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ELSEVIER Applied Mathematics and Computation 169 (2005) 854–865
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

An analytic approach to solve multiple solutions of a strongly nonlinear problem

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Abstract

Based on a new kind of analytic method, namely the homotopy analysis method, an analytic approach to solve multiple solutions of strongly nonlinear problems is described by using Gelfand equation as an example. Its validity is verified by comparing the approximation series with the known exact solution. And different from perturbation techniques, this approach is independent upon any small/large perturbation quantities. So, the basic ideas of this approach can be employed to search for multiple solutions of strongly nonlinear problems in science and engineering.
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Keywords: Gelfand problem; Multiple solutions; Strong nonlinearity; Bifurcation; Heat transfer; Homotopy analysis method

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doi:10.1016/j.amc.2004.09.066

1. Introduction

The Gelfand equation [1,2]

$$\begin{cases} \Delta u + \lambda e^u = 0, & \mathbf{x} \in \Omega, \\ u = 0, & \mathbf{x} \in \partial\Omega \end{cases} \tag{1}$$

is well-known for the existence and multiplicity of its nontrivial solutions, where parameter λ denotes the reaction term, $\mathbf{x}$ is a spatial variable, Ω and $\partial\Omega$ denote the spatial domain and its boundary, respectively. This equation contains the exponent term $\exp(u)$ and thus has very strong nonlinearity. It comes from the theory of combustion, and is used as a model for the thermal reaction process such as that when a combustible medium is placed in a vessel whose walls are maintained at a fixed temperature. The Gelfand equation represents the steady state of diffusion and transfer of heat conduction [2,3]. There exists steady-state heat transfer for small λ. As λ is greater than a critical value, the reaction will lead to explosion so that Eq. (1) has no solutions. We state the critical λ as λ^*. For details about Gelfand's equation, please refer to [4–8].

It is hard to solve the Gelfand equation in a general domain Ω. The classical Gelfand problem is with radially symmetric domain, i.e. $u = u(r)$. In this case, one has

$$\begin{cases} u'' + \frac{N-1}{r} u' + \lambda e^u = 0, \quad r \in (0,1], \quad N = 1,2,3,\ldots, \\ u'(0) = u(1) = 0, \end{cases} \tag{2}$$

where $N = 1, 2$, and 3 correspond to the infinite slab, infinite circular cylinder, and sphere, respectively. In 1973, Joseph and Lundgren [9] obtained the numerical solutions for all $N = 1, 2, 3, \ldots$ for the domain of a unit ball. He made an interesting conclusion that all solutions of Eq. (2) lie on a unique curve in $(\lambda, u(0))$ plane. When $N = 1$, Eq. (2) is equivalent to the equation

$$\begin{cases} u'' + \lambda e^u = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{3}$$

where λ is a physical parameter, and the prime denotes the differentiation with respect to x. Obviously, u has the maximum value at $x = 1/2$, denoted by μ. In 1853, Liouville [10] found an analytic expression between μ and λ for $N = 1$. When $0 < \lambda < \lambda^*$, where $\lambda^* \simeq 3.51383$, there exist two values of μ, as shown in Fig. 4. Thus, Eq. (3) has multiple solutions.

Generally speaking, it is difficult to get multiple solutions of a nonlinear problem, especially by means of analytic methods. Perturbation techniques are dependent upon the existence of small or large parameters. This greatly restricts their applications. Currently, a new analytic technique, namely the homotopy analysis method [11,12], is developed. Different from perturbation techniques, the homotopy analysis method does not depend upon any small

or large parameters and thus is valid for more problems in science and engineering. Besides, it logically contains other nonperturbation techniques such as Lyapunov small parameter method [13], the δ-expansion method [14], and Adomian decomposition method [15]. The homotopy analysis method has been successfully applied to many nonlinear problems such as nonlinear vibration [16], nonlinear water waves [17], viscous flows of nonNewtonian fluids [18], Thomas–Fermi's equation [19], nonlinear heat transfer [20], a third grade fluid past a porous plate [21], the flow of an Oldroyd 6-constant fluid [22], and so on. In this paper, based on the homotopy analysis method, a new approach is proposed to solve multiple solutions of strongly nonlinear problems by using Gelfand equation (3) as an example.

2. Homotopy analysis solution

Under the transformation

$$w(x) = \mathrm{e}^{-u}. \tag{4}$$

Eq. (3) becomes

$$w(x)w''(x) - [w'(x)]^2 - \lambda w(x) = 0, \quad x \in (0,1), \tag{5}$$

subject to the boundary conditions

$$w(0) = w(1) = 1. \tag{6}$$

From (6), it is obvious that $w(x)$ can be expressed by the power series

$$w(x) = 1 + \sum_{n=1}^{+\infty} a_n x^n,$$

where a_n is a coefficient. So, we can further make the transformation

$$w(x) = 1 + \gamma s(x), \tag{7}$$

where γ is unknown and is dependent on λ. Here, γ is introduced to search for the multiple solutions, as described later. Using the above transformation, Eq. (5) becomes

$$\gamma[1 + \gamma s(x)]s''(x) - \gamma^2[s'(x)]^2 - \lambda[1 + \gamma s(x)] = 0, \quad x \in (0,1), \tag{8}$$

subject to the boundary conditions

$$s(0) = s(1) = 0. \tag{9}$$

From (9), it is obvious that $s(x)$ can be expressed by the power series

$$s(x) = \sum_{n=1}^{+\infty} b_n x^n, \tag{10}$$

where b_n is a coefficient. This provides us with the so-called Rule of Solution Expression.

According to the Rule of Solution Expression denoted by (10) and the boundary conditions (9), it is straightforward to choose the initial solution

$$s_0(x) = 4x(1-x), \tag{11}$$

which satisfies $s_0(1/2) = 1$. According to the Rule of Solution Expression denoted by (10) and from Eq. (8), it is obvious to choose the auxiliary linear operator

$$\mathscr{L}\phi = \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}. \tag{12}$$

According to Eq. (8), we define the nonlinear operator

$$\mathscr{N}[\phi(x;q), \Gamma(q)] = \Gamma(q)[1 + \Gamma(q)\phi(x;q)]\frac{\partial^2\phi(x;q)}{\partial x^2} - \Gamma^2(q)\left[\frac{\partial\phi(x;q)}{\partial x}\right]^2 - \lambda[1 + \Gamma(q)\phi(x;q)], \tag{13}$$

where $q \in [0,1]$ is an embedding parameter. Let $H(x) \neq 0$ denote an auxiliary function and $\hbar \neq 0$ an auxiliary parameter. We construct the zero-order deformation equation

$$(1-q)\mathscr{L}[\phi(x;q) - s_0(x)] = \hbar H(x)\mathscr{N}[\phi(x;q), \Gamma(q)], \quad x \in (0,1), \tag{14}$$

subject to the boundary conditions

$$\phi(0;q) = \phi(1;q) = 0. \tag{15}$$

When $q = 0$ and 1, the above equation has the solution

$$\phi(x;0) = s_0(x) \tag{16}$$

and

$$\phi(x;1) = s(x), \quad \Gamma(1) = \gamma, \tag{17}$$

respectively.

Assume that the auxiliary function $H(x)$ and the auxiliary parameter $\hbar$ are so properly chosen that $\phi(x;q)$ and $\Gamma(q)$ can be expressed by the Taylor series

$$\phi(x;q) = \phi(x;0) + \sum_{n=1}^{+\infty} s_n(x)q^n, \tag{18}$$

$$\Gamma(q) = \gamma_0 + \sum_{n=1}^{+\infty} \gamma_n q^n, \tag{19}$$

where

$$s_n(x) = \frac{1}{n!}\left.\frac{\partial^n \phi(x;q)}{\partial q^n}\right|_{q=0}, \quad \gamma_n = \frac{1}{n!}\left.\frac{\partial^n \Gamma(q)}{\partial q^n}\right|_{q=0}, \tag{20}$$

and besides that the above two series are convergent at $q = 1$. Then, using (16) and (17), we have

$$s(x) = s_0(x) + \sum_{n=1}^{+\infty} s_n(x), \tag{21}$$

$$\gamma = \gamma_0 + \sum_{n=1}^{+\infty} \gamma_n. \tag{22}$$

For the sake of simplicity, define the vectors

$$\vec{s}_m(x) = \{s_0(x), s_1(x), s_2(x), \ldots, s_m(x)\}$$

and

$$\vec{\gamma}_m = \{\gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_m\}.$$

Differentiating the zero-order deformation Eqs. (14) and (15) n times with respect to the embedding parameter q, then setting $q = 0$, and finally dividing by $n!$, we have the nth-order deformation equation

$$\mathscr{L}[s_n(x) - \chi_n s_{n-1}(x)] = \hbar H(x) R_n[\vec{s}_{n-1}(x), \vec{\gamma}_{n-1}], \tag{23}$$

subject to the boundary conditions

$$s_n(0) = s_n(1) = 0, \tag{24}$$

where

$$R_n[\vec{s}_{n-1}(x), \vec{\gamma}_{n-1}] = \sum_{i=0}^{n-1} (1 - \chi_{i+1} + A_i) D_{n-1-i} - \sum_{i=0}^{n-1} B_i B_{n-1-i} - \lambda(1 - \chi_n + A_{n-1}) \tag{25}$$

under the definitions

$$A_m = \sum_{i=0}^{m} \gamma_{m-i} s_i(x), \tag{26}$$

$$B_m = \sum_{i=0}^{m} \gamma_{m-i} s_i'(x), \tag{27}$$

$$D_m = \sum_{i=0}^{m} \gamma_{m-i} s_i''(x), \tag{28}$$

and

$$\chi_k = \begin{cases} 0, & k \leqslant 1, \\ 1, & k > 1. \end{cases} \tag{29}$$

Obviously, the solution of Eq. (23) is

$$s_n(x) = \chi_n s_{n-1}(x) + \hbar \int_0^x \int_0^\eta H(\xi) R_n[\vec{s}_{n-1}(\xi), \vec{\gamma}_{n-1}] \mathrm{d}\xi \mathrm{d}\eta + C_1 + C_2 x, \tag{30}$$

where the integral constants C_1 and C_2 are determined by the boundary condition (24).

Due to the Rule of Solution Expression denoted by (10), $H(x)$ can be in the form

$$H(x) = x^\kappa, \quad \kappa \geqslant 0.$$

For simplicity, we choose here

$$H(x) = 1. \tag{31}$$

Note that, $s_1(x)$ contains the unknown γ_0. In general, $s_n(x)$ contains the unknown γ_{n-1} for $n \geqslant 1$. Obviously, $\phi(x;q)$ is dependent on $\Gamma(q)$. Thus, the convergence of the series (21) is dependent on $\Gamma(q)$ and also the corresponding series (22). Note that we have great freedom to choose $\gamma_0 = \Gamma(0)$. Therefore, although $\Gamma(1) = \gamma$ is determined by λ, the function $\Gamma(q)$ can be quite different. Let

$$\beta_0 + \sum_{n=1}^{+\infty} \beta_n \tag{32}$$

denote a known convergent series, where

$$\beta_0 = s_0(1/2) = 1 \tag{33}$$

and β_n ($n \geqslant 1$) is a given constant. We can choose $\Gamma(q)$ in such a way that the series

$$s_0(x) + \sum_{n=1}^{+\infty} s_n(x)$$

is convergent at $x = 1/2$, and besides

$$s_n(1/2) = \beta_n, \quad n \geqslant 1. \tag{34}$$

Solving above algebraic equation, we obtain the unknown γ_{n-1}.

An infinite number of convergent series can be employed as the series (32). We provide here two approaches. First, we can choose such a convergent series

$$\beta_n = 0, \quad n \geqslant 1, \tag{35}$$

which gives the algebraic equation

$$s_n(1/2) = 0, \quad n \geqslant 1 \tag{36}$$

for γ_{n-1}. In this case, it holds

$$w(1/2) = 1 + \gamma.$$

When $n = 1$, we have

$$\left(\frac{7}{6}\right)\gamma_0^2 + \left(1 + \frac{5\lambda}{48}\right)\gamma_0 + \frac{\lambda}{8} = 0, \tag{37}$$

which has two branches of solutions

$$\gamma_0 = -\frac{5\lambda + 48 - \sqrt{25\lambda^2 - 864\lambda + 2304}}{112} \tag{38}$$

and

$$\gamma_0 = -\frac{5\lambda + 48 + \sqrt{25\lambda^2 - 864\lambda + 2304}}{112}, \tag{39}$$

when

$$\lambda < \frac{48}{25}\left(9 - 2\sqrt{14}\right) \approx 2.91204,$$

as shown in Fig. 1. Each of them corresponds to a different solution. When $n \geqslant 2$, Eq. (35) is a linear algebraic equation of γ_{n-1}. Note that, the expressions (38) and (39) give complex number when

$$\lambda > \frac{48}{25}\left(9 - 2\sqrt{14}\right) \approx 2.91204.$$

For $\lambda > 2.91204$, we employ the second approach and simply set γ_0 a proper value. In this case, $s_1(x)$ is determined by the chosen value of γ_0, and $\beta_1 = s_1(1/2)$ ($n \geqslant 2$) might be nonzero. The unknown γ_n is determined by

$$s_n(1/2) = 0,$$

and it holds

$$w(1/2) = 1 + \gamma(1 + \beta_1).$$

The key of the above mentioned method is to ensure the convergence of the series (21) and (32). Fortunately, we have great freedom to choose the auxiliary parameter $\hbar$. As mentioned by Liao [11], it is $\hbar$ that provides us with a simple way to control the convergence of the series given by the homotopy analysis method. A convenient method is provided by Liao [11] to choose the value of $\hbar$. Simply speaking, one can plot the so-called $\hbar$-curve of γ via $\hbar$ at high enough order of approximations and then choose a proper value of $\hbar$ in a region

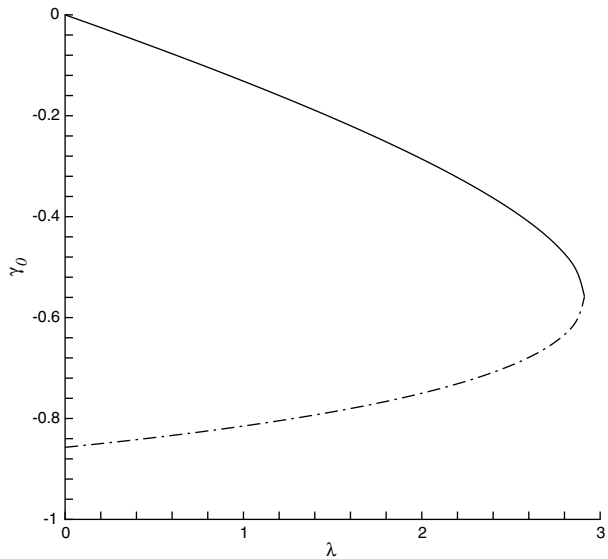

Fig. 1. The two branches of γ_0. Solid line: γ_0 given by (38); dashed line: γ_0 given by (39).

corresponding to a horizontal line segment of γ via $\hbar$. For details, please refer to Liao [11].

Using above-mentioned method, we successfully obtain two different solutions. When $0 < \lambda < 2.9$, the first approach is used. For $\lambda \geqslant 2.9$, we employ the second approach. There exist two branches of solutions with two different values $\mu = u(1/2)$ at $x = 1/2$. Our results agree well with exact solutions, as shown for example in Tables 1 and 2 for $\lambda = 2$. In general,

Table 1
Comparison of the HAM approximation of $\mu = u(1/2)$ with the exact solution $\mu = 0.328952421$ when $\lambda = 2$, $\hbar = 3$, and $\gamma_0 = -285\,714/1\,000\,000$

Order of approximation	μ	Relative error (%)
1th	0.336472022	2.285923588
2th	0.330769779	0.552468161
3th	0.329470463	0.157482219
4th	0.329114926	0.049400632
5th	0.329006596	0.016468937
6th	0.328971255	0.005725281
7th	0.328959172	0.002052217
8th	0.328954898	0.000753005
9th	0.328953347	0.000281436
10th	0.328952773	0.000106769
11th	0.328952556	4.1008E–05
20th	0.328952421	1.0414E–08

Table 2
Comparison of the HAM approximation of $u'(0)$ with the exact solution $u'(0) = 1.248217518$ when $\lambda = 2$, $\hbar = 3$, and $\gamma_0 = -285\,714/1\,000\,000$

Order of approximation	$u'(0)$	Relative error (%)
1th-order	1.230319866	−1.433856809
2th-order	1.239432039	−0.703841947
3th-order	1.244475951	−0.299752804
4th-order	1.246673947	−0.123662013
5th-order	1.247584963	−0.05067667
6th-order	1.247958115	−0.020781873
7th-order	1.248110816	−0.008548345
8th-order	1.248173465	−0.003529228
9th-order	1.248199263	−0.001462474
10th-order	1.248209927	−0.000608161
11th-order	1.248214351	−0.000253718
20th-order	1.248217516	−1.07838E–07

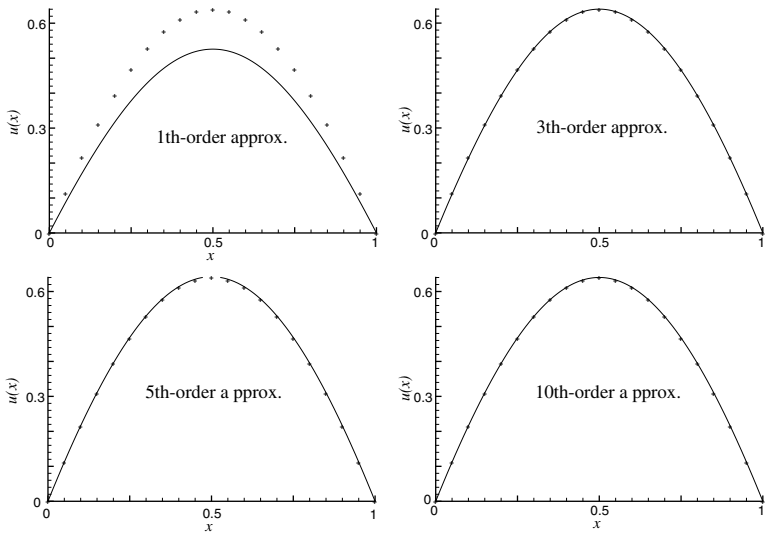

Fig. 2. Comparisons of the HAM approximation with the exact solution for smaller μ when $\lambda = 3$. Symbol: exact solution; solid line: the HAM approximation given by $\gamma_0 = -\frac{9}{26}$ and $\hbar = -3$.

the series with smaller value of μ converges more quickly than the series with larger value of μ, as shown in Figs. 2 and 3 for $\lambda = 3$. The whole curve of μ via λ agrees well with the exact ones, as shwon in Fig. 4. And it is found that the series diverge when $\lambda > \lambda^*$, indicating that there exist no solutions for $\lambda > \lambda^*$. All of these verify that our analytic approach is valid

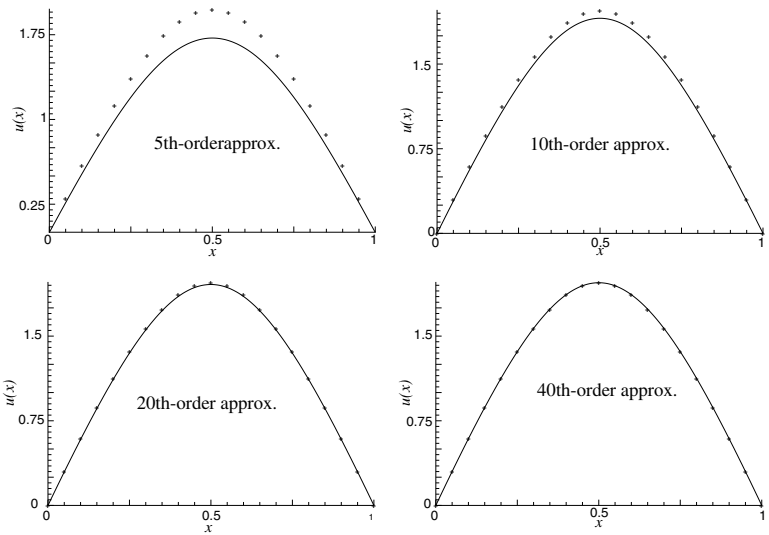

Fig. 3. Comparisons of the HAM approximation with the exact solution for the larger μ when $\lambda = 3$. Symbol: the exact solution; solid line: the HAM approximation given by $\gamma_0 = -\frac{4}{5}$ and $\hbar = -3$.

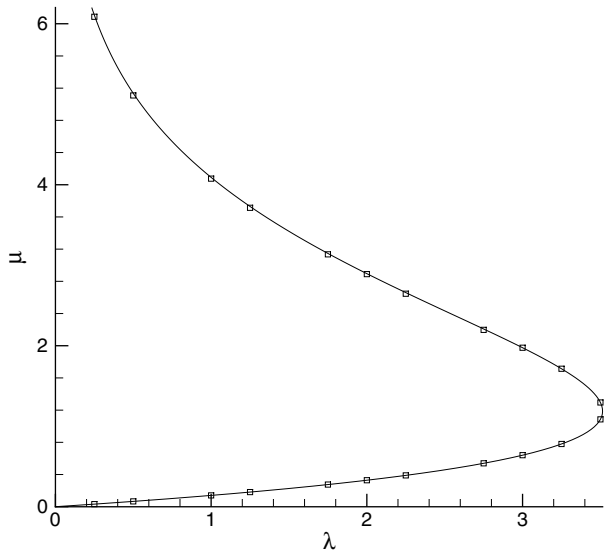

Fig. 4. The comparison between the exact value of μ with the HAM approximations. Symbol: the HAM approximation; solid line: the exact result.

for searching for multiple solutions of strongly nonlinear problems such as Gelfand problem.

3. Conclusion

Based on a new kind of analytic method, namely the homotopy analysis method [11], an analytic approach of searching for multiple solutions of strongly nonlinear problems is described by using Gelfand equation as an example. The approach is convenient and efficient. Its validity is verified by comparing the approximation series with the known exact solution. Note that the Gelfand equation contains an exponent term $\exp(u)$ and thus has very strong nonlinearity. And different from perturbation techniques, this approach is independent upon any small/large perturbation quantities. So, the basic ideas of this approach can be employed to search for multiple solutions of strongly nonlinear problems in science and engineering.

Acknowledgement

This work is supported by the "National Science Fund for Distinguished Young Scholars" (Approval No. 50125923) of Natural Science Foundation of China for the financial support.

References

[1] I.M. Gelfand, Some problems in the theory of quasi-linear equations, Am. Math. Soc. Transl. Ser. 2 (29) (1963) 295–381.
[2] J. Jacobsen, The Liouville–Bratu–Gelfand problem for radial operators, J. Diff. Equat. 184 (2002) 283–298.
[3] D.A. Frank-Kamenetskii, Diffusion and Heat Transfer in Chemical Kinetics, Princeton University Press, Princeton,NJ, 1955.
[4] P. Korman, An accurate computation of the global solution curve for the Gelfand problem through a two point approximation, Appl. Math. Computat. 139 (2003) 363–369.
[5] J.S. McGough, Numerical continuation and the gelfand problem, Appl. Math. Computat. 89 (1998) 225–239.
[6] M. Plum, C. Wieners, New solution of the Gelfand problem, J. Math. Anal. Appl. 269 (2002) 588–606.
[7] M. Plum, Computer-assisted enclosure methods for elliptic differential equations, Linear Algebra Appl. 324 (2001) 147–187.
[8] E. Balakrishnan, A. Swift, G.C. Wake, Critical values for some non-class a geometries in thermal ignition theory, Math. Comput. Model. 24 (8) (1996) 1–10.
[9] D.D. Joseph, T.S. Lundgren, Quasilinear dirichlet problem driven by positive source, Arch. Rat. Mech. Anal. 49 (1973) 241–269.

[10] J. Liouville, Sur i'équation aux différence partielles $\frac{d^2 \log \lambda}{du\, dv} \pm \lambda 2a^2 = 0$, J. Math Pures Appl. 18 (1853) 71–72.
[11] S.J. Liao, Beyond Perturbation: Introduction to Homotopy Analysis Method, Chapman & Hall/ CRC Press, Boca Raton, 2003.
[12] S.J. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Computat. 147 (2004) 499–513.
[13] A.M. Lyapunov, (1892). General Problem on Stability of Motion, London, Taylor & Francis, 1992 (English translation).
[14] A.V. Karmishin, A.T. Zhukov, V.G. Kolosov, Methods of Dynamics Calculation and Testing for Thin-walled Structures, Mashinostroyenie, Moscow, 1990 (in Russian).
[15] G. Adomian, Nonlinear stochastic differential equations, J. Math. Anal. Appl. 55 (1976) 441–452.
[16] S.J. Liao, An analytic approximate approach for free oscillations of self-excited systems, Int. J. Non-Linear Mech. 39 (2) (2004) 271–280.
[17] S.J. Liao, K.F. Cheung, Homotopy analysis of nonlinear progressive waves in deep water, J. Eng. Math. 45 (2) (2003) 105–116.
[18] S.J. Liao, On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet, J. Fluid Mech. 488 (2003) 189–212.
[19] S.J. Liao, An explicit analytic solution to the Thomas–Fermi equation, Appl. Math. Computat. 144 (2003) 495–506.
[20] C. Wang, et al., On the explicit analytic solution of Cheng–Chang equation, Int. J. Heat Mass Transfer 46 (10) (2003) 1855–1860.
[21] M. Ayub, A. Rasheed, T. Hayat, Exact flow of a third grade fluid past a porous plate using homotopy analysis method, Int. J. Eng. Sci. 41 (2003) 2091–2103.
[22] T. Hayat, M. Khan, M. Ayub, On the explicit analytic solutions of an Oldroyd 6-constant fluid, Int. J. Eng. Sci. 42 (2004) 123–135.