SCIENCE CHINA
Physics, Mechanics & Astronomy

• **Article** •

November 2014 Vol. 57 No. 11: 2121–2126
doi: 10.1007/s11433-014-5563-5

On the stability of the three classes of Newtonian three-body planar periodic orbits†

LI XiaoMing[2] & LIAO ShiJun[1,2,3*]

[1] *Ministry-of-Education Key Laboratory in Scientific Computing, Shanghai 200240, China;*
[2] *School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai 200240, China;*
[3] *Nonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University (KAU), Jeddah, Saudi Arabia*

Received May 23, 2014; accepted July 15, 2014; published online August 18, 2014

Currently, the fifteen new periodic orbits of Newtonian three-body problem with equal mass were found by Šuvakov and Dmitrašinović [Phys Rev Lett, 2013, 110: 114301] using the gradient descent method with double precision. In this paper, these reported orbits are checked stringently by means of a reliable numerical approach (namely the “Clean Numerical Simulation”, CNS), which is based on the arbitrary-order Taylor series method and data in arbitrary-digit precision with a procedure of solution verification. It is found that seven among these fifteen orbits greatly depart from the periodic ones within a long enough interval of time, and are thus most possibly unstable at least. It is suggested to carefully check whether or not these seven unstable orbits are the so-called “computational periodicity” mentioned by Lorenz in 2006. This work also illustrates the validity and great potential of the CNS for chaotic dynamic systems.

three body problem, periodic orbit, stability, computational reliability, Clean Numerical Simulation (CNS)

PACS number(s): 96.12.Pc, 96.30.ld

Citation: Li X M, Liao S J. On the stability of the three classes of Newtonian three-body planar periodic orbits. Sci China-Phys Mech Astron, 2014, 57: 2121–2126, doi: 10.1007/s11433-014-5563-5

1 Introduction

It is a common knowledge that orbits of the famous three-body problem [1] are not integrable generally. Although chaotic orbits of three-body problems are common, however, three families of periodic orbits were found:

1. the Lagrange-Euler family, dating back to the analytical solutions in the 18th century (one recent orbit was given by Moore [2]);

2. the Broucke-Hadjidemetriou-Hénon family, dating back to the mid-1970s [3–8];

3. the Figure-8 family, discovered in 1993 by Moore [2] and extended to the rotating cases [9–12].

Note that nearly all of these reported periodic orbits are planar. Currently, Šuvakov and Dmitrašinović [13] found by means of the gradient descent method (in normal precision) that there exist four classes of planar periodic orbits of Newtonian three body with equal mass, with the above three families belonging to one class. Besides, they reported three new classes of planar periodic orbits and gave a few initial conditions for each class with the 5-digit precision. At first, they found around 50 different regions containing candidates for periodic orbits, at return proximity of 10^{-1} in the phase space, while evaluating this section of the initial conditions space. Then, they further refined these initial conditions to the level of return proximity of less then 10^{-6} by means of the gradient descent method. For the details of their 15 planar “periodic” orbits, please refer to the gallery (http://suki.ipb.ac.rs/3body/). Especially, Šuvakov and Dmi-

*Corresponding author (email: sjliao@sjtu.edu.cn)
†Contributed by LIAO ShiJun (Associate Editor)

trašinović [13] expected their solutions "to be either stable or marginally unstable, as otherwise they probably would not have been found" by their numerical method.

Let the vector $\boldsymbol{r}_i(t)$ denote the orbit of three body with equal mass, where t denotes the time and $i = 1, 2, 3$, respectively. If the orbit is periodic with the period T, it holds $\boldsymbol{r}_i(t) = \boldsymbol{r}_i(t + nT)$ for arbitrary time $t \geqslant 0$ and arbitrary integer $n \geqslant 1$. If, given a tiny disturbance (for example at $t = 0$), the three-bodies greatly depart their periodic orbits after a prolonged time, then the corresponding orbits are unstable.

Šuvakov and Dmitrašinović [13] used the gradient descent method to search for the initial conditions of the periodic orbits of three-body of equal mass in the accuracy of 10^{-6}. Currently, they obtained the more accurate initial conditions in the 15-digit precision. However, it is well-known that orbits of three-body problem are often chaotic, i.e., very sensitive to initial conditions [14–16]. As reported by Lorenz [14], many traditional numerical methods in single or double precision often lead to the so-called "computational periodicity" (CP) of chaotic dynamic systems: when the exact solution is chaotic, computed solutions are, however, periodic within a range of time step. Thus, it is very important to gain reliable orbits of the three-body problem. Note that Šuvakov and Dmitrašinović [13] employed a traditional numerical method only in the double precision. Thus, it is necessary to check their reported periodic orbits very carefully using a more reliable approach.

To gain mathematically reliable numerical simulations of orbits of Newtonian three body problem, we use here the so-called "Clean Numerical Simulation" (CNS) [17–19], which is based on the arbitrary-order Taylor series method (TSM) and the arbitrary precision library of C with a procedure of solution verification. The TSM [20–22] can trace back to Newton, Euler, Liouville and Cauchy. It has an advantage that its formula at arbitrary order can be easily expressed in the same form. So, from viewpoint of numerical simulations, it is rather easy to use the TSM at very high order so as to deduce the truncation error to a required level. Besides, the round-off error can be reduced to arbitrary level by means of the multiple precision (MP) library [23]. Let M denote the order of TSM and N_s the number of significant digits of multiple-precision data, respectively. Unlike other numerical approaches, the CNS enforces that N_s increases together with M, such as $N_s = 2M$ as illustrated by Liao [17] who gained, for the first time, a reliable chaotic solution in a long interval [0, 1000] of Lorenz equation by means of $M = 400$ and $N_s = 800$. More importantly, unlike other methods, the CNS has a procedure of solution verification: the reliability of one CNS simulation in a given finite but long enough interval must be guaranteed by means of other better CNS simulations using larger M and/or smaller time step Δt. In this way, the numerical noises can be decreased to such a small level that both truncation and round-off errors are negligible in a given finite but sufficiently long interval. For example, using the 400th-order TSM and the high floating point precision (800-digits), Liao [17] gained a reliable chaotic solution of Lorenz equation in a long interval [0, 1000] of time, whose reliability was confirmed currently by Kehlet and Logg [24] using a 200th-order finite element method with the high floating point precision (400 digits). In addition, Liao [18] employed the CNS to accurately and reliably simulate the propagation of physical uncertainty of initial positions (caused by the so-called wave-particle duality of de Broglie, even at the infinitesimal dimensionless level 10^{-60}) of the chaotic Hamiltonian Hénon-Heiles system for motions of stars in a plane about the galactic center. Recently, using 1200 CPUs of the National Supercomputer TH-A1 and the modified parallel integral algorithm based on the CNS with the 3500th-order TSM and the 4180-digit multiple-precision data, Liao and Wang [25], for the first time, obtained a mathematically reliable simulation of chaotic solution of Lorenz equation in a rather long interval [0, 10000]. Such kind of reliable, convergent chaotic solution of Lorenz equation has never been reported. All of these indicate that the CNS can indeed provide us a safe way to gain mathematically reliable simulations of chaotic dynamic systems in a finite but adequately long interval.

Currently, Liao [19] successfully applied the CNS to accurately investigate the influence of the micro-level physical uncertainty of initial position of three-body problem with equal mass on its chaotic trajectories, and found that the micro-level physical uncertainty might transfer into macroscopic randomness. This further validates the utility of the CNS for three-body problem. So, in this paper, we use the same CNS approach to check and verify the periodic orbits reported by Šuvakov and Dmitrašinović [13]. All numerical simulations reported below are obtained by the CNS with high enough order of TSM and accurate enough multiple-precision data with a procedure of solution verification, say, whose validity in a given long enough interval is further confirmed by better CNS simulations using higher-order TSM, and/or more accurate MP data, and/or smaller time step Δt. For the detailed numerical algorithm, please refer to Liao [19].

2 Stability of the newly found periodic orbits

Šuvakov and Dmitrašinović [13] reported the initial conditions of the newly found periodic orbits in the 5-digit precision. Currently, they obtained the more accurate initial conditions in the 15-digit precision for their periodic orbits. Using these initial conditions, we simulate the orbits by means of the CNS. According to our highly accurate numerical results in a long enough interval, at least seven orbits among them (listed in Table 1) are most possibly unstable.

Without loss of generality, let us consider the case of BUTTERFLY-I, i.e., the Class I.A.1, defined by Šuvakov and Dmitrašinović [13]. Using the 20th to 50th order TSM and the 300-digit multiple precision data with the time-step $\Delta t = 10^{-5}$, we obtain the convergent trajectories of the three bodies in the interval [0, 200]: all of these trajectories agree

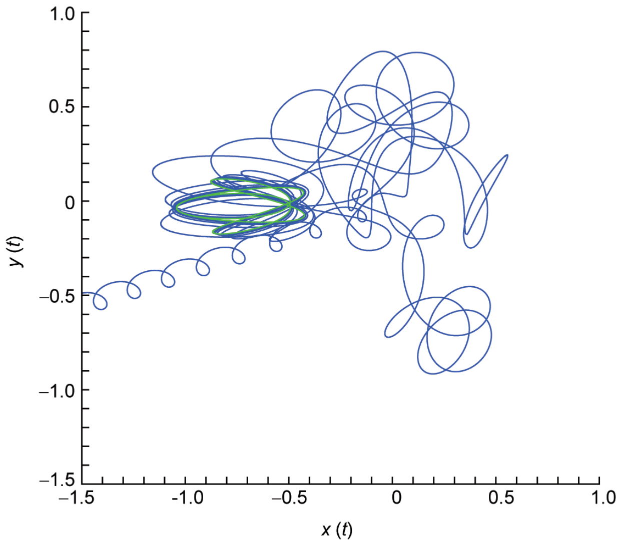

Figure 1 Orbits of Body-1 in case of BUTTERFLY-I in the interval [0, 200] gained by means of the CNS using 40th-order TSM and 300-digit multiple precision with time step $\Delta t = 10^{-5}$. The initial condition in 15-digit precision in Table 1 is used. Green line: periodic orbit reported in ref. [13].

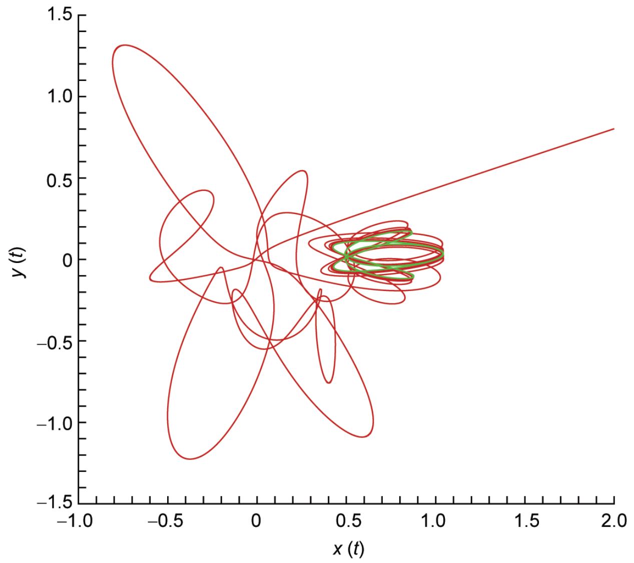

Figure 2 Orbits of Body-2 in case of BUTTERFLY-I in the interval [0, 200] gained by means of the CNS using 40th-order TSM condition in 15-digit precision in Table 1 is used. Green line: periodic orbit reported in ref. [13].

Table 1 The initial conditions of the 7 periodic orbits in 15-digit precision, given by Šuvakov in a private email communication

Class, number, name	$\dot{x}_1(0)$	$\dot{y}_1(0)$
I.A.1 BUTTERFLY I	0.306892758965492	0.125506782829762
I.B.4 MOTH III	0.383443534851074	0.377363693237305
I.B.5 GOGGLES	0.0833000564575194	0.127889282226563
I.B.7 DRAGONFLY	0.080584285736084	0.588836087036132
II.B.1 YARN	0.559064247131347	0.349191558837891
II.C.2a YIN-YANG I	0.513938054919243	0.304736003875733
II.C.2b YIN-YANG I	0.282698682308198	0.327208786129952

Table 2 The position (x_1, y_1) of Body-1 at $t = 200$ in case of BUTTERFLY-I given by the different orders of TSM and 300-digit multiple-precision data with the different time steps. The initial condition is listed in Table 1

Order	Δt	$x_1(200)$	$y_1(200)$
20	10^{-5}	–33.498137	–12.017376
25	10^{-5}	–33.498137957	–12.017376712
30	10^{-5}	–33.49813795772	–12.01737671265
40	10^{-5}	–33.49813795771996	–12.01737671265596
45	10^{-5}	–33.49813795771996	–12.01737671265596
50	10^{-5}	–33.49813795771996	–12.01737671265596
12	10^{-6}	–33.49813795771996	–12.01737671265596
15	10^{-6}	–33.49813795771996	–12.01737671265596
20	10^{-6}	–33.49813795771996	–12.01737671265596
12	10^{-7}	–33.49813795771996	–12.01737671265596

well in the whole interval [0, 200] at least in 7 digits, for example as shown in Table 2 for the position of Body-1 at $t = 200$. Note that, at the arbitrary order $M \geqslant 40$, the CNS results given by the Mth-order TSM with $\Delta t = 10^{-5}$ have at least the 14 significant digits in the whole interval [0, 200], whose reliability is further confirmed by using the M'th-order ($M' \geqslant 12$) TSM with a smaller time step $\Delta t = 10^{-6}$. Therefore, all of these CNS numerical simulations are convergent to the same result in the interval [0, 200] and thus are reliable mathematically. However, the corresponding orbits of the three bodies are approximately periodic only up to about $t = 130$, but thereafter depart from the periodic ones far and far away, as shown in Figures 1–3. According to our reliable numerical simulations in the interval [0, 200], we are quite sure that the orbits are completely non-periodic after $t > 130$, as shown in Figure 4: Body-1 and Body-3 escape together to become a binary-body system, while Body-2 escapes in the opposite direction. This counter-example clearly indicates that the initial condition in the 15-digit precision of

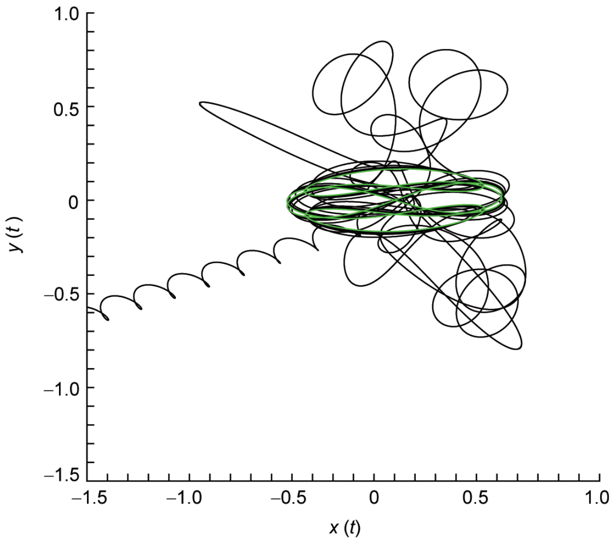

Figure 3 Orbits of Body-3 in case of BUTTERFLY-I in the interval [0, 200] gained by means of the CNS using 40th-order TSM and 300-digit multiple precision with time step $\Delta t = 10^{-5}$. The initial condition in 15-digit precision in Table 1 is used. Green line: periodic orbit reported in ref. [13].

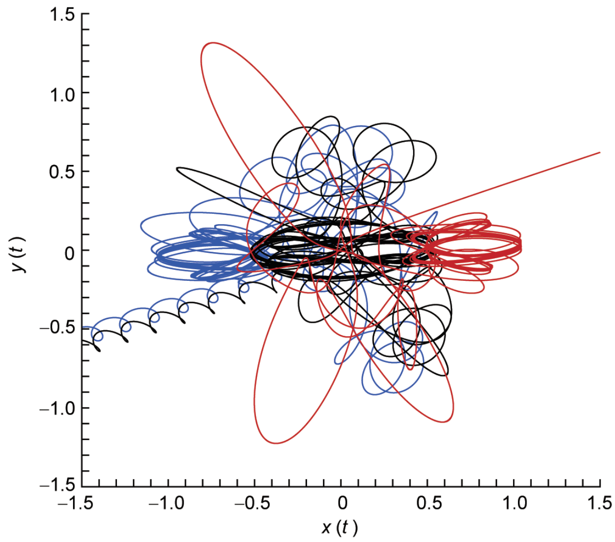

Figure 4 Orbits of three bodies in case of BUTTERFLY-I in the interval [0, 200] gained by means of the CNS using 40th-order TSM and 300-digit multiple precision with time step $\Delta t = 10^{-5}$. The initial condition in 15-digit precision in Table 1 is used. Blue line: orbit of Body-1; Red line: orbit of Body-2; Black line: orbit of Body-3.

the periodic orbit BUTTERFLY-I given by ref. [13] is not accurate enough to guarantee periodic orbits. Note that, for chaotic dynamic systems, exact initial conditions of periodic orbits should be irrational numbers, as illustrated by Viswanath [26] who reported the initial conditions of periodic solutions of Lorenz equation in accuracy of 500 significant digits. So, the initial condition in the 15-digit precision given by Šuvakov and Dmitrašinović [13] can be regarded as the exact initial condition plus a small disturbance at the level of 10^{-15}. Our accurate simulations given by the CNS indicate that the orbit related to BUTTERFLY-I departs from its periodic ones due to this very small disturbance, and thus is unstable.

The reliable numerical simulations of orbits by means of the seven initial conditions are listed in Table 1. The reliable simulations by means of the CNS with the 300-digit multiple precision data (we will not repeat this point thereafter), high enough orders of TSM and small enough time step are obtained similarly. Every numerical simulation given by the CNS is guaranteed to be convergent and reliable in a finite but long enough interval. However, it is found that all of these seven initial conditions can not guarantee periodic orbits: all of them depart from periodic orbits after a long enough time and thus are most possibly unstable, as mentioned below.

In case of the MOTH-III, the orbits in the interval [0, 590] gained by means of the initial condition listed in Table 1 and CNS (with the 20th-order TSM, the 300-digit multiple precision and the time step $\Delta t = 10^{-5}$) agree well in the 13 significant digits with those obtained by the CNS with 25th-order TSM and the same time step. The orbits are approximately periodic up to $t = 560$, i.e., about 22 periods ($T = 25.8406180475758$) of MOTH-III, but thereafter depart from the periodic ones far and far away.

In the case of GOGGLES, the orbits in the interval [0, 90] given by the CNS with the 30th-order TSM and the time step $\Delta t = 10^{-5}$ agree well in the 8 significant digits with those given by the 15th-order TSM and the smaller time step $\Delta t = 10^{-6}$. It is found that the orbits are almost periodic only up to $t = 55$ (i.e., a little more than 5 periods of GOGGLES reported by Šuvakov and Dmitrašinović [13]), but thereafter depart from the periodic ones far and far away.

In the case of DRAGONFLY, the orbits in the interval [0, 950] given by the CNS with the 25th-order TSM and the time step $\Delta t = 10^{-5}$ agree well in the 4 significant digits with those given by the 30th-order TSM and the same time step. The orbits are approximately periodic only up to $t = 720$, but thereafter drift far apart the periodic ones.

In the case of YARN, the orbits in the interval [0, 560] given by the CNS with the 25th-order TSM and the time step $\Delta t = 10^{-5}$ agree well in the 7 significant digits with those given by the 30th-order TSM and the same time step. The orbits are approximately periodic only up to $t = 440$, but thereafter depart far apart from the periodic ones.

In the case of YIN-YANG I (II.C.2a), the orbits in the interval [0, 320] given by the CNS with the 25th-order TSM and the time step $\Delta t = 10^{-5}$ agree well in the 16 significant digits with those given by the 30th-order TSM and the same time step. The orbits are approximately periodic only up to $t = 250$, however thereafter depart from the periodic ones far and far away.

In the case of YIN-YANG I (II.C.2b), the orbits in the interval [0, 190] given by the CNS with the 25th-order TSM and the time step $\Delta t = 10^{-5}$ agree well in the 6 significant digits with those given by the 30th-order TSM and the same time step. The orbits are approximately periodic only up to $t = 135$, but thereafter depart from the periodic ones further apart, as shown in Figure 5.

Besides, it is found that these orbits are sensitive to the initial conditions: adding a small disturbance at the level 10^{-17} to the initial conditions in Table 1, we gain a non-periodic orbit that departs considerably from the original non-periodic ones after a prolonged time. This confirms that the orbits given by the seven initial conditions in Table 1 are most possibly unstable.

Using the original initial conditions (in 5-digit precision) of the considered seven orbits reported by Šuvakov and Dmitrašinović [13], we gain qualitatively the same conclusion: the seven corresponding orbits become non-periodic after a long enough time and thus are most possibly unstable.

In summary, using the CNS with the initial conditions of the seven orbits reported by Šuvakov and Dmitrašinović [13] (see Table 1), we can gain convergent and reliable numerical results of the orbits, which however become non-periodic after a long enough interval of time. This suggests that at least the seven (listed in Table 1) of the periodic orbits reported by Šuvakov and Dmitrašinović [13] are most possibly unstable.

It is found that the other 8 periodic orbits (see Table 3) reported by Šuvakov and Dmitrašinović [13] do not greatly

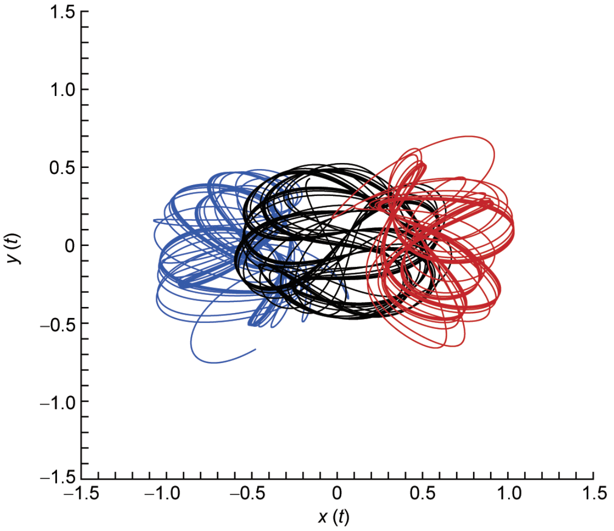

Figure 5 Orbits of three bodies in case of II.C.2b (YIN-YANG I) in the interval [0, 190] gained by means of the CNS using 25th-order TSM and 300-digit multiple precision with time step $\Delta t = 10^{-5}$. The initial condition in 15-digit precision in Table 1 is used. Blue line: orbit of Body-1; red line: orbit of Body-2; black line: orbit of Body-3.

Table 3 The initial conditions (in 15-digit precision) of the 8 stable periodic orbits given by Šuvakov and Dmitrašinović [13] in a private communication via email

Class, number, name	$\dot{x}_1(0)$	$\dot{y}_1(0)$
I.A.2 BUTTERFLY II	0.392955223941802	0.0975792352080344
I.A.3 BUMBLEBEE	0.184278506469727	0.587188195800781
I.B.1 MOTH I	0.464445237398184	0.396059973403921
I.B.2 MOTH II	0.439165939331987	0.452967645644678
I.B.3 BUTTERFLY III	0.405915588857606	0.230163127422333
I.B.6 BUTTERFLY IV	0.350112121391296	0.0793394773483276
II.C.3a YIN-YANG II	0.416822143554688	0.330333312988282
II.C.3b YIN-YANG II	0.417342877101898	0.313100116109848

depart from periodic orbits even though at $t = 10000$ (after hundreds of periods).

3 Concluding remarks and discussions

The fifteen periodic orbits of the Newtonian three-body problem currently reported by Šuvakov and Dmitrašinović [13] were checked very carefully by means of the CNS, a numerical technique based on the arbitrary order of Taylor series and the arbitrary precision of data with a procedure of solution verification. In all cases, convergent and reliable numerical results are obtained in a long enough interval of time. The convergent CNS results indicate that at least seven orbits (listed in Table 1) among them greatly depart from the periodic ones after a long enough interval of time, and thus are most possibly unstable. The other eight orbits (listed in Table 3) do not greatly depart from the periodic ones even in a rather long interval of time [0, 10000] after hundreds of periods.

It is well-known that dynamic systems related to three-body problem are often chaotic, i.e. very sensitive to initial condition. As pointed out by Lorenz [14], many traditional numerical approaches might lead to the so-called "computational periodicity", say, when the exact solution is chaotic, computed solutions seem, however, to be periodic within a range of time. It is an open question whether the seven unstable periodic orbits listed in Table 1 are "computational periodicity" or not. It is suggested to use periodic base functions to search for periodic solutions of complicated nonlinear dynamic systems like three-body problems, since such kind of solutions are periodic even for arbitrarily large time. Besides, stability analysis is very important, since an unstable dynamic system can not exist in nature.

This work also illustrates the validity and great potential of the CNS (Clean Numerical Simulation) for complicated nonlinear dynamic systems.

The authors would like to express their sincere acknowledgements to Dr. Šuvakov M for providing us the more accurate initial conditions (listed in Table 1 of the 15 periodic orbits found in ref. [13]. This work was partly supported by the National Natural Science Foundation of China (Grant No. 11272209) and the Deanship of Scientific Research (DSR), King Abdulaziz University (KAU) (Grant No. 37-130-35-HiCi).

1 Valtonen M, Karttunen H. The Three Body Problem. New York: Cambridge University Press, 2005
2 Moore C. Braids in classical dynamics. Phys Rev Lett, 1993, 70: 3675–3679
3 Broucke R, Boggs D. Periodic orbits in the planar general three body problem. Celest Mech, 1975, 11: 13–38
4 Hadjidemetriou J D, Christides T. Families of periodic orbits in the planar three-body problem. Celest Mech, 1975, 12: 175–187
5 Hadjidemetriou J D. Stability of periodic orbits in 3-body problem. Celest Mech, 1975, 12: 255–276
6 Broucke R. On relative periodic solutions of the planar general three body problem. Celest Mech, 1975, 12: 439–462
7 Hénon M. A family of periodic solutions of the planar three-body problem and their stability. Celest Mech, 1976, 13: 267–285
8 Hénon M. Stability of interplay motions. Celest Mech, 1977, 15: 243–261
9 Nauenberg M. Periodic orbits for three particles with finite angular momentum. Phys Lett A, 2001, 292: 93–99
10 Fejoz J, Chenciner A, Montgomery R. Rotating eights: I. The three families. Nonlinearity, 2005, 18: 1407–1424
11 Elipe A, Broucke R, Riaguas A. On the figure-8 periodic solutions in the three-body problem. Chaos Solitons Fractals, 2006, 30: 513–520
12 Nauenberg M. Continuity and stability of families of figure eight orbits with finite angular momentum. Celest Mech, 2007, 97: 1–15
13 Šuvakov M, Dmitrašinović V. Three classes of Newtonian three-body planar periodic orbits. Phys Rev Lett, 2013, 110: 114301
14 Lorenz E N. Computational periodicity as observed in a simple system. Tellus-A, 2006, 58: 549–557
15 Zeng Q G, Li J P, Chou J F. Computational uncertainty principle in nonlinear ordinary differential equations (ii): Theoretical analysis. Sci China Ser E-Technol Sci, 2001, 44: 55–74
16 Wang P F, Li J P, Li Q. Computational uncertainty and the application

of a high-performance multiple precision scheme to obtaining the correct reference solution of lorenz equations. Numer Algorithms, 2012, 59: 147–159

17 Liao S J. On the reliability of computed chaotic solutions of non-linear differential equations. Tellus-A, 2009, 61: 550–564

18 Liao S J. On the numerical simulation of propagation of micro-level inherent uncertainty for chaotic dynamic systems. Chaos Solitons Fractals, 2013, 47: 1–12

19 Liao S J. Physical limit of prediction for chaotic motion of three-body problem. Commun Nonlinear Sci Numer Simul, 2014, 19: 601–616

20 Corliss G F, Chang Y F. Solving ordinary differential equations using Taylor series. ACM Trans Math Software, 1982, 8: 114–144

21 Blesa F, Barrio R, Lara M. VSVO formulation of the Taylor method for the numerical solution of ODEs. Comput Math Appl, 2005, 50: 93–111

22 Abadb A, Barrio R, Rodrguez M, et al. Uncertainty propagation or box propagation. Math Comput Model, 2011, 54: 2602–2615

23 Oyanarte P. MP—A multiple precision package. Comput Phys Commun, 1990, 59: 345–358

24 Kehlet B, Logg A. Quantifying the computability of the Lorenz system. 2013, arXiv: 1306.2782

25 Liao S J, Wang P F. On the mathematically reliable long-term simulation of chaotic solutions of lorenz equation in the interval [0, 10000]. Sci China-Phys Mech Astron, 2014, 57: 330–335

26 Viswanath D. The fractal property of the lorenz attractor. Phys D, 2004, 190: 115–128