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Dual solutions of boundary layer flow over an upstream moving plate

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Abstract

The homotopy analysis method is applied to study the boundary layer flow over a flat plate which has a constant velocity opposite in direction to that of the uniform mainstream. The dual solutions in series expressions are obtained with the proposed technique, which agree well with numerical results. Note that, by introducing a new auxiliary function $\beta(z)$, the bifurcation of the solutions is obtained. This indicates that the homtopy analysis method is a open system, in the framework of this technique, we have great freedom to choose the auxiliary parameters or functions. As a result, complicated nonlinear problems may be resolved in a simple way. The present work shows that the homotopy analysis method is an effective tool for solving nonlinear problems with multiple solutions.

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Keywords: Boundary layer flow; Dual solution; Crocco variable formulation

1. Introduction

Consider the boundary layer flows over a upstream moving flat plate, governed by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

subject to the boundary conditions

$$u=-U_w,\quad v=0\quad \text{at } y=0, \tag{3}$$

$$u=U_\infty\quad \text{as } y\to\infty, \tag{4}$$

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where x and y are two spatial independent variables along and perpendicular to the plate, u and v denote the velocity components in the x- and y-directions, ν the kinematic viscosity coefficient of the fluid, U_w the speed of the moving plate, U_∞ the fluid velocity of the mainstream far away from the plate, respectively. Note that $U_w > 0$ when the plate surface moves in the direction opposite to the mainstream.

Let ψ denote the stream function. Following Blasius [1], one uses the following similarity transformations:

$$f(\eta) = \frac{\psi(x,y)}{\sqrt{2\nu x U_\infty}}, \quad \eta = y\sqrt{\frac{U_\infty}{2\nu x}}. \tag{5}$$

Then, Eq. (2) becomes

$$f'''(\eta) + f(\eta)f''(\eta) = 0, \tag{6}$$

subject to the boundary conditions

$$f(0) = 0, \quad f'(0) = -\alpha, \quad f'(\infty) = 1, \tag{7}$$

where $\alpha = U_w/U_\infty$ is the ratio of the speed of the plate surface to the velocity of the free stream.

The original differential equation arises from a classical similarity transformation of the boundary-layer equations (see [2]). Weyl [3] established the existence and uniqueness of the solution of Eq. (6) for the case $\alpha = 0$. In the case of $0 < \alpha \leqslant 0.354$, Merkin [4] reported the dual solutions by using both the perturbation techniques and the numerical method, and discussed the stability of those solutions. Riley and Weidman [5] made analysis and presented the numerical solutions when $\alpha < 0.3541$. Hussaini et al. [6] proved that a solution exists only if the parameter α does not exceed a certain critical value and gave numerical evidences that the solution is non-unique. It is known that the problem of flat plate can be expressed in alternative analytical forms by employing various transformations of both dependent and independent variables. Crocco [7,8] developed such a transformation that the sheer stress is taken as a primary dependent variable while the velocity component u, paralleled to the plate, is taken as an independent variable to replace y. Callegari and Friedman [9], Callegari and Nachman [10] found that it is expedient to work with the Crocco variable formulation, i.e. using the shear stress $g(u) = f''(\eta)$ as the dependent variable and tangential velocity $u = f'(\eta)$ as the independent variable. In this way, Eqs. (6) and (7) become

$$g(u)g''(u) + u = 0, \quad -\alpha < u < 1, \tag{8}$$

$$g'(-\alpha) = 0, \quad g(1) = 0. \tag{9}$$

We use the transformations $g^*(z) = g(u)$ and $z = u + \alpha$ to map the interval $-\alpha < u < 1$ into $0 < z < 1 + \alpha$. Dropping star for convenience, Eq. (8) becomes

$$g(z)g''(z) + z - \alpha = 0, \quad 0 < z < 1 + \alpha, \tag{10}$$

subject to the boundary conditions

$$g'(0) = 0, \quad g(1 + \alpha) = 0. \tag{11}$$

2. Homotopy analysis solution

Many nonlinear problems have multiple solutions. It is not easy to find out all multiple solutions of a nonlinear problem even by means of numerical techniques, say nothing of analytic methods. Recently, a kind of new analytic technique, namely the homotopy analysis method [11], is developed for strongly nonlinear problems. Different from perturbation techniques [12], the homotopy analysis method does not depend upon any small or large parameters and thus is valid for most of nonlinear problems in science and engineering. Besides, it logically contains other non-perturbation techniques such as Lyapunov's small parameter method [13], the δ-expansion method [14], and Adomian's decomposition method [15]. The homotopy analysis method has been successfully applied to many nonlinear problems [16–25]. In this paper, we apply the homotopy analysis method to obtain the series expressions of the dual solutions of Eqs. (10) and (11).

Using the transformation

$$g(z) = \delta s(z) + \beta(z), \tag{12}$$

where $\delta = g(0) = f''(0)$ is an unknown constant and $\beta(z)$ is a function to be determined later, satisfying

$$\beta(0) = 0, \quad \beta'(0) = 0, \quad \beta(1+\alpha) = 0, \tag{13}$$

we have

$$[\delta s(z) + \beta(z)][\delta s''(z) + \beta''(z)] + z - \alpha = 0, \tag{14}$$

subject to boundary conditions

$$s(0) = 1, \quad s'(0) = 0, \quad s(1+\alpha) = 0. \tag{15}$$

2.1. Zeroth-order deformation equation

Obviously, $s(z)$ can be expressed by a set of base functions

$$\{z^m | m \geqslant 0\} \tag{16}$$

in the following form:

$$s(z) = \sum_{m=0}^{+\infty} A_m z^m, \tag{17}$$

where A_m is a coefficient to be determined later. Thus, all approximations of $s(z)$ should be expressed in the above form. This is the so-called *Rule of Solution Expression* of $s(z)$.

There are a lot of functions $\beta(z)$ satisfying the boundary conditions (13). Under the *Rule of Solution Expression* denoted by (17) and from (13), we simply choose

$$\beta(z) = -z^2(z - 1 - \alpha). \tag{18}$$

Similarly, under the *Rule of Solution Expression* denoted by (17) and using the boundary condition (15), it is straightforward to choose

$$s_0(z) = 1 - \frac{z^2}{(1+\alpha)^2} + \epsilon\left[\frac{z^3}{(1+\alpha)^3} - \frac{z^2}{(1+\alpha)^2}\right] \tag{19}$$

as the initial approximation of $s(z)$, where ϵ is a parameter to be determined later. Besides, under the *Rule of Solution Expression* denoted by (17) and from the governing equation (14), we choose

$$\mathscr{L}\phi = \frac{\partial^2 \phi}{\partial z^2} \tag{20}$$

as our auxiliary linear operator, which has the following property:

$$\mathscr{L}[C_1 + C_2 z] = 0, \tag{21}$$

where C_1 and C_2 are integral constants. Let $q \in [0,1]$ denote an embedding parameter. Based on Eq. (14), we define a nonlinear operator

$$\mathscr{N}[\phi(z;q), \Lambda(q)] = [\Lambda(q)\phi(z;q) + \beta(z)]\left[\Lambda(q)\frac{\partial^2\phi(z;q)}{\partial z^2} + \beta''(z)\right] + z - \alpha, \tag{22}$$

where $\phi(z;q)$ is a kind of mapping of $s(z)$, and $\Lambda(q)$ is a kind of mapping of δ, respectively, which are defined below.

Let $\hbar$ denote a non-zero auxiliary parameter. We construct the zeroth-order deformation equation

$$(1-q)\mathscr{L}[\phi(z;q) - s_0(z)] = q\hbar\mathscr{N}[\phi(z;q), \Lambda(q)], \tag{23}$$

subject to boundary conditions

$$\phi(0;q) = 1, \quad \left.\frac{\partial \phi(z;q)}{\partial z}\right|_{z=0} = 0, \quad \phi(1+\alpha;q) = 0, \tag{24}$$

where $q \in [0,1]$ is an embedding parameter. When $q = 0$ and $q = 1$, the above zeroth-order deformation equations (23) and (24) have the solutions

$$\phi(z;0) = s_0(z), \quad \Lambda(0) = \delta_0 \tag{25}$$

and

$$\phi(z;1) = s(z), \quad \Lambda(1) = \delta, \tag{26}$$

respectively, where δ_0 is an initial approximation of δ to be determined later. Thus, as the embedding parameter q increases from 0 to 1, the mapping $\phi(z, q)$ varies (or deform) from the initial guess $s_0(z)$ to the solution $s(z)$ of the original equations (14) and (15), so does $\Lambda(q)$ from the initial guess δ_0 to the exact value of δ.

Expanding $\phi(z; q)$ and $\Lambda(q)$ in Taylor's series with respect to the embedding parameter q, we have

$$\phi(z;q) = \phi(z,0) + \sum_{m=1}^{+\infty} s_m(z) q^m, \tag{27}$$

$$\Lambda(q) = \Lambda(0) + \sum_{m=1}^{+\infty} \delta_m q^m, \tag{28}$$

where

$$s_m(z) = \frac{1}{m!} \left.\frac{\partial^m \phi(z;q)}{\partial q^m}\right|_{q=0}, \quad \delta_m = \frac{1}{m!} \left.\frac{\partial^m \Lambda(q)}{\partial q^m}\right|_{q=0}, \tag{29}$$

respectively. Obviously, it is important to ensure that the above series are convergent at $q = 1$. Fortunately, there exists an auxiliary parameter $\hbar$ in the zeroth-order deformation equations (23) and (24). If the auxiliary parameters $\hbar$ is properly chosen so that the series (27) and (28) are convergent at $q = 1$, we have from (25) and (26) that

$$s(z) = s_0(z) + \sum_{m=1}^{+\infty} s_m(z) \tag{30}$$

and

$$\delta = \delta_0 + \sum_{m=1}^{+\infty} \delta_m, \tag{31}$$

respectively. The unknown $s_m(z)$ and δ_m will be determined in the way described below.

2.2. High-order deformation equation

For the sake of simplicity, define

$$\vec{s}_m(z) = \{s_0(z), s_1(z), s_2(z), \ldots, s_m(z)\} \tag{32}$$

and

$$\vec{\delta}_m = \{\delta_0, \delta_1, \delta_2, \ldots, \delta_m\}. \tag{33}$$

Differentiating the zeroth-order deformation equations (23) m times with respect to q, then setting $q = 0$, and finally dividing it by $m!$, we obtain the mth-order deformation equation

$$\mathscr{L}[s_m(z) - \chi_m s_{m-1}(z)] = \hbar R_m(z, \vec{s}_{m-1}, \vec{\delta}_{m-1}), \tag{34}$$

subject to the boundary conditions

$$s_m(0) = 0, \quad s_m(1+\alpha) = 0, \quad s'_m(0) = 0, \tag{35}$$

where

$$R_m(z,\vec{s}_{m-1},\vec{\delta}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(z;q),\Lambda(q)]}{\partial q^{m-1}}\bigg|_{q=0}$$
$$= \sum_{n=0}^{m-1}\Gamma_n\Omega_{m-1-n} + \beta''(z)\Gamma_{m-1} + \beta(z)\Omega_{m-1} + (1-\chi_m)[\beta(z)\beta''(z)+z-\alpha] \tag{36}$$

under the definitions

$$\Gamma_n = \sum_{i=0}^{n}\delta_{n-i}s_i(z), \quad \Omega_n = \sum_{i=0}^{n}\delta_{n-i}s_i''(z) \tag{37}$$

and

$$\chi_m = \begin{cases} 0, & m=1, \\ 1, & m>1. \end{cases} \tag{38}$$

Note that we have now a unknown function $s_m(z)$ and a unknown parameter δ_{m-1} for $m = 1, 2, 3, \ldots$ Let $s_m^*(z)$ denote a particular solution of Eq. (34). From (21), the corresponding general solution reads

$$s_m(z) = s_m^*(z) + C_1 + C_2 z, \tag{39}$$

where the integral constants C_1 and C_2 are determined by the first two conditions of (35). Note that, the above expression of $s_m(z)$ contains the unknown parameter δ_{m-1}, which is then determined by the third condition of (35), i.e.

$$s_m'(0) = 0. \tag{40}$$

In this way, one can obtain all s_m and δ_{m-1} one after the other in the order $m = 1, 2, 3, \ldots$ At the Mth-order of approximation, we have

$$s(z) \approx s_0(z) + \sum_{n=1}^{M} s_n(z), \tag{41}$$

$$\delta \approx \delta_0 + \sum_{n=1}^{M-1}\delta_n. \tag{42}$$

From (12), the Mth-order approximation of $g(z)$ reads

$$g(z) = \left(\sum_{n=0}^{M-1}\delta_n\right)\left[\sum_{n=0}^{M}s_n(z)\right] + \beta(z). \tag{43}$$

As $M \to \infty$, we have the series solutions of Eqs. (10) and (11).

3. Analysis of results

3.1. The dual solutions

When $m = 1$, we have a nonlinear algebraic equation about δ_0 as

$$\left(-\frac{25}{3}+\frac{2\epsilon}{3}-\frac{\epsilon^2}{3}\right)\delta_0^2 + \left(\frac{2}{3}+2\alpha+2\alpha^2+\frac{2}{3}\alpha^3\right)(1-\epsilon)\delta_0 + \frac{4}{3} - 2\alpha - 10\alpha^2 - 10\alpha^3 - 5\alpha^4 - 2\alpha^5 - \frac{\alpha^6}{3} = 0, \tag{44}$$

which has two different solutions

$$\delta_0 = -\frac{(-1+\epsilon)(1+3\alpha+3\alpha^2+\alpha^3)}{25-2\epsilon+\epsilon^2} \pm \frac{\sqrt{(1+\alpha)^2[101-490\alpha-96\alpha^3-24\alpha^4+(1-2\alpha)(5\epsilon^2-10\epsilon)]}}{25-2\epsilon+\epsilon^2}. \quad (45)$$

When $m \geqslant 2$, Eq. (40) becomes a linear algebraic equation of δ_{m-1}, and thus can be easily solved. Using the above two different expressions of δ_0, the dual solutions of the considered problem can be determined.

3.2. The choice of ϵ

The parameter ϵ is used to optimize the initial approximation of $s(z)$. From (45), we know that ϵ becomes a complex number when

$$\epsilon < \left|\frac{-5+10\alpha-2\sqrt{30}\sqrt{-4+22\alpha-22\alpha^2-8\alpha^3-\alpha^4-2\alpha^5}}{5(-1+2\alpha)}\right|. \quad (46)$$

In order to determine the suitable ϵ, we define the error function $\mathscr{E}(\epsilon)$ as

$$\mathscr{E}(\epsilon) = \int_0^{\infty} \{[\delta s_0(z)+\beta(z)][\delta s_0''(z)+\beta''(z)]+z-\alpha\}^2 \mathrm{d}z. \quad (47)$$

When $\mathscr{E}(\epsilon)$ has the smallest value, we obtain the best value of ϵ, i.e. $\epsilon = -6$ for $0 < \alpha < 0.3541$, corresponding to the best initial approximation $s_0(z)$ of $s(z)$.

3.3. The choice of $\hbar$

Note that our series solutions contain the auxiliary parameter $\hbar$, which provides us with a simple way to control and adjust the convergence of the series (30) and (31), as pointed by Liao [11]. Following Liao [11], we can choose an appropriate value of $\hbar$ by means of taking $\hbar$ as a variable and plotting the so-called $\hbar$-curve of $f''(0) \sim \hbar$. It is found that the solution series converge when $-4 \leqslant \hbar < 0$ for the upper branch of solutions and $-30 \leqslant \hbar < -5$ for the lower ones. In this way, for a given α, we can always find a proper value of $\hbar$ to ensure that the solution series converge.

Table 1
Comparison of numerical results with analytical approximations of the upper branch of $f''(0)$ and the corresponding results given by the $[m,m]$ homotopy-Páde technique

α	50 order	60 order	70 order	[40,40] approximations	[50,50] approximations	Numerical results
0.30	0.35520	0.35632	0.35660	0.35609	0.35662	0.35664
0.31	0.34302	0.34415	0.34428	0.34317	0.34428	0.34426
0.32	0.32877	0.32881	0.32885	0.32817	0.32885	0.32885
0.33	0.31163	0.31183	0.31184	0.31175	0.31184	0.31184
0.34	0.28980	0.28985	0.28985	0.28983	0.28985	0.28985
0.35	0.25779	0.25729	0.25758	0.25755	0.25758	0.25759

Table 2
Comparison of numerical results with analytical approximations of the lower branch of $f''(0)$ and the corresponding results given by the $[m,m]$ homotopy-Páde technique

α	50 order	60 order	70 order	[40,40] approximations	[50,50] approximations	Numerical results
0.30	0.08418	0.08541	0.08489	0.08382	0.08486	0.08487
0.31	0.09675	0.09612	0.09617	0.09619	0.09615	0.09617
0.32	0.10894	0.10956	0.10941	0.10925	0.10941	0.10941
0.33	0.12590	0.12593	0.12544	0.12532	0.12544	0.12543
0.34	0.14645	0.14628	0.14628	0.14623	0.14628	0.14628
0.35	0.17890	0.17899	0.17899	0.17885	0.17899	0.17899

3.4. Validity of the series solutions

As $\epsilon = -6$, Eq. (45) gives a real number of δ_0 when $\alpha \leqslant 0.381501$. However, it is found that no solution exists when $\alpha > 0.3541$. Our analytic approximations of $f''(0)$ agree well with numerical ones, as shown in Tables 1 and 2, and also in Fig. 1. It is found that higher-order approximations are necessary for the lower

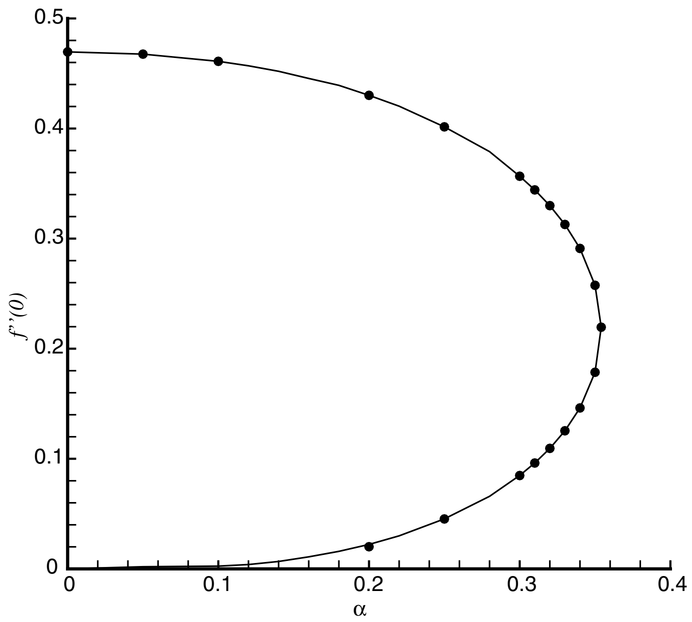

Fig. 1. Comparison of numerical solutions with analytic approximations of $f''(0)$: (circle) 70th-order homotopy analysis results; (solid line) numerical results.

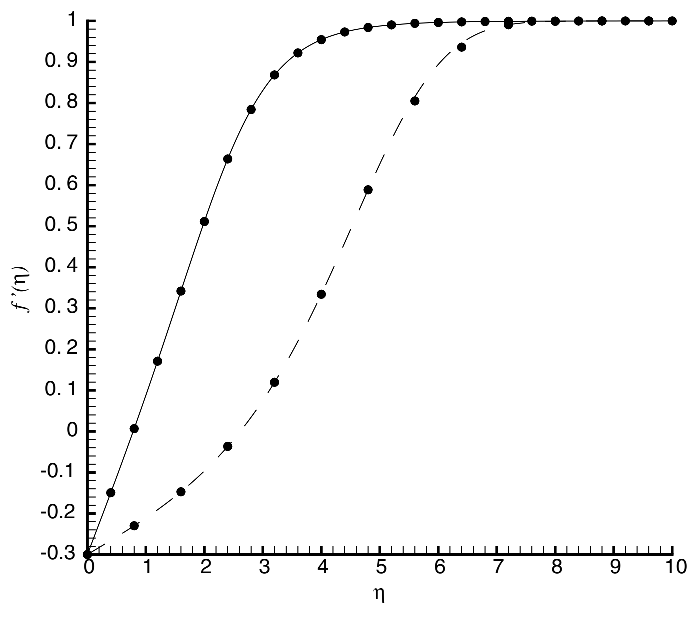

Fig. 2. Comparison of numerical results with analytic approximations of $f'(\eta)$ when $\alpha = 0.3$: (circle) 70th-order homotopy analysis results; (solid line) upper branch numerical solutions; (dashed line) lower branch numerical solutions.

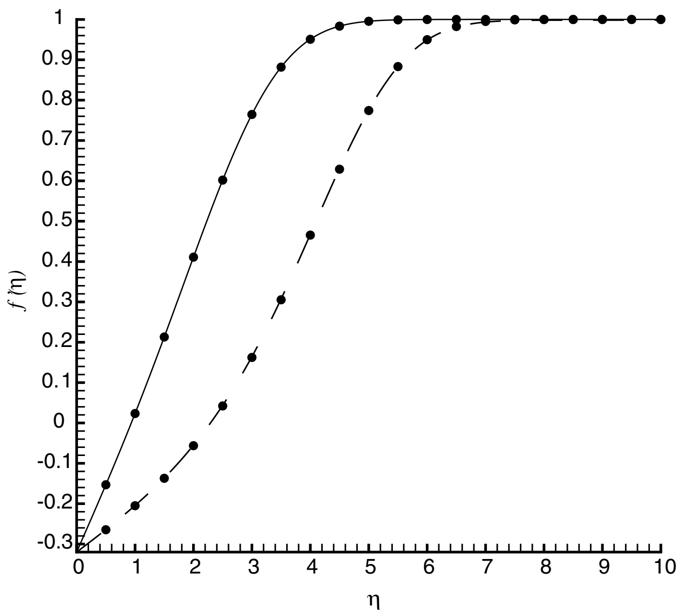

Fig. 3. Comparison of numerical results with analytic approximations of $f'(\eta)$ when $\alpha = 0.32$: (circle) 70th-order homotopy analysis results; (solid line) upper branch numerical solutions; (dashed line) lower branch numerical solutions.

branch of solutions than the upper ones. This agrees with Merkin's conclusions [4]. Note that, homotopy-Páde technique [11] is used to accelerate the convergence of the series solutions, when necessary.

According to the definitions of $g(u)$ and u, it holds

$$g(u) = \frac{\mathrm{d}u}{\mathrm{d}\eta}, \tag{48}$$

which gives

$$\eta = \int_{-\alpha}^{u} \frac{1}{g(u)} \mathrm{d}u. \tag{49}$$

From above expressions, it is easily to obtain the graph of $f'(\eta)$. All of our analytic approximations agree well with numerical ones, as shown in Figs. 2 and 3.

4. Conclusions

In this paper, the dual series solutions of the boundary layer flow over an upstream moving flat plate are obtained with the help of the homotopy analysis method. The validity of our solutions is verified by the numerical results. It is worth mentioning that most our previous work was only considered the nonlinear problems with unique solution. while in this work, by introducing a new auxiliary function $\beta(z)$, we successfully obtain the dual solutions of the considered problem. Thus, it can be deemed as a kind of innovation of the homotopy analysis method. This work indicates that the homotopy analysis method is still valid for nonlinear problems with multiple solutions. The series solutions might find wide applications in engineering, such as the migration of moisture through the air contained in fibrous insulations and grain storage installations, and dispersion of chemical contaminants through water-saturated soil. The analytic approach described in this paper can be employed to other nonlinear problems with multiples solutions in the similar way.

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