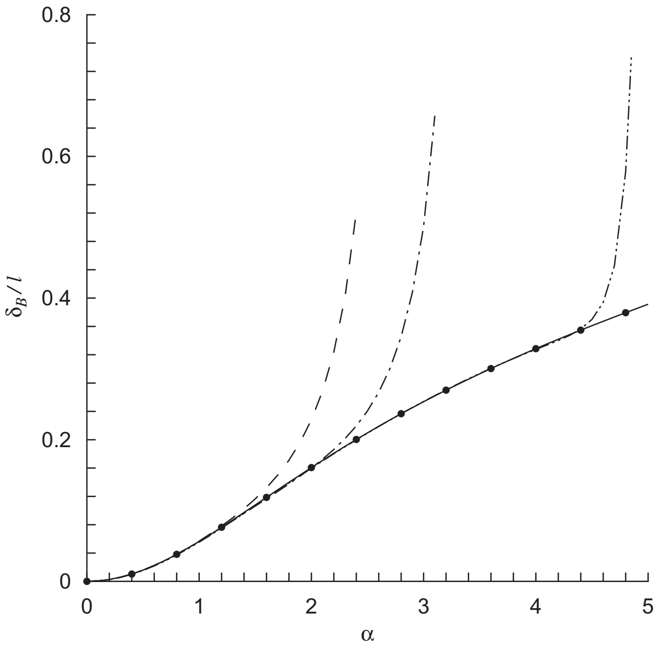

Fig. 4. The dimensionless horizontal displacement δ_B/l at the free tip. Dashed line: HAM solution based on (31); dash-dotted line: HAM solution based on (32); dash-dot-dotted line: HAM result based on (33); solid line: HAM result based on (34); and filled circle: exact solution.

is convergent in the range $0 \leqslant \alpha \leqslant 5$, as shown in Fig. 2, where

$$\begin{aligned} f(\alpha) = {} & 1 + 3.98575 \times 10^{-2}\alpha^2 - 5.41174 \times 10^{-2}\alpha^4 + 5.72575 \times 10^{-3}\alpha^6 \\ & + 3.79533 \times 10^{-4}\alpha^8 - 8.87896 \times 10^{-6}\alpha^{10} + 2.63041 \times 10^{-8}\alpha^{12} \\ & - 1.51429 \times 10^{-11}\alpha^{14} - 2.29142 \times 10^{-15}\alpha^{16} \\ & - 3.45006 \times 10^{-21}\alpha^{18} - 7.00678 \times 10^{-28}\alpha^{20}, \end{aligned} \tag{35}$$

$$\begin{aligned} g(\alpha) = {} & 1 + 0.131524\alpha^2 - 5.99231 \times 10^{-2}\alpha^4 + 2.34466 \times 10^{-3}\alpha^6 \\ & + 9.90299 \times 10^{-4}\alpha^8 - 1.37001 \times 10^{-5}\alpha^{10} + 3.44172 \times 10^{-8}\alpha^{12} \\ & - 1.45098 \times 10^{-11}\alpha^{14} - 2.26721 \times 10^{-15}\alpha^{16} \\ & - 3.50731 \times 10^{-21}\alpha^{18} - 7.00678 \times 10^{-28}\alpha^{20}. \end{aligned} \tag{36}$$

We can clearly see from Figs. 2 to 4 that the explicit expression of θ_B given in (32)–(33) are accurate for a large range of dimensionless parameter α. Especially, (32) is very simple but accurate for $\alpha \leqslant 2$. From Fig. 3, $\alpha = 2$ corresponds to the vertical displacement at the free end almost as half of the original length of the beam. Clearly, the explicit, simple expression (32) of θ_B is enough for nearly all kinds of practical engineering problems. For even larger parameter α, the explicit expressions (33) and (34) are straightforward to calculate.

3. Discussions

The linear solution is valid only for smaller α. For larger α, the nonlinear effect of the equation is noticeable, and it cannot be neglected as a result. Using the HAM, we obtain a family of *explicit* solutions for the large deformation of a cantilever beam with a point load at the free end. It is shown that our *explicit* solutions for different parameters and solution schemes agree well with the exact solutions from elliptical integrals. However, our solutions given by the

HAM, especially (32), is simple and straightforward in comparison to the exact solution from the elliptical integrals which requires to solve a transcendental equation. Clearly, our solution is easy to calculate with the explicit polynomial expressions. This is especially convenient for practical engineering applications with minimum requirements on calculation and computation.

The HAM is a new technique to solve nonlinear differential equations which are common in many engineering and scientific fields. Although many practical problems involving nonlinear equations have been successfully solved by traditional methods such as the perturbation method, there are stringent requirements for applications of these methods which may fail especially when the nonlinear effect is strong. In this case, the HAM is an excellent alternative for problems which cannot be effectively solved before. In fact, the HAM [11–15,4–6,8,17–19,1,2] is capable of solving a wide range of nonlinear problems, particularly when the nonlinearity is strong. By solving the exact bending equation of a cantilever beam, we demonstrated the usefulness and effectiveness of the HAM to a class of equations in solid mechanics. For example, it is easy to apply this analytic approach to solve large deformation of a cantilever beam under distributed load or periodic point loads. Our experiences from this study and related researches show that the HAM can also be utilized to solve many strongly nonlinear problems in structural analysis.

Acknowledgments

This work is supported by a grant from The Ning Bo Residents' Association (Hong Kong) Limited. The authors would like to thank the anonymous reviewers for their valuable comments and suggestions.

Appendix A.

A.1. Mathematica code

```
(*********************************************************************************
Mathematica code for large deformation of a beam under point load at the free tip.
To get free electronic version of this code, please email a message to the authors
*********************************************************************************)
<<Calculus`Pade`;
<<Graphics`Graphics`;
(*********************************************************************************
Define chi[m] given by (25)
*********************************************************************************)
chi[ m_ ] := If[ m <= 1, 0, 1 ];
(*********************************************************************************
Define R_m given by (24)
*********************************************************************************)
GetR[ Norder_ ] := Module[{ temp, EQ, q, S, Szz},
Print[" — Get the term R[n] — "];
S = Sum[ s[ k ] * q ^ k, { k, 0, Norder }];
Szz = Sum[ szz[ k ] * q ^ k, {k, 0, Norder }];
EQ[1] = Szz + alpha * Cos[ q * S ];
   R[1] = EQ[ 1 ] /. q -> 0;
For[ n = 1, n <= Norder -1, n = n + 1,
Print[ " n = ", n + 1 ];
EQ[ n + 1 ] = D[ EQ[ n ], q ];
   R[ n + 1 ] = EQ[ n + 1 ]/n! /. q -> 0 //Expand;
];
];
```

```
(*******************************************************************************
This module defines the right-hand side term of Eq. (22)
*******************************************************************************)
GetRHS[ m_ ] := Module[{ }, RHS[ m ] = Expand[ hbar * R[ m ] ]; ];
(*******************************************************************************
Define initial guess (12)
*******************************************************************************)
GetInitial = Module[{ },
   s[ 0 ] = alpha * z * ( 1 −z/2 );
   S[ 0 ] = s[ 0 ];
Smax[ 0 ] = S[ 0 ] /. z → 1;
];
(*******************************************************************************
This module defines the auxiliary linear operator (14)
*******************************************************************************)
   L[ f_ ] :=D[f, { z, 2 } ];
(*******************************************************************************
Define inverse operator L^-1 of the equation L[u] = f
*******************************************************************************)
Linv[ f_ ] :=Module[{ temp, solution },
temp[ 1 ] = Integrate[ f, z ]; temp[ 2 ] = − temp[ 1 ] /. z → 1;
temp[ 3 ] = temp[ 1 ] + temp[ 2 ] //Expand;
solution = Integrate[ temp[ 3 ], z ];
Expand[ solution ]
];
(*******************************************************************************
The property of the inverse operator of L^-1 : L^-1[a f + b g] = a L^-1[f] + b L^-1[g]
*******************************************************************************)
Linv[ p_Plus ] :=Map[ Linv, p ];
Linv[ c_ * f_ ] :=c * Linv[ f ] /; FreeQ[ c, z ];
(*******************************************************************************
This module gives special solutions of the high-order deformation Eq. (22)
*******************************************************************************)
GetSPECIAL[ m_ ] :=Module[ { temp },
temp = Expand[ RHS[ m ] ];
SPECIAL = Linv[ temp ];
];
(*******************************************************************************
Define the exact solution by means of solving Eq. (10)
*******************************************************************************)
Exact[ z_ ] :=Module[{ temp, k, eq, phi },
k = (1 + Sin[ x ] )/2;
phi = ArcSin[1 / Sqrt[ 2 * k ]];
eq = EllipticK[ k ] − EllipticF[ phi, k ] − Sqrt[ z ];
temp[ 1 ] = FindRoot[ eq == 0, { x, 0.01 } ];
temp[ 2 ] = x /. temp[ 1 ];
Expand[ temp[ 2 ] ]
];
(*******************************************************************************
Define δ_b by (6)
*******************************************************************************)
deltaB[ alpha_ , thetaB_ ] :=1− Sqrt[ 2 * Sin[ thetaB ]/alpha ];
```

```
(*********************************************************************************
Define f_b by (4)
*********************************************************************************)
fB[ alpha_ , thetaB_ ] :=Module[{ temp, mu, phi },
mu = ( 1 + Sin[ thetaB ] )/2;
phi = ArcSin[ 1 / Sqrt[ 2 * mu ] ];
temp = 1 − 2/Sqrt[ alpha ]*( EllipticE[ mu ] − EllipticE[ phi, mu ]);
Expand[ temp ]
];
(*********************************************************************************
This module gives the [m,n] homotopy-Pàde approximation of a series
For details, please refer to §2.3.7 of Liao's book [6]: Beyond Perturbation, CRC Press.
*********************************************************************************)
hp[ F_ , m_ , n_ ] :=Block[ {i, k, dF, temp, q },
dF[ 0 ] = F[ 0 ];
For[ k = 1, k <= m + n, k = k + 1, dF[ k ] = Expand[ F[ k ] − F[ k − 1 ] ] ];
temp = dF[ 0 ] + Sum[ dF[ i ] * q ∧ i, {i, 1, m+n }];
Pade[temp, {q, 0, m, n }] /. q → 1
];
(*********************************************************************************
Main code
*********************************************************************************)
ham[ m0_ , m1_ ] :=Module[ {temp , k , variable },
Print[" — Get the kth-order HAM approximation — "];
For[ k = Max[ 1 , m0 ] , k < = m1, k = k + 1,
Print[" k = ", k ];
szz[ k − 1 ] = D[ s[ k − 1 ], { z , 2 }];
GetRHS[ k ];
GetSPECIAL[ k ];
  s[ k ] = SPECIAL + chi[ k ] * s[ k − 1 ] // Expand;
  S[ k ] = S[ k − 1 ] + s[ k ] //Expand;
Smax[ k ] = S[ k ] /. z → 1 //Expand;
If[ PRN == 1,
Print[" ThetaB = ", N[ Smax[ k ] , 20 ] ,
" delta = ", N[ Smax[ k ] − chi[ k ] * Smax[ k − 1 ], 20]];
];
];
Print["Successful !"];
];
(*********************************************************************************
Would you like to print more information to check the convergence ?
1 corresponding to YES
0 corresponding to NO
*********************************************************************************)
PRN = 0;
(*********************************************************************************
Parameters definitions
*********************************************************************************)
hbar = −1/2;
alpha = a;
(* Get 10th-order HAM approximation *)
```

```
GetR[ 10 ];
ham[ 1 , 10 ];
hp[ Smax , 4 , 4 ];
```

References

[1] S. Abbasbandy, The application of the homotopy analysis method to nonlinear equations arising in heat transfer, Phy. Lett. A 360 (2006) 109–113.
[2] S. Abbasbandy, The application of the homotopy analysis method to solve a generalized Hirota–Satsuma coupled KdV equation, Phy. Lett. A 361 (2007) 478–483.
[3] G. Adomian, Nonlinear stochastic differential equations, J. Math. Anal. Appl. 55 (1976) 441–452.
[4] S. Asghar, M.M. Gulzar, M. Ayub, Effect of partial slip on flow of a third grade fluid, Acta Mech. Sinica 22 (2006) 195–198.
[5] M. Ayub, A. Rasheed, T. Hayat, Exact flow of a third grade fluid past a porous plate using homotopy analysis method, Internat. J. Eng. Sci. 41 (2003) 2091–2103.
[6] J. Cheng, S.J. Liao, I. Pop, Analytic series solution for unsteady mixed convection boundary layer flow near the stagnation point on a vertical surface in a porous medium, Transp. Porous Media 61 (2005) 365–379.
[7] J.M. Gere, S.P. Timoshenko, Mechanics of Materials, PWS Publishing Company, Boston, 1997.
[8] T. Hayat, M. Khan, S. Asghar, Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid, Acta Mech. 168 (2004) 213–232.
[9] J.-H. He, Homotopy perturbation techique, Comput. Methods Appl. Mech. Eng. 178 (1999) 257–262.
[10] A.V. Karmishin, A.T. Zhukov, V.G. Kolosov, Methods of Dynamics Calculation and Testing for Thin-walled Structures, Mashinostroyenie, Moscow, 1990 (in Russian).
[11] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, Ph.D. Thesis, Shanghai Jiao Tong University, 1992.
[12] S.J. Liao, An explicit, totally analytic approximation of Blasius viscous flow problems, Internat. J. Non-Linear Mech. 34 (4) (1999) 759–778.
[13] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman & Hall/CRC Press, Boca Raton, 2003.
[14] S.J. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 147 (2004) 499–513.
[15] S.J. Liao, Series solutions of unsteady boundary-layer flows over a stretching flat plate, Stud. Appl. Math. 117 (3) (2006) 2529–2539.
[16] A.H. Nayfeh, Perturbation Methods, Wiley, New York, 2000.
[17] M. Sajid, T. Hayat, S. Asghar, On the analytic solution of the steady flow of a forth grade fluid, Phy. Lett. A 355 (2006) 18–26.
[18] M. Sajid, T. Hayat, S. Asghar, Comparison between the HAM and HPM solutions of tin film flows of non-Newtonian fluids on a moving belt, Nonlinear Dynam., in press.
[19] Y.R. Shi, et al., Application of the homotopy analysis method to solve nonlinear evalution equations, Acta Phys. Sinica 55 (2006) 1555–1559.
[20] T.X. Yu, L.Z. Zhang, Theory of Plastic Bending and its Applications, Science Publishing House, Beijing, 1992.