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Some notes on the general boundary element method for highly nonlinear problems

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Abstract

We give a short review of the so-called general boundary element method for strongly nonlinear problems in heat and viscous flow and a brief discussion about opportunity and challenge of the boundary element method as a numerical tool, compared with other numerical techniques.

Keywords: General boundary element method; Nonlinear partial differential equations; Homotopy analysis method

1. Introduction

The boundary element method (BEM) [1,2] is rather efficient to solve linear problems governed by homogeneous equation

$$\mathscr{L}u = 0, \tag{1}$$

subject to linear boundary conditions, provided the fundamental solution of the linear operator $\mathscr{L}$ *exists* and can be *found*. In this case the BEM replaces the original linear differential equations with a set of integral equations defined on the boundary *only* and hence reduces the dimension of the problem by one. This is the major advantage of the BEM over other domain numerical

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techniques such as finite difference method (FDM), finite element method (FEM), finite volume method (FVM) and so on, which in principle require discretization of full domain.

However, this advantage of the BEM is lost even for linear problems governed by inhomogeneous linear equation

$$\mathscr{L}u = f, \tag{2}$$

where f is a known function, often regarded as body force term. This is mainly because additional domain-integral terms appear. Domain integrals can be evaluated by cell integration approach [3], which however makes the BEM lose its boundary-only nature, resulting in a numerical scheme several orders of magnitude more time consuming than other domain techniques mentioned above.

This disadvantage of the BEM becomes more prominent for nonlinear equations

$$\mathscr{A}u = f, \tag{3}$$

where $\mathscr{A}$ is a nonlinear operator. Generally speaking, the traditional nonlinear BEM schemes [3–6] are based on iteration techniques, which regard nonlinear terms as"pseudo-body force" moved to the right-hand side of differential equations, i.e.

$$\mathscr{L}_0 u = f - \mathscr{N}_0 u, \tag{4}$$

where $\mathscr{L}_0$ is a linear operator, $\mathscr{N}_0$ is a nonlinear operator and

$$\mathscr{A} = \mathscr{L}_0 + \mathscr{N}_0. \tag{5}$$

If the fundamental solution of the linear operator $\mathscr{L}_0$ *exists* and can be *found*, the right-hand side term

$$f - \mathscr{N}_0 u$$

can be regarded as a known "pseudo-body force" term and the BEM can be employed to solve the linear inhomogeneous equation (4) at each iteration. Obviously, domain integrals appear due to the right-hand side term

$$f - \mathscr{N}_0 u,$$

which varies at different step of iteration.

Some techniques are developed to overcome the numerical inefficiency of the BEM due to domain integrals. For some special inhomogeneous linear equations it is possible to find a particular solution, which can be employed to transform the domain integrals to boundary integrals [6–8]. However, the shortage of closed form of particular solution makes a general implementation difficult. More general approaches are the dual reciprocity method [9–14] and the domain decomposition technique [15,16]. The dual reciprocity method reduces domain integrals to boundary integrals by using basis functions such as radial basis functions [11], global shape functions [12], hybrid functions [13] and so on, to approximate the inhomogeneity. By means of the domain decomposition technique one divides the original domain into sub-regions, and in each of them, a full integral representation formula is given. The unknown functions and their derivatives are enforced to be continuous at the common interface between adjacent sub-regions. This leads to block sparse systems with one block for each sub-region and overlapping blocks when two sub-regions have a common interface, while matrices arising from a single domain formulation are fully populated. The dual reciprocity method and domain decomposition technique can be combined to increase the efficiency of the nonlinear BEM schemes [17–20]. To our knowledge, the efficiency and effectiveness of the dual reciprocity method and domain decomposition method for three-dimensional nonlinear problems are barely reported [21].

Second, the traditional nonlinear BEM schemes are generally valid only for weakly nonlinear problems. For example, many researchers applied the BEM to solve viscous flow problems governed by Navier–Stokes equations [16–20,22–28], among which the two-dimensional driven cavity flow might be the simplest. However, "the BEM approaches based on convective velocity free kernels fail to converge" for high and even moderate Reynolds number, and "neither severe underrelaxation nor Newton–Raphson iteration are remedies", as pointed out by Grigoriev and Dargush [16]. For two-dimensional driven cavity flow, many researchers cannot give BEM solutions at Reynolds number greater than 1000, except Grigoriev and Dargush [16] who applied poly-region technique and Oseen fundamental solution to obtain convergent results at Reynolds number up to 5000. Note that Ghia et al. [29] employed the finite difference method to obtain convergent steady-state solutions of the driven cavity flow with Reynolds number up to 7500. Note also that Oseen fundamental solution has physical meanings only for viscous flow problems and hence it seems difficult to apply Grigoriev and Dargush's [16] BEM approach to other sorts of problems with strong nonlinearity.

Third, it seems to be neglected that the above-mentioned nonlinear BEM approaches are based on two *assumptions*. First of all, there *must* exist such a linear operator $\mathscr{L}_0$ that $\mathscr{A}$ can be divided as (5). If all terms of a nonlinear differential equation are nonlinear, i.e. $\mathscr{A} = \mathscr{N}_0$, nothing is left on the left-hand side of Eq. (4), if we regard nonlinear terms as "pseudo-body force" and move them to the right-hand side of equations. Unfortunately some problems with very strong nonlinearity indeed do not contain any linear terms at all. Furthermore, the linear operator $\mathscr{L}_0$ must be simple enough that the fundamental solution of $\mathscr{L}_0$ *exists* and can be *found*. Otherwise, (4) cannot be replaced by integration equations. Thus, different from domain numerical techniques such as FDM, FEM, FVM and so on, the traditional nonlinear BEM approaches cannot be applied to highly nonlinear problems without linear terms.

In summary, the traditional nonlinear BEM schemes have the following limitations:

- Inefficiency: it is numerically inefficient due to domain integrals;
- Divergence: iteration often diverges for strongly nonlinear problems;
- Restricted application: it cannot be applied to nonlinear equations without any linear terms.

The last two ones are related to the ability of the BEM and thus greatly restrict the application regions of the BEM as a numerical tool.

Liao and his co-authors generalized the traditional nonlinear BEM schemes and proposed the so-called general boundary element method [30–40], which can overcome two of the above-mentioned limitations of the traditional nonlinear boundary element method. In the following sections the basic ideas of the general boundary element method are briefly described and some examples are given to show its validity.

2. Homotopy analysis method

The general boundary element method is based on an analytic technique for nonlinear problems, namely the homotopy analysis method [41–50]. Unlike perturbation techniques [51,52], the artificial small parameter method [53], the δ-expansion method [54] and the decomposition method [55], the homotopy analysis method *itself* provides us with a convenient way to *control* the convergence of approximation series and *adjust* convergence regions when necessary. Briefly speaking, the homotopy analysis method has the following advantages:

1. it is valid even if a given nonlinear problem does *not* contain any small/large parameters *at all*;
2. it *itself* can provide us with a convenient way to *control* the convergence of approximation series and *adjust* convergence regions when necessary;
3. it provides us with great freedom to choose auxiliary linear operators so that we can approximate a nonlinear problem by *choosing* different sets of base functions.

The basic ideas of the homotopy analysis method are very simple. For example, let us consider the nonlinear equation (3). Selecting a proper, familiar auxiliary linear operator $\mathscr{L}$, whose fixed point is zero, say $\mathscr{L}(0) = 0$, we construct a homotopy $U(\vec{r}, p, \hbar)$: $\Omega \times [0, 1] \times R_0 \to R$, which satisfies

$$(1 - p)\{\mathscr{L}[U(\vec{r}, p, \hbar)] - \mathscr{L}[u_0(\vec{r})]\} = p\hbar\{\mathscr{A}[u(\vec{r}, p, \hbar)] - f\}, \tag{6}$$

where $R_0 = (-\infty, 0) \bigcup (0, \infty)$, $u_0(\vec{r})$ is an initial approximation, $U(\vec{r}, p, \hbar)$ is a function of the variables $\vec{r} \in \Omega$, $p \in [0, 1]$ and $\hbar \neq 0$. Obviously, from Eq. (6), the following two expressions:

$$U(\vec{r}, 0, \hbar) = u_0(\vec{r}), \tag{7}$$

$$U(\vec{r}, 1, \hbar) = u(\vec{r}), \tag{8}$$

hold, where $u(\vec{r})$ is the solution of Eq. (3). Therefore, $U(\vec{r}, p, \hbar)$ varies continuously from $u_0(\vec{r})$ to $u(\vec{r})$ as the the parameter p increases from 0 to 1. It is more precise to say, $u_0(\vec{r})$ and $u(\vec{r})$ are homotopic. In topology, this kind of continuous variation is called deformation. So, we call Eq. (6) the zeroth-order deformation equation.

Assume that the continuous deformation $U(\vec{r}, p, \hbar)$ is smooth enough about p so that

$$U^{[m]}(\vec{r}, p, \hbar) = \frac{\partial^m U(\vec{r}, p, \hbar)}{\partial p^m}, \quad m = 1, 2, 3, \ldots, \tag{9}$$

called mth-order deformation derivatives, exist. Then according to the theory of Taylor's series, we have from (7) and (8) that

$$U(\vec{r}, p, \hbar) = u_0 + \sum_{m=1}^{\infty} U^{[m]}(\vec{r}, 0, \hbar)\left(\frac{p^m}{m!}\right) = u_0 + \sum_{m=1}^{\infty} u_m p^m, \tag{10}$$

where $U^{[m]}(\vec{r}, 0, \hbar)$ denotes the mth-order deformation derivatives $U^{[m]}(\vec{r}, p, \hbar)$ at $p = 0$, and

$$u_m = \frac{U^{[m]}(\vec{r}, 0, \hbar)}{m!}, \tag{11}$$

If $\hbar$ is so properly selected that the convergence radius of the series (10) is not less than 1, we have by (8) that

$$u = u_0 + \sum_{m=1}^{\infty} u_m. \tag{12}$$

The equation of u_m can be obtained in the following way. Differentiating Eq. (6) m times with respect to p, and then setting $p = 0$ and finally dividing each side by $m!$, we get

$$\mathscr{L}[u_m - \chi_m u_{m-1}] = \hbar R_m, \quad m \geqslant 1, \tag{13}$$

where

$$R_m = \frac{1}{(m-1)!}\left\{\frac{\mathrm{d}^{m-1}}{\mathrm{d}p^{m-1}}\left[\mathscr{A}u(\vec{r},p,\hbar) - f\right]\right\}\Big|_{p=0}, \tag{14}$$

under the definition

$$\chi_k = \begin{cases} 0, & k \leqslant 1, \\ 1, & k > 1. \end{cases} \tag{15}$$

The linear equation (13) can be solved either analytically (by means of symbolic software such as Mathematica, Maple, MathLab and so on) or numerically (by means of FDM, FEM, BEM and so on). For details, please refer to Liao [42,48].

3. General boundary element method

For simplicity, we write

$$\mathscr{L}\bar{u}_m = \hbar R_m, \tag{16}$$

where

$$\bar{u}_m = u_m - \chi_m u_{m-1}. \tag{17}$$

Note that the right-hand term of (16) is a known function, i.e., (16) is a linear equation. So it can be easily solved by boundary element method as it can be written in the form

$$c(\vec{r})\bar{u}_m(\vec{r}) = \int_\Gamma [\bar{u}_m \mathscr{B}(\omega) - \omega\mathscr{B}(\bar{u}_m)]\mathrm{d}\Gamma + \int_\Omega \hbar R_m \omega\, \mathrm{d}\Omega, \tag{18}$$

where $c(\vec{r})$ is the geometric factor which depends on the location of the position vector $\vec{r}$, ω is the fundamental solution of the operator $\mathscr{L}$, $\mathscr{B}$ is its boundary operator, Γ denotes the boundary of the domain Ω.

The convergence radius ρ of the series (12) depends on the auxiliary linear operator $\mathscr{L}$, initial approximation u_0 and the value of $\hbar$. If they are properly selected, the approximation at considerable high-order may be accurate enough and no iterations are necessary. Otherwise, we can use the Nth-order approximation

$$u \approx u_0 + \sum_{m=1}^{N} u_m \tag{19}$$

as the new initial solution so that we obtain the high-order iterative formulation

$$u^{i+1} \leftarrow u_0^i + \sum_{m=1}^{N} u_m^i, \tag{20}$$

The homotopy analysis method provides us with great freedom and flexibility. First, we have great freedom to choose the auxiliary linear operator $\mathscr{L}$, *no matter* whether the original nonlinear equation (3) contains any linear terms or not. Obviously, we can *choose* such an auxiliary linear operator $\mathscr{L}$ that its fundamental solution exists and is known and that the linear sub-problems governed by (13) can be solved by the traditional BEM for inhomogeneous linear equations. In this way, one can apply the BEM to nonlinear problems that even do *not* contain any linear terms at all, as verified by Liao [32–34]. Therefore, the general BEM overcomes the 3rd limitation of the foregoing traditional nonlinear BEM schemes.

Second, we have freedom to choose the initial guess u_0, and the high-order iterative formulation can be applied to all numerical techniques. It is easy to prove that many well-known iterative schemes are only special cases of (19) when $N = 1$. It is found [36] that, at each iteration and for a given initial guess, the higher the order N of iteration formula, the better the approximate result. Thus, the higher-order iterative formula (with larger N) has better property of convergence.

Besides, there exists an auxiliary parameter $\hbar$ in each linear sub-problem governed by (13), which provides us with a simply way to adjust and control the convergence region and rate of the series (12), as verified by Liao and his co-authors [42–48]. Thus, by means of choosing a proper value of the auxiliary parameter $\hbar$ and high enough order iterative formula (19), one can obtain convergent results of nonlinear problems with very strong nonlinearity.

Recently, Zhao and Liao [40] applied the general BEM to obtain convergent results of viscous flow in a driven square cavity at Reynolds number up to 7500, which agree well with FDM results given by Ghia et al [29]. It should be emphasized that it is the *first* time that such convergent

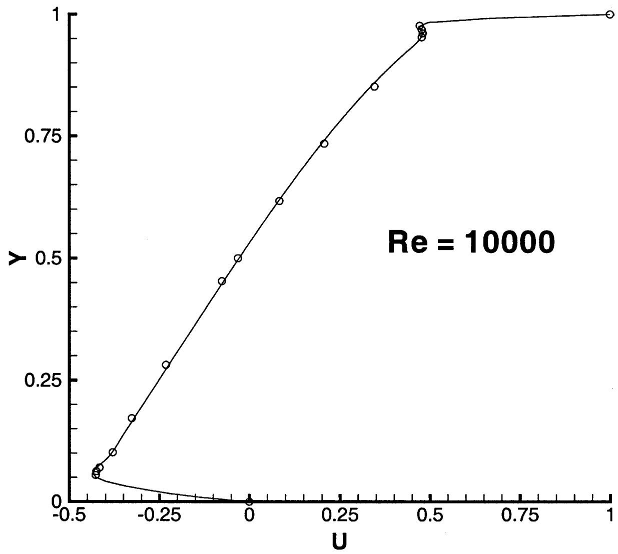

Fig. 1. Profiles of velocity u at $x = 1/2$ for $Re = 10,000$. Solid line: current result; circle: results given by Ghia et al. [29].

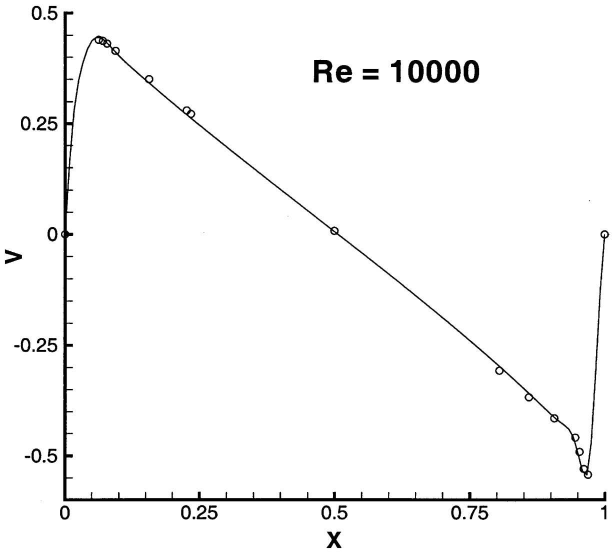

Fig. 2. Profiles of velocity v at $y = 1/2$ for $Re = 10,000$. Solid line: current result; circle: results given by Ghia et al. [29].

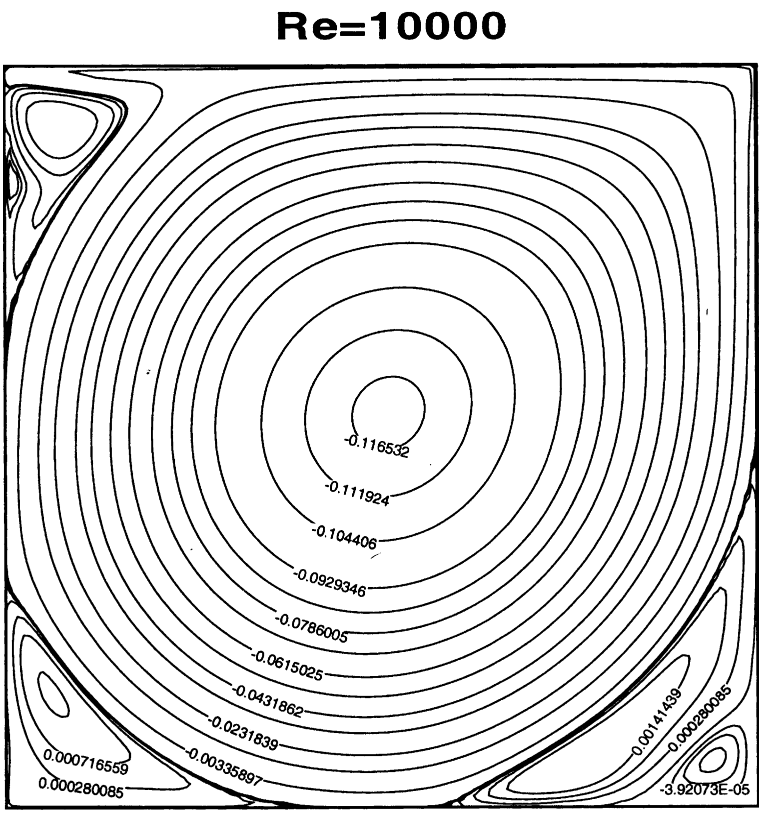

Fig. 3. Contour of the stream-function ψ when $Re = 10,000$.

results at Reynolds number up to 7500 are given by the *boundary element method*. As the continuation of this work, we ran the code and obtained the results at $Re = 10,000$. The velocity profiles of u at $x = 1/2$ and v at $y = 1/2$, compared with the results given by Ghia et al. [29], are shown in Figs. 1 and 2, respectively. The contours of the stream-function are shown in Fig. 3. All of our numerical results agree well with the solutions provided by Ghia et al.[29] by mean of the finite difference method.

Furthermore, the general BEM has been successfully applied to both of hyperbolic and parabolic unsteady nonlinear heat transfer problems, as verifies by Liao [56] and Liao and Chwang [38], respectively. These are good examples to illustrate the validity and great potential of the general BEM for strongly nonlinear problems. All of these indicate that the general boundary element method can overcome the 2nd limitation of the traditional nonlinear BEM schemes mentioned above.

In summary the general BEM can overcome the last two limitations of the traditional nonlinear BEM schemes listed in the first section.

4. Opportunity and challenge of BEM

Overcoming the last two limitations of the foregoing traditional nonlinear BEM schemes, the general BEM can be applied to much more of nonlinear problems in science and engineering. This certainly enhances the ability of the boundary element method in the competition with other numerical techniques such as FDM, FEM and so on, which require discretization of full domain.

However, domain integral is still a large obstacle for the BEM to solve nonlinear problems. Especially, it is unknown if current techniques for domain integrals are efficient enough for three-dimensional nonlinear problems. Thus, it is necessary to develop some new, more efficient techniques for domain integrals, while further improving dual reciprocity method, domain decomposition method and so on. Note that an integral over a domain can be evaluated *independently* over many sub-domains, thus it is quite easy and efficient to employ parallel computation to integrals. So, it is much easier to employ parallel computation to the BEM scheme than other domain numerical techniques such as FDM, FEM and so on. This is an advantage of the BEM, which might become an important factor in the competition between the BEM and other numerical methods. The parallel computation technique can greatly enhance the efficiency of the BEM, even if cell integration is employed to calculate the domain integrals, as verified by Zhao and Liao [40]. So, the combination of the general BEM with the parallel computation techniques, dual reciprocity method and domain decomposition method might, we hope, provide us with an efficient, widely applied BEM approach for strongly nonlinear problems. Besides, more attentions should be given to complicated, practical three-dimensional nonlinear problems in engineering.

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