On a generalized Taylor theorem: a rational proof of the validity of the Homotopy Analysis Method

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ABSTRACT. A generalized Taylor series of a complex function was derived and some related theorems about its convergence region were given. The generalized Taylor theorem can be applied to greatly enlarge convergence regions of approximation series given by other traditional techniques. The rigorous proof of the generalized Taylor theorem also provides us with a rational base of the validity of a new kind of powerful analytic technique for nonlinear problems, namely the homotopy analysis method.

Key words: Taylor series, convergence and summability of series, homotopy analysis method.

Mathematics Subject Classification: 41A58, 40A05, 26A06.

1. Introduction

In 17th century Isaac Newton [1] gave such a binomial expression for fractional and negative exponents $(1+t)^{\alpha}$, i.e.

$$1+\sum_{k=1}^{+\infty}\left[\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}\right]t^{k} \qquad (\alpha\neq 0,1,2,\cdots)\,, \tag{1}$$

whose convergence radius is one. Furthermore, the classical Taylor series (see [2])

$$\lim_{m\to+\infty}\sum_{k=0}^{m}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k} \tag{2}$$

of a complex function $f(z)$ at $z=z_0$ is valid mostly in a restricted convergence region

$$|z-z_0|<\min_{k\,\in I}|\xi_k-z_0|,$$

where $f^{(k)}(z_0)$ denotes the kth order derivative of $f(z)$ at $z=z_0$, ξ_k $(k\in I)$ are all of the singularities of $f(z)$, I denotes the set of index.

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Liao and his co-authors developed a new kind of analytic technique for nonlinear differential equations, namely, the homotopy analysis method [3-14]. Different from perturbation techniques [15], the validity of the homotopy analysis method does *not* depend upon whether a nonlinear problem contains small/large parameters or not. Besides, unlike **all** other perturbation techniques and non-perturbative methods such as artificial small parameter method [16], δ - expansion method [17], Adomian decomposition method [18-21] and so on, the homotopy analysis method provides us with a convenient way to **control** the convergence of approximation series and **adjust** convergence regions when necessary (see [10-14]). The homotopy analysis method has been successfully applied to solve many kinds of nonlinear problems in applied mathematics and mechanics. Unfortunately, it is well known that hundreds of examples would not be better than a rigorous logical proof. In the frame of the homotopy analysis method Liao [5] obtained the so-called generalized Taylor series and illustrated that it can be applied to greatly enlarge convergence regions of approximation series given by perturbation techniques [6]. In this paper, without applying the homotopy analysis method, we rigorously derive the so-called generalized Taylor series and logically prove some related theorems about convergence regions. This, in the same time, can provide us with a solid rational base of the validity of the homotopy analysis method, although indirectly.

2. The generalized Taylor theorem

THEOREM 1. *Let $\hbar$ be a complex number. If a complex function $f(z)$ is analytic at $z = z_0$, the so-called generalized Taylor series*

$$\lim_{m\to+\infty}\sum_{k=0}^{m}\left[\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k\right]\phi_{m,k}(\hbar) \tag{3}$$

converges to $f(z)$ in the region

$$\bigcap_{k\in I}\left|1+\hbar-\hbar\left(\frac{z-z_0}{\xi_k - z_0}\right)\right|<1, \qquad |1+\hbar|<1, \tag{4}$$

where

$$\phi_{m,n}(\hbar)=\begin{cases}1 & n\le 0\\ (-\hbar)^n\sum_{k=0}^{m-n}\binom{k+n-1}{k}(1+\hbar)^k & 1\le n\le m\end{cases} \tag{5}$$

and ξ_k $(k\in I)$ are singularities of $f(z)$.

PROOF. Let $p, z, z_0, \hbar \neq 0$ and

$$\tau = z_0 - \frac{\hbar\, p(z-z_0)}{1-(1+\hbar)p}$$

be complex numbers and $f(\tau)$ a complex function. We have $\tau = z_0$ when $p = 0$ and $\tau = z$ when $p = 1$, respectively. Writing $G(p) = f(\tau)$, we have $G(0) = f(z_0)$ and $G(1) = f(z)$, respectively. If

$$\left|1-(1+\hbar)p\right| < 1,$$

the Maclaurin series of $G(p) = f(\tau)$ about p is

$$\begin{aligned}
& f(z_0) + \sum_{k=1}^{+\infty}\left[\frac{-\hbar\ p}{1-(1+\hbar)\,p}\right]^k \left[\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k\right] \\
&= f(z_0) + \sum_{k=1}^{+\infty} (-\hbar)^k p^k \left[1 + \sum_{r=1}^{+\infty}\binom{k+r-1}{r}(1+\hbar)^r\ p^r\right]\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \\
&= \sum_{n=0}^{+\infty}\sigma_n\ p^n
\end{aligned} \tag{6}$$

where

$$\sigma_0 = f(z_0),\quad \sigma_1 = (-\hbar)f'(z_0)(z-z_0)$$

and

$$\begin{aligned}
\sigma_n &= (-\hbar)^n\left[\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\right] \\
&\quad + \sum_{k=1}^{n-1}\binom{n-1}{n-k}(-\hbar)^k(1+\hbar)^{n-k}\left[\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k\right]
\end{aligned}$$

for $n \ge 2$. So, the *m*th-order Maclaurin expansion of $G(p) = f(\tau)$ about p is

$$\psi_m(z, z_0, \hbar, p) = \sum_{n=0}^{m}\sigma_n\ p^n,$$

which gives at $p = 1$ that

$$
\begin{aligned}
\psi_m(z,z_0,\hbar,1) &= \sum_{n=0}^{m}\sigma_n \\
&= \sum_{n=0}^{m}(-\hbar)^n\left[\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\right] + \sum_{n=2}^{m}\sum_{k=1}^{n-1}\binom{n-1}{n-k}(-\hbar)^k(1+\hbar)^{n-k}\left[\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k\right] \\
&= \sum_{n=0}^{m}(-\hbar)^n\left[\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\right] + \sum_{k=1}^{m-1}\sum_{n=k+1}^{m}\binom{n-1}{n-k}(-\hbar)^k(1+\hbar)^{n-k}\left[\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k\right] \\
&= \sum_{n=0}^{m}(-\hbar)^n\left[\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\right] + \sum_{n=1}^{m-1}\sum_{k=n+1}^{m}\binom{k-1}{k-n}(-\hbar)^n(1+\hbar)^{k-n}\left[\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\right] \\
&= \sum_{n=0}^{m}\left[\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\right]\phi_{m,n}(\hbar)
\end{aligned}
$$

where

$$\phi_{m,0}(\hbar) = 1$$

and

$$\phi_{m,n}(\hbar) = \sum_{k=n}^{m}\binom{k-1}{k-n}(-\hbar)^n(1+\hbar)^{k-n} = (-\hbar)^n\sum_{k=0}^{m-n}\binom{k+n-1}{k}(1+\hbar)^k$$

for $1 \le n \le m$. The convergence radius of the Maclaurin series of $G(p) = f(\tau)$, i.e.

$$\lim_{m\to+\infty}\psi_m(z,z_0,\hbar,p) = \sum_{n=0}^{+\infty}\sigma_n\, p^n$$

is equal to the distance from the nearest singular points $\overline{p}_k$ $(k \in I)$ of $G(p) = f(\tau)$ to the point $p = 0$. Let ξ_k $(k \in I)$ be all of the singularities of $f(z)$. Then, for given z, z_0 and $\hbar$, the corresponding singularities $\overline{p}_k$ $(k \in I)$ of $G(p) = f(\tau)$ is determined by

$$\xi_k = z_0 - \frac{\hbar(z-z_0)\overline{p}_k}{1-(1+\hbar)\overline{p}_k}, \qquad (\hbar \ne 0, k \in I)$$

say,

$$\overline{p}_k = \frac{1}{1+\hbar-\hbar\left(\dfrac{z-z_0}{\xi_k - z_0}\right)}, \qquad (\hbar \ne 0, k \in I).$$

Furthermore, there exists an additional singularity dependent only on $\hbar$, i.e.

$$\bar{p}_0 = \frac{1}{1+\hbar}.$$

For a given z_0, if z and $\hbar$ are properly selected so that all above singularities are out of the region $|p| \le 1$, i.e.

$$|\bar{p}_0| = \frac{1}{|1+\hbar|} > 1, \quad |p_k| = \frac{1}{\left|1+\hbar-\hbar\left(\frac{z-z_0}{\xi_k - z_0}\right)\right|} > 1 \qquad (\hbar \ne 0, k \in I),$$

or more precisely to say,

$$\bigcap_{k\in I}\left|1+\hbar-\hbar\left(\frac{z-z_0}{\xi_k - z_0}\right)\right| < 1, \qquad |1+\hbar| < 1,$$

the Maclaurin series (6) converges to $G(1) = f(z)$ at $p = 1$, say,

$$f(z) = \lim_{m\to+\infty} \psi_m(z, z_0, \hbar, 1) = \lim_{m\to+\infty} \sum_{n=0}^{m}\left[\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\right]\phi_{m,n}(\hbar).$$

This completes the proof.

THEOREM 2. *Let* $\rho_k > 0$, γ_k, $0 \le \beta < 1$ *and* $\alpha \in [-\pi, \pi]$ *be real numbers and* $\xi_k = z_0 + \rho_k \exp(i\gamma_k)$ $(k \in I)$ *denote all singularities of a complex function* $f(z)$. *The general Taylor series* (3) *converges to* $f(z)$ *in the region*

$$\Omega = \left\{ z = z_0 + \rho \exp(i\theta) : \theta \in [-\pi, \pi], 0 \le \rho \le \min \rho_k \mu_k, k \in I \right\}$$

where

$$\mu_k = \frac{\beta[\cos(\theta-\gamma_k-\alpha) - \beta\cos(\theta-\gamma_k)] + \sqrt{\Theta_k}}{1-2\beta\cos\alpha+\beta^2},$$

$$\Theta_k = \beta^2[\cos(\theta-\gamma_k-\alpha) - \beta\cos(\theta-\gamma_k)]^2 + (1-\beta^2)(1-2\beta\cos\alpha+\beta^2).$$

Proof: Write $\hbar = -1+\beta\exp(i\alpha)$, where $\alpha \in [-\pi, \pi]$ and $0 \le \beta < 1$, thus $|1+\hbar| < 1$ holds. Write $z = z_0 + \rho\exp(i\theta)$, where $\rho \ge 0$ and $\theta \in [-\pi, \pi]$. Owing to (4), the general Taylor series (3) converges to $f(z)$ when

$$\left|\ \beta\, e^{i\alpha} + e^{i(\theta-\gamma_k)}(1-\beta e^{i\,\alpha})\rho/\rho_k\ \right| < 1,$$

i.e.

$$(1-2\beta\cos\alpha+\beta^2)\left(\frac{\rho}{\rho_k}\right)^2 - 2\beta\left[\cos(\theta-\gamma_k-\alpha)-\beta\cos(\theta-\gamma_k)\right]\left(\frac{\rho}{\rho_k}\right) - (1-\beta^2) < 0$$

for all $k \in I$. Clearly,

$$\Delta_k = 4\Theta_k \geq 0\,,$$

where

$$\Theta_k = \beta^2\left[\cos(\theta-\gamma_k-\alpha)-\beta\cos(\theta-\gamma_k)\right]^2 + (1-\beta^2)(1-2\beta\cos\alpha+\beta^2)\ .$$

Thus, we have

$$\chi_k < \frac{\rho}{\rho_k} < \mu_k$$

for all $k \in I$, where

$$\chi_k = \frac{\beta\left[\cos(\theta-\gamma_k-\alpha)-\beta\cos(\theta-\gamma_k)\right]-\sqrt{\Theta_k}}{1-2\beta\cos\alpha+\beta^2} < 0\,,$$

$$\mu_k = \frac{\beta\left[\cos(\theta-\gamma_k-\alpha)-\beta\cos(\theta-\gamma_k)\right]+\sqrt{\Theta_k}}{1-2\beta\cos\alpha+\beta^2} > 0\ .$$

Note that it holds $\rho/\rho_k \geq 0$. So, we have

$$0 \leq \rho < \mu_k\,\rho_k\,, \qquad k \in I\ .$$

Thus, the general Taylor series (3) converges to $f(z)$ in the region

$$\Omega = \left\{\ \mathrm{z} = \mathrm{z}_0 + \rho\exp(i\theta) : \theta \in [-\pi,\pi], 0 \leq \rho \leq \min \rho_k\,\mu_k, k \in I\ \right\}.$$

This completes the proof.

COROLLARY 1. *If a complex function* $f(z)$ *has only one singularity* $\xi_1 = \mathrm{z}_0 + \rho_1\exp(i\gamma_1)$, *then, when* $\hbar = -1+\beta\exp(i\alpha)$ *tends to* 0 *along the negative real axis, the series* (3) *converges to* $f(z)$ *in an infinite region*

$$\left\{z = z_0 + \rho \exp(i\theta) : 0 \le \rho < \rho_1\, \mu,\ \theta \in [-\pi, \pi]\right\},$$

where

$$\mu = \begin{cases} \sec(\theta - \gamma_1) & \text{when } \ \theta - \gamma_1 \in [-\pi/2, \pi/2] \\ +\infty & \text{otherwise} \end{cases}$$

Proof: When $\alpha = 0$, say, $\hbar = -1 + \beta \ \ (0 \le \beta < 1)$, we have

$$\mu_1 = \frac{1+\beta}{\sqrt{1-\beta^2 \sin^2(\theta - \gamma_k)} + \beta \cos(\theta - \gamma_1)}.$$

Clearly, it holds

$$\mu = \lim_{\alpha=0, \beta \to 1} \mu_1 = \frac{2}{|\cos(\theta - \gamma_1)| + \cos(\theta - \gamma_1)}$$

which gives $\mu = \sec(\theta - \gamma_1)$ when $\theta - \gamma_k \in [-\pi/2, \pi/2]$, but $\mu = +\infty$ otherwise. This completes the proof.

COROLLARY 2. *If all singularities of a complex function* $f(z)$ *are on the left side of the imaginary axis, then, when* $\hbar = -1 + \beta \exp(i\alpha)$ *tends to* 0 *along the negative real axis, the general Taylor series* (3) *converges to* $f(z)$ *on the whole positive real axis.*

Proof: owing to COROLLARY 1, this corollary is obviously right.

LEMMA 1. *For* $0 \le n \le m$, *it holds* $\phi_{m,n}(-1) = 1$.

Proof: Owing to the definition of $\phi_{m,n}(\hbar)$, $\phi_{m,0}(-1) = 1$ holds. Furthermore, when $1 \le n \le m-1$, it holds

$$\phi_{m,n}(\hbar) = (-\hbar)^n + (-\hbar)^n \sum_{k=1}^{m-n} \binom{k+n-1}{k} (1+\hbar)^k,$$

which gives

$$\phi_{m,n}(-1) = 1^n + 0 = 1.$$

Besides, it holds $\phi_{m,m}(\hbar) = (-\hbar)^m$ so that we have $\phi_{m,m}(-1) = 1$. This ends the proof.

LEMMA 2. *Let $n \geq 0$ be finite integer and $\hbar$ a complex number. When $|1+\hbar|<1$, it holds* $\lim_{m->+\infty} \phi_{m,n}(\hbar)=1$.

Proof: Owing to the definition of $\phi_{m,n}(\hbar)$, $\phi_{m,0}(\hbar)=1$ holds. Moreover, when $|1+\hbar|<1$, we have

$$(-\hbar)^{-n} = \left[1-(1+\hbar)\right]^{-n} = \sum_{k=0}^{+\infty}\binom{n+k-1}{k}(1+\hbar)^k .$$

Thus, for finite integer $n \geq 1$, it holds

$$\lim_{m->+\infty}\phi_{m,n}(\hbar)=(-\hbar)^n \lim_{m->+\infty}\sum_{k=0}^{m-n}\binom{n+k-1}{k}(1+\hbar)^k$$
$$=(-\hbar)^n\sum_{k=0}^{+\infty}\binom{n+k-1}{k}(1+\hbar)^k = (-\hbar)^n(-\hbar)^{-n} = 1$$

This ends the proof.

LEMMA 3. *It holds $\Phi_{m,n}(\hbar)=\phi_{m,n}(\hbar)$, where*

$$\Phi_{m,n}(\hbar)=\begin{cases}1, & n \leq 0\\ (-\hbar)^n\sum_{k=0}^{m-n}\binom{k+n-1}{k}\binom{m}{m-n-k}\hbar^k, & 1 \leq n \leq m\end{cases}$$

Proof: When $1 \leq n \leq m$, it holds

$$\phi_{m,n}(\hbar)=(-\hbar)^n\sum_{k=0}^{m-n}\binom{k+n-1}{k}(1+\hbar)^k = (-\hbar)^n\sum_{k=0}^{m-n}\binom{k+n-1}{k}\sum_{r=0}^{k}\binom{k}{r}\hbar^r$$
$$=(-\hbar)^n\sum_{r=0}^{m-n}\sum_{k=r}^{m-n}\binom{k+n-1}{k}\binom{k}{r}\hbar^r = (-\hbar)^n\sum_{k=0}^{m-n}\sum_{r=k}^{m-n}\binom{r+n-1}{r}\binom{r}{k}\hbar^k$$

So, we need to show

$$\sum_{r=k}^{m-n}\binom{r+n-1}{r}\binom{r}{k}=\binom{k+n-1}{k}\binom{m}{m-n-k}.$$

Noticing that in the relevant ranges, it holds

$$\binom{r+n-1}{r}\binom{r}{k}=\binom{k+n-1}{k}\binom{r+n-1}{r-k},$$

which reduces to showing that

$$\binom{m}{m-n-k}=\sum_{r=k}^{m-n}\binom{r+n-1}{r-k}=\sum_{s=0}^{m-n-k}\binom{s+n+k-1}{s}.$$

Writing $r=n+k$, the above expression is

$$\binom{m}{r}=\sum_{s=0}^{m-r}\binom{s+r-1}{s}=\sum_{s=0}^{m-r}\binom{s+r-1}{r-1}=\sum_{j=r-1}^{m-1}\binom{j}{r-1}.$$

In order to prove this, we use the following formula in The Handbook of Mathematics (G.A. Kurn & T. M. Kurn, §21.5-1)

$$\binom{N+1}{n+1}=\sum_{j=n}^{N}\binom{j}{n}.$$

Setting $N=m-1, n=r-1$, it gives

$$\binom{m}{r}=\sum_{j=r-1}^{m-1}\binom{j}{r-1}.$$

This completes the proof.

REMARK 1. Owing to the LEMMA 3, $\phi_{m,n}(\hbar)$ is just the so-called approaching function $\Phi_{m,n}(\hbar)$ obtained by Liao [5] in the frame of the homotopy analysis method.

COROLLARY 3. *The classical Taylor series*

$$f(z_0)+\sum_{k=1}^{+\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

is a special case of the general Taylor series (3) *in case of* $\hbar=-1$.

Proof: Owing to LEMMA 1, it holds $\phi_{m,n}(-1)=1$ for $n\le m$. Moreover, when $\hbar=-1$, the convergence region (4) becomes $|z-z_0|<\min_{k\in I}|\xi_k-z_0|$. This ends the proof.

THEOREM 3. *Let* $t, \hbar$ and α *be real numbers. Then, the general Newtonian binomial expression*

$$1+\lim_{m\to+\infty}\sum_{n=1}^{m}\left[\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}t^{n}\right]\phi_{m,n}(\hbar),\quad (\alpha\neq 0,1,2,3,\cdots) \tag{7}$$

converges to $(1+t)^{\alpha}$ *in the region*

$$-1<t<\frac{2}{|\hbar|}-1 \qquad (-2<\hbar<0)\,, \tag{8}$$

where $\phi_{m,n}(\hbar)$ *is defined by* (5).

PROOF: The complex function $(1+z)^{\alpha}$ $(\alpha\neq 0,1,2,3,\cdots)$ has only one singularity $\xi_1=-1$. So, by Theorem 1, the series (7) converges to $(1+t)^{\alpha}$ in the region

$$|1+\hbar+\hbar t|<1, \qquad (-2<\hbar<0)\,,$$

which leads to (8). This completes the proof.

3. Conclusion

We rigorously derived the so-called generalized Taylor series (3) and logically proved some related theorems about convergence regions. These theorems indicate that, by multiplying each term of a series by the so-called approaching function $\phi_{m,n}(\hbar)$ defined by (5) and selecting a proper value of the auxiliary parameter $\hbar$, one can greatly enlarge its convergence region. Due to the fact that the so-called generalized Taylor series (3) was first obtained by Liao [5] in the frame of the homotopy analysis method, the proof of the generalized Taylor theorem also provides us with a solid rational base of the validity of the homotopy analysis method, although indirectly.

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