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# A direct boundary element approach for unsteady non-linear heat transfer problems

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**Abstract**

Unsteady non-linear problems are usually solved based on the time marching scheme together with iterative methods. In this paper, we employed an example to illustrate a new direct boundary element technique, which can provide accurate solutions of unsteady non-linear problems without the need for iterations. The principle and validity of the technique is demonstrated by considering an unsteady non-linear hyperbolic heat transfer problem, together with a fully implicit difference scheme in the time domain. It illustrates that the proposed non-iterative boundary element approach is numerically rather efficient for unsteady non-linear problems. Thus, it provides an alternative to traditional iterative methods for unsteady non-linear problems. © 2001 Elsevier Science Ltd. All rights reserved.

*Keywords*: Heat transfer; Non-linear problems; Damping effect; General boundary element method

## 1. Introduction

Many physical problems in nature are governed by unsteady non-linear differential equations. In most numerical analyses, these equations are solved based on a time marching technique. The principle of this technique is to use the solution at the present time step as a starting point to get the solution at the next time step. In theory, the procedure can be repeated over any desired period of time. In practice, however, its success very much depends on the ratio of the step used in the time marching to the size of the cell used to discretise the space domain. In particular, when the so-called explicit scheme is used, which usually gives a linear equation at each time step, the numerical result may become inherently unstable. Even when it is stable, an extremely small time step may be needed, which usually means prohibitive computational requirements. Thus, in many cases the unsteady non-linear differential equations are solved by the fully implicit scheme that gives a non-linear equation at each time step. The most commonly used method to solve a non-linear equation is based on iteration, and iterative methods have been quite successful in many applications. However, its drawback is that it does not always converge, and even when it does converge, the rate of convergence is extremely slow in some cases.

In the present work, we will solve the unsteady non-linear differential equation using a direct boundary element approach based on homotopy analysis method [5,6]. The method has been used in some problems, which are governed by the steady non-linear differential equations [3,4]. Here, we further consider an unsteady non-linear heat transfer problem governed by the hyperbolic heat conduction equations [1,7,8]. We will show that this non-iterative boundary element approach is especially efficient and accurate for unsteady non-linear problems so that it can become an alternative to traditional iterative methods.

## 2. Governing equation and solution procedure

To show how the scheme described in the introduction works, we consider here an unsteady non-linear hyperbolic heat transfer problem governed by

$$\frac{1}{c^2}\frac{\partial \theta^2}{\partial t^2} + \frac{1}{\nu}\frac{\partial \theta}{\partial t} = \nabla[k(\theta)\nabla\theta], \tag{2.1}$$

where $\theta$ is temperature, $t$ denotes time, $\nu$ is thermal diffusivity and $c$ is thermal propagation speed. All these parameters have been non-dimensionalised in a conventional manner [1,7]. Eq. (2.1) can model the propagation of thermal waves, which have initially a sharp front and with a damping effect. Details can be found in the works of Baumeister et al. [1], Bialecki et al. [2] and Tzou [8]. For the simplicity, we consider here the unsteady heat transfer process in a two-dimensional unit square with $c = \nu = 1$

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and $k(\theta) = 1 + \alpha\theta$. Eq. (2.1) becomes

$$\frac{\partial\theta^2}{\partial t^2} + \frac{\partial\theta}{\partial t} = (1 + \alpha\theta)\nabla^2\theta + \alpha\left(\theta_x^2 + \theta_y^2\right), \quad r = (x, y) \in \Omega \tag{2.2}$$

with boundary conditions

$$\theta(r, t) = 1, \quad r \in \Gamma_1, \tag{2.3}$$

$$\frac{\partial\theta(r, t)}{\partial\mathbf{n}} = 0, \quad r \in \Gamma_2, \tag{2.4}$$

and initial conditions

$$\theta(r, 0) = 0, \quad r \in \Omega, \tag{2.5}$$

$$\left.\frac{\partial\theta}{\partial t}\right|_{t=0} = 0, \quad r \in \Omega, \tag{2.6}$$

where $\Gamma_1$ is the half centre part of the left side of the unit square, $\Gamma_2$ denotes the other sides with heat insulation, $\mathbf{n}$ is the outward unit normal vector to the boundary, and $\Omega = (0, 1) \times (0, 1)$ is the spatial domain.

The above problem will be solved by the time marching method with a constant time step $\Delta t$. Let $\theta^n(r) = \theta(r, n\Delta t)$ denote the temperature at the $n$th time step. From Eqs. (2.5) and (2.6), we have the results at the first two time steps, namely

$$\theta^0(r) = \theta^1(r) = \begin{cases} 1, & r \in \Gamma_1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

For the purpose of numerical stability, we discretise Eq. (2.2) in the time domain in terms of a fully implicit form with the backward finite-difference method for both the first-order and second-order time derivatives. This gives

$$(1 + \alpha\theta^n)\nabla^2\theta^n + \alpha\left[\left(\frac{\partial\theta^n}{\partial x}\right)^2 + \left(\frac{\partial\theta^n}{\partial y}\right)^2\right] - (\gamma + \gamma^2)\theta^n + (\gamma + 2\gamma^2)\theta^{n-1} - \gamma^2\theta^{n-2} = 0, \tag{2.8}$$

with boundary condition

$$\theta^n(r) = 1, \quad r \in \Gamma_1, \tag{2.9}$$

$$\frac{\partial\theta^n(r)}{\partial\mathbf{n}} = 0, \quad r \in \Gamma_2, \tag{2.10}$$

where $\gamma = 1/\Delta t$ and $\theta^{n-1}(r), \theta^{n-2}(r)$ are known temperature distributions at the $(n-1)$th and $(n-2)$th time steps, respectively. These equations present a non-linear boundary-value problem at each time step, which is usually solved using iterative methods.

Here, we shall use a different method. Let us define a non-linear differential operator

$$\mathscr{A}\Theta = (1 + \alpha\Theta)\nabla^2\Theta + \alpha\left[\left(\frac{\partial\Theta}{\partial x}\right)^2 + \left(\frac{\partial\Theta}{\partial y}\right)^2\right] - (\gamma + \gamma^2)\Theta + (\gamma + 2\gamma^2)\theta^{n-1} - \gamma^2\theta^{n-2}, \tag{2.11}$$

where $n \geq 2$. We then construct the so-called zeroth-order deformation equation

$$(1 - p)\mathscr{L}[\Theta(r, \hbar, p) - \theta_0(r)] = \hbar p\mathscr{A}[\Theta(r, \hbar, p)], \quad p \in [0, 1],\ r \in \Omega,\ \hbar \neq 0, \tag{2.12}$$

with boundary conditions

$$\Theta(r, \hbar, p) = 1, \quad r \in \Gamma_1,\ p \in [0, 1],\ \hbar \neq 0, \tag{2.13}$$

$$\frac{\partial\Theta(r, \hbar, p)}{\partial\mathbf{n}} = 0, \quad r \in \Gamma_2,\ p \in [0, 1],\ \hbar \neq 0, \tag{2.14}$$

where $\theta_0(r)$ is an initial approximation which satisfies the boundary conditions Eqs. (2.9) and (2.10), $\Theta(r, \hbar, p)$ is the homotopy of $\theta^n(r)$ and the auxiliary linear operator $\mathscr{L}$ is a second-order differential operator to be specified later. When $p = 0$, we have from Eqs. (2.12)–(2.14) that

$$\Theta(r, \hbar, 0) = \theta_0(r). \tag{2.15}$$

When $p = 1$, Eqs. (2.12)–(2.14) are the same as Eqs. (2.8)–(2.10), and therefore we have

$$\Theta(r, \hbar, 1) = \theta^n(r). \tag{2.16}$$

Thus, as $p$ increases from 0 to 1, $\Theta(r, \hbar, p)$ varies from the initial approximation $\theta_0(r)$ to $\theta^n(r)$, the temperature at the $n$th time step, and Eq. (2.12) constructs a kind of continuous mapping. Due to continuous mapping theory, the variation (or the so called deformation) $\Theta(r, \hbar, p)$ is commonly continuous.

Assume that $\theta_0(r), \hbar$ and the auxiliary linear operator $\mathscr{L}$ are properly selected so that the deformation $\Theta(r, \hbar, p)$ is smooth enough and there exists

$$\theta_0^{[m]}(r, \hbar) = \frac{1}{m!}\left.\frac{\partial^m\Theta(r, \hbar, p)}{\partial p^m}\right|_{p=0} \quad (m \geq 1), \tag{2.17}$$

the so called $m$th-order deformation derivatives. Then, together with Eq. (2.15), the Maclaurin series of the deformation $\Theta(r, \hbar, p)$ is given as

$$\theta_0(r) + \sum_{m=1}^{+\infty}\theta_0^{[m]}(r, \hbar)p^m, \quad r \in \Omega. \tag{2.18}$$

When $p = 1$, the above series becomes

$$\theta_0(r) + \sum_{m=1}^{+\infty}\theta_0^{[m]}(r, \hbar), \quad r \in \Omega. \tag{2.19}$$

When this Maclaurin series is convergent, due to condition (2.16), we get the $M$th-order approximation of $\theta^n(r)$, namely

$$\theta^n(r) \approx \sum_{m=0}^{M}\theta_0^{[m]}(r, \hbar), \quad r \in \Omega, \tag{2.20}$$

where $\theta_0^{[0]}(r, \hbar) = \theta_0(r)$. It should be noted that whether the series (2.19) converges or not depends upon both $\mathscr{L}$ and $\hbar$, and it is a crucial part of the scheme to properly select both of them.

Eq. (2.20) gives a relationship between the initially guessed $\theta_0(r)$ and the solution required at the $n$th time step, $\theta^n(r)$, through the deformation derivatives $\theta_0^{[m]}(r)$ ($m \geq 1$). These derivatives can be obtained from the following procedure. Differentiating Eqs. (2.12)–(2.14) $m$ times with respect to $p$, dividing both sides of the result by $m!$ and then setting $p = 0$, we obtain the so-called $m$th-order deformation equation

$$\mathscr{L}\theta_0^{[m]} = f_m(r,\hbar), \quad r \in \Omega, \hbar \neq 0, \tag{2.21}$$

with boundary conditions

$$\theta_0^{[m]}(r,\hbar) = 0, \quad r \in \Gamma_1, \tag{2.22}$$

$$\frac{\partial \theta_0^{[m]}(r,\hbar)}{\partial \mathbf{n}} = 0, \quad r \in \Gamma_2, \tag{2.23}$$

where

$$f_1(r,\hbar) = \hbar \mathscr{A}\theta_0, \tag{2.24}$$

$$f_m(r,\hbar) = \mathscr{L}\theta_0^{[m-1]} + \frac{\hbar}{(m-1)!} \left. \frac{\partial^{m-1}\mathscr{A}[\Theta(r,p,\hbar)]}{\partial p^{m-1}} \right|_{p=0}, \quad m \geq 2, \tag{2.25}$$

and

$$\frac{1}{(m-1)!} \left. \frac{\partial^{m-1}\mathscr{A}[\Theta(r,p,\hbar)]}{\partial p^{m-1}} \right|_{p=0} = \nabla^2\theta_0^{[m-1]} - (\gamma + \gamma^2)\theta_0^{[m-1]} + \alpha \sum_{i=0}^{m-1} \left( \theta_0^{[m-1-i]}\nabla^2\theta_0^{[i]} + \frac{\partial \theta_0^{[m-1-i]}}{\partial x} \frac{\partial \theta_0^{[i]}}{\partial x} + \frac{\partial \theta_0^{[m-1-i]}}{\partial y} \frac{\partial \theta_0^{[i]}}{\partial y} \right). \tag{2.26}$$

The solution procedure at each time step starts from choosing $\theta_0$. If $\theta_0$ is not accurate, then $\mathscr{A}\theta_0$ will give a significant error. This error will be corrected first through using Eqs. (2.21)–(2.24). If the correction is not sufficient, further higher-order approximations governed by Eqs. (2.21) –(2.23), (2.25) and (2.26) will be given, and this can be repeated until the desired accuracy has been achieved. Thus, to get an approximation at $m$th-order, one has to use all of the approximations at lower orders. This is somewhat similar to some analytic techniques but essentially different from traditional iterative methods, which need only *one* initial guess.

It has been found that in some cases, the error in $\theta_0$ can be overcorrected through Eqs. (2.21)–(2.26), which may lead to a divergent Maclaurin series. For this reason, $-1 \leq \hbar < 0$ has been introduced in the scheme to make each correction milder so that the series converges. More details on the role of $\hbar$ can be found in the work of Liao [5]. In addition, we emphasise here that the $m$th-order ($m \geq 1$) deformation Eqs. (2.21)–(2.23) are linear. Thus, due to the expression (2.20), we can obtain the $M$th-order approximation of the temperature $\theta^n(r)$ by solving $M$ linear PDEs successively, even though the temperature $\theta^n(r)$ itself is governed by a non-linear PDE.

In the original non-linear Eq. (2.8) there exists a linear component

$$\nabla^2\theta^n - (\gamma + \gamma^2)\theta^n. \tag{2.27}$$

Thus, we select an auxiliary linear operator $\mathscr{L}$ such that

$$\mathscr{L}\phi = \nabla^2\phi - (\gamma + \gamma^2)\phi, \tag{2.28}$$

which is a 2D modified Helmholtz operator. It is related to the fundamental solution

$$\omega(r,r') = -\frac{1}{2\pi} K_0\left(\Lambda\sqrt{(x-x')^2 + (y-y')^2}\right), \tag{2.29}$$

where $\Lambda = \sqrt{\gamma + \gamma^2}$, $r' = (x', y')$ is the source point, and $K_0$ is the modified Bessel function of the second kind of order zero. The Green's identity then gives

$$\theta_0^{[m]}(r) = \oint_\Gamma \left[ \omega(r,r') \frac{\partial \theta_0^{[m]}(r')}{\partial n} - \frac{\partial \omega(r,r')}{\partial n} \theta_0^{[m]}(r') \right] \mathrm{d}\Gamma - \int\int_\Omega \omega(r,r') f_m(r',\hbar) \mathrm{d}\Omega. \tag{2.30}$$

Substituting Eqs. (2.22) and (2.23) into Eq. (2.30), we obtain the integral equation

$$\theta_0^{[m]}(r) = \oint_{\Gamma_1} \left[ \omega(r,r') \frac{\partial \theta_0^{[m]}(r')}{\partial n} \right] \mathrm{d}\Gamma - \oint_{\Gamma_2} \left[ \frac{\partial \omega(r,r')}{\partial n} \theta_0^{[m]}(r') \right] \mathrm{d}\Gamma - \int\int_\Omega \omega(r,r') f_m(r',\hbar) \mathrm{d}\Omega. \tag{2.31}$$

Substituting the above expression into the boundary conditions (2.22) and (2.23), we obtain equations for the unknown $\partial\theta_0^{[m]}(r)/\partial n$ on the boundary $\Gamma_1$ and $\theta_0^{[m]}(r)$ on the boundary $\Gamma_2$. Discretising these equations, using the traditional BEM, we get a set of linear algebraic equations. It should be noted that the coefficient matrices of the set of algebraic equations are the same for all $m \geq 1$ at every time step. Thus, we need to calculate the matrix and its inverse only once, and this makes the proposed approach quite efficient.

## 3. Numerical results

In the numerical computation, we divide each boundary

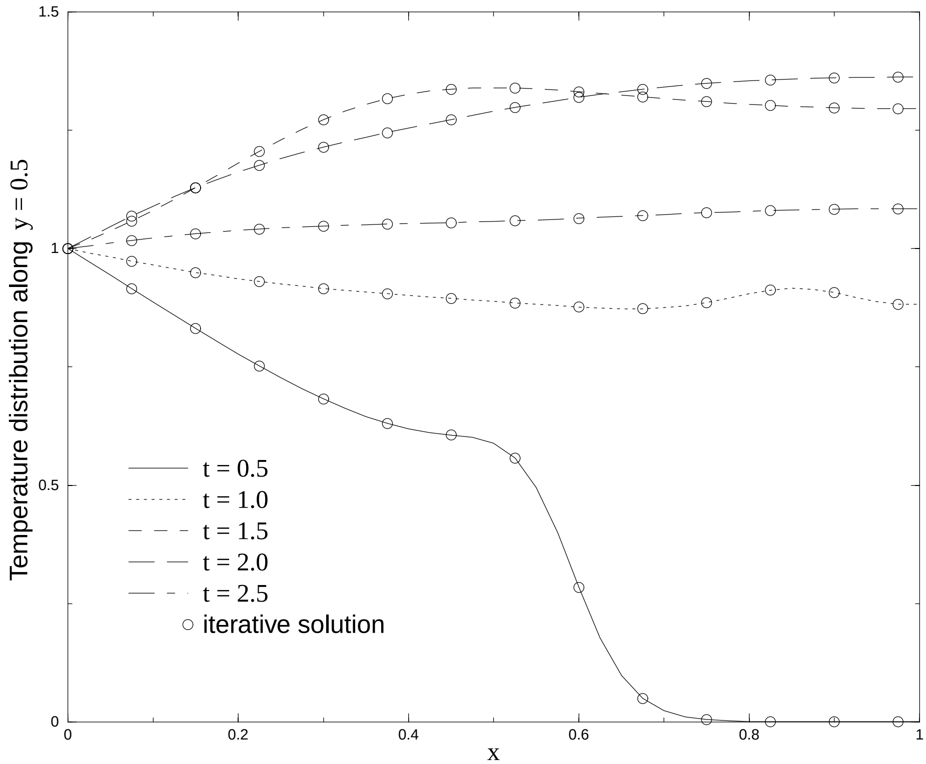


Fig. 1. Comparison of the 3rd-order approximation (at $y = 0.5$) with iterative results in the case of $\alpha = 1$, $\Delta t = 0.01$ and $\hbar = -1$.

side into $N$ equal boundary elements. At each corner of the domain, two points, which are located on the different sides but very close to the corner, are used to deal with the discontinuity. Within each element, a linear variation for the unknown is assumed and the Gauss-integral is used to numerically calculate the domain integral in Eq. (2.31). In most cases, four-point formula is sufficient, but when $\Delta t$ is small, more points are needed because of the sharp variation of the fundamental solution $\omega$ defined by Eq. (2.29). To get the unknown $\theta^n(r)$ at $t = n\Delta t$, we use the solution at the previous step $\theta^{n-1}(r) = \theta(n\Delta t - \Delta t)$ as the initial guess $\theta_0(r)$.

We consider two cases here, namely $\alpha = 1$ and $\alpha = 10$, and in both cases we set $N = 40$. However, we chose different time steps, i.e. $\Delta t = 0.01$ when $\alpha = 1$ and $\Delta t = 0.0025$ when $\alpha = 10$, respectively. This is mainly because in the case of $\alpha = 10$ the thermal wave propagates faster so that smaller $\Delta t$ should be used. All these parameters can give converged results, as shown in Figs. 1 and 2. Once these parameters have been chosen, the convergence rate of the series in Eq. (2.19) solely depends on the value of $\hbar$. Our experience, through numerous calculations, indicates that when $-1 \le \hbar < 0$ the series is always convergent at all time steps for both $\alpha = 1$ and $\alpha = 10$. In the calculation, $M$ is chosen in such a way that it can guarantee sufficiently accurate results throughout the time steps. In the case of $\alpha = 1$ and $\hbar = -1$, it is found that even the 3rd-order approximation, i.e. $M = 3$ in eq. (2.20), gives sufficiently accurate results, compared with those obtained from an iterative boundary element approach [3,4], as shown in Fig. 1. In the case of $\alpha = 10$ and $\hbar = -1$, the 5th-order approximation ($M = 5$) gives sufficiently accurate results, as shown in Fig. 2. In the case of $\hbar = -3/4$, the 5th-order approximations give the same results for both $\alpha = 1$ and $\alpha = 10$ as those in the case of $\hbar = -1$. Figs. 1 and 2 show a very good agreement between the results given by the present direct method and those given by the iterative method. This clearly illustrates that the present non-iterative boundary element method can accurately solve the unsteady non-linear problem.

Since all the PDEs Eqs. (2.21)–(2.23) are governed by the same auxiliary linear operator, the proposed approach is numerically efficient. For example, in the case of $\alpha = 1$, $\Delta t = 0.01$, $\hbar = -3/4$ and $M = 5$, the calculations takes only about 49 s on a PC (PII 400) to give a detailed description of the thermal propagation over the period $0 \le t \le 6$. In the case of $\alpha = 10$, $\Delta t = 0.0025$, $\hbar = -3/4$ and $M = 5$, the calculation takes about 108 s.

## 4. Conclusions

A direct (non-iterative) boundary element method for solving unsteady non-linear problems is presented. To describe its basic ideas and illustrate its validity, a non-linear hyperbolic heat transfer problem is employed. Comparisons of our numerical results with those given by traditional iterative methods indicate that the proposed

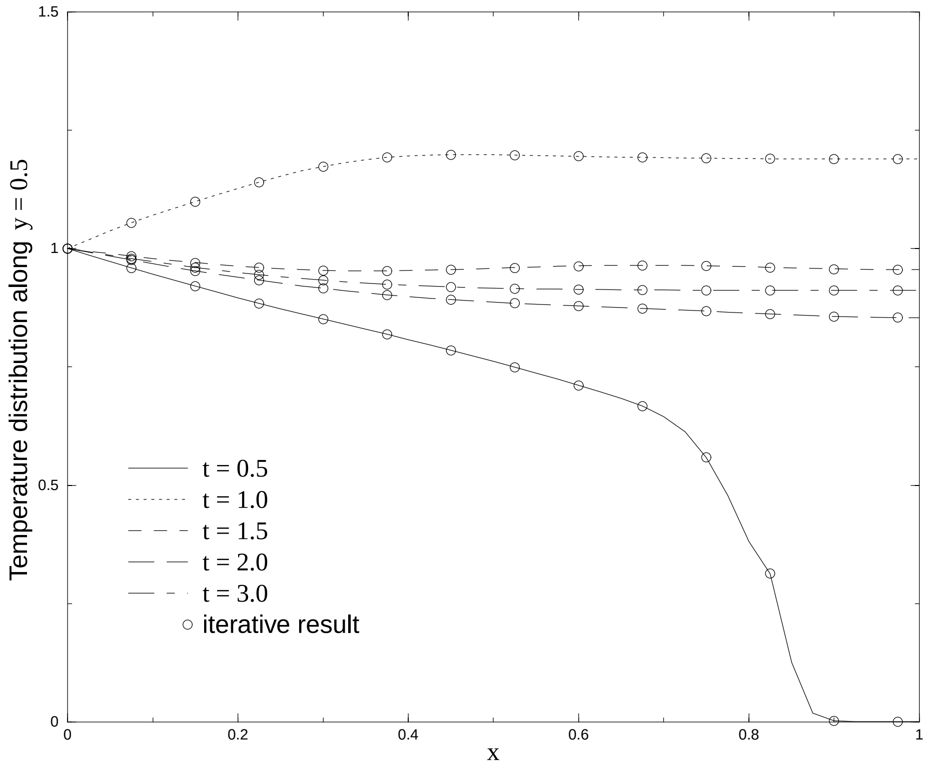


Fig. 2. Comparison of the fifth order approximations (at $y = 0.5$) with iterative results in the case $\alpha = 10$, $\Delta t = 0.0025$ and $\hbar = -1$.

non-iterative boundary element method can provide rather accurate solutions. It should be noted that the approximations, even at third or fifth order, are accurate for the non-linear heat transfer problem under consideration. This illustrates that the direct boundary element approach is especially suitable and numerically efficient for *unsteady* non-linear problems. Therefore, our approach provides an alternative to traditional iterative methods, which are commonly used for unsteady non-linear problems.

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