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Application of Homotopy Analysis Method in Nonlinear Oscillations

In this paper, we apply a new analytical technique for nonlinear problems, namely the Homotopy Analysis Method (Liao 1992a), to give two-period formulas for oscillations of conservative single-degree-of-freedom systems with odd nonlinearity. These two formulas are uniformly valid for any possible amplitudes of oscillation. Four examples are given to illustrate the validity of the two formulas. This paper also demonstrates the general validity and the great potential of the Homotopy Analysis Method.

1 Introduction

Nonlinear problems are more difficult to solve than linear ones. So far, the most widely applied nonlinear analytical technique is the perturbation method (Nayfeh 1981, 1985; Nayfeh and Mook 1979). The perturbation method is valid in principle only for problems containing small (or large) parameters. Its basic idea is to transform, by means of small parameters, a nonlinear problem into an infinite number of linear subproblems, or a complicated linear problem into an infinite number of simpler ones. Therefore, the small parameter plays a very important role in the perturbation method. It determines not only the accuracy of the perturbation approximations but also the validity of the perturbation method itself. Therefore, it is the small parameter that greatly restricts the applications of the perturbation method. However, in both science and engineering, there exist many nonlinear problems which do not contain any small parameters, especially those with strong nonlinearity. Thus, it is necessary to develop and improve some nonlinear analytical techniques which are independent of small parameters.

There exist some alternative analytical approaches, such as the harmonic balance method (Denman et al. 1964, 1965; Delamotte 1993), the Krylov-Bogoliubov-Mitropolsky (KBM) method (Nayfeh 1981, 1985), and the weighted linearization method (Agrwal and Denman 1985), which sometimes may give analytical approximations even for large parameters. For nonlinear oscillators governed by

$$u'' + F(u) = 0, \tag{1}$$

the abovementioned techniques can produce first approximations of the period of nonlinear oscillations, which are valid even for rather large amplitudes. However, it is usually rather difficult to apply them to produce higher order approximations.

Liao (1992a) proposed a new analytical method called the Homotopy Analysis Method (HAM). The HAM is in principle based on homotopy (Wang and Kao 1991) which is an important part of topology (Duld, 1972; Nash and Sen, 1983; Papy, 1970) so that it has a solid mathematical base. The basic ideas of the HAM are simple. First of all, one introduces an imbedding parameter to construct a homotopy and then analyzes it by the Taylor formula. Subsequently, by means of a linear property of homotopy described by Liao (1992a), one can transform a nonlinear problem into an infinite number of linear subproblems, whether the nonlinear problem contains small parameters or not. Therefore, unlike the perturbation method, the HAM is independent of small parameters and can overcome the restrictions of the perturbation methods. Liao (1992b, c, 1995) and Liao and Chwang (1996) successfully applied the HAM to solve some simple nonlinear problems. These successful applications of the HAM demonstrate the validity and the great potential of the HAM, although the HAM is still in development and further improvements.

For example, let us consider a periodic oscillator governed by

$$(1 + 4q^2u^2)\frac{d^2u}{dt^2} + \kappa u + 4q^2\left(\frac{du}{dt}\right)^2 u = 0 \quad (\kappa > 0) \tag{2}$$

with two initial conditions

$$u(0) = A, \quad u'(0) = 0. \tag{3}$$

Liao (1995) applied the HAM to give two approximate formulas of period T,

$$\frac{T}{T_0} = \sqrt{1 + 2q^2A^2} \tag{4}$$

and

$$\frac{T}{T_0} = \sqrt{5 + 9q^2A^2 - \sqrt{16 + 56q^2A^2 + 53q^4A^4}}, \tag{5}$$

where $T_0 = 2\pi/\sqrt{\kappa}$. As pointed out by Liao (1995), the maximum relative errors of the above two formulas with respect to the exact period

$$\frac{T}{T_0} = \frac{2}{\pi}\sqrt{1 + 4q^2A^2}E(k) \tag{6}$$

are, respectively, ten percent and three percent in the *whole* interval $(qA)^2 \in [0, \infty)$, where $k = 2qA/\sqrt{1 + 4q^2A^2}$ and $E(k)$ is the complete elliptic integral of the second kind. Although Eq. (2) is only a special case, the basic ideas of the HAM are

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general so that it can be applied to solve more nonlinear oscillation problems as shown in this paper.

Let us further consider the free oscillation (undamped and unforced) of a conservative nonlinear system governed by

$$u''(t) + F(u) = 0 \tag{7}$$

with initial conditions

$$u(0) = \beta, \quad u'(0) = 0, \tag{8}$$

where $u(t)$ is a dimensionless variable and $F(u)$ is an odd function of u so that there does not exist the so-called drift or steady-streaming. Suppose that $u = 0$ is a stable equilibrium point, i.e., a center of the oscillation. If $F(u)$ is a nonlinear function, both period T and frequency $\omega = 2\pi/T$ of the corresponding oscillation are dependent upon the amplitude of oscillation β. Usually, period T and frequency ω are functions of β.

In this paper, we first apply the HAM to derive two general formulas for period T of a conservative single-degree-of-freedom system with odd nonlinearity. Then, four examples are given to illustrate that these two period formulas are uniformly valid for *any* possible amplitudes of oscillation.

2 Basic Ideas of the HAM and Main Derivation

Following the approach of Liao (1992a), we construct a one-parameter family of equations

$$\frac{\partial^2 U(t;p)}{\partial t^2} + F[U(t;p)] = (1-p)\left\{\frac{d^2 u_0(t)}{dt^2} + \left[\frac{\Omega^2(p)}{\omega_0^2}\right] F[u_0(t)]\right\}$$

$$t \in [0, \infty), \quad p \in [0, 1] \tag{9}$$

with initial conditions

$$U(t;p) = \beta, \quad \frac{\partial U(t;p)}{\partial t} = 0, \quad \text{for} \quad t = 0, \quad p \in [0, 1], \tag{10}$$

where $p \in [0, 1]$ is an imbedding parameter, $u_0(t)$, which satisfies Eq. (10), is an initial approximation of $u(t)$, and ω_0 denotes the frequency of $u_0(t)$. Note that $U(t; p)$ is now a function of both t and p so that its frequency, denoted by $\Omega(p)$ in Eq. (9), is naturally dependent upon the value of p and moreover, $\Omega(0) = \omega_0$.

At $p = 0$, we have obviously $U(t; 0) = u_0(t)$. At $p = 1$, Eqs. (9) and (10) are exactly the same as Eqs. (7) and (8), respectively, so that $U(t; 1) = u(t)$ which implies $\Omega(1) = \omega$, where ω is the frequency of $u(t)$. Note that $u(t)$ and its frequency $\omega = 2\pi/T$ are exactly what we want to know. As the imbedding parameter p varies from zero to one, $U(t; p)$ varies continuously from $u_0(t)$ to $u(t)$ and $\Omega(p)$ varies from ω_0 to ω. This kind of continuous variations is called deformation in topology. The continuous deformations of $U(t; p)$ and $\Omega(p)$ are completely governed by Eqs. (9) and (10). Therefore, we call them the zeroth-order deformation equations, which represent the changing or becoming process of $U(t; p)$ and $\Omega(p)$. Here, let us emphasize that, in modern science, becoming is considered more important than being, as pointed out by Prigogine and Stengers (1984).

Let

$$\Lambda(p) = \frac{1}{\Omega(p)}, \quad p \in [0, 1] \tag{11}$$

$\lambda_0 = 1/\omega_0$ and $\lambda = 1/\omega$. Then, we have

$$\lambda_0 = \frac{1}{\omega_0} = \frac{1}{\Omega(0)} = \Lambda(0) \tag{12}$$

and

$$\lambda = \frac{1}{\omega} = \frac{1}{\Omega(1)} = \Lambda(1). \tag{13}$$

Introducing a transformation $\tau = \Omega(p)t$ in Eq. (9) and then dividing both sides by $\Omega^2(p)$, we have

$$\frac{\partial^2 U(\tau;p)}{\partial \tau^2} + \Lambda^2(p) F[U(\tau;p)] = (1-p)\left\{\frac{d^2 u_0(\tau)}{d\tau^2} + \lambda_0^2 F[u_0(\tau)]\right\}$$

$$\tau \in [0, \infty), \quad p \in [0, 1] \tag{14}$$

with two initial conditions

$$U(\tau;p) = \beta, \quad \frac{\partial U(\tau;p)}{\partial \tau} = 0, \text{ for } \tau = 0, \ p \in [0, 1], \tag{15}$$

which are also called the zeroth-order deformation equations.

Here, we emphasize that neither small nor large parameters are necessary in constructing the zeroth-order deformation Eqs. (9) and (10), or (14) and (15). In fact, whether or not Eq. (7) contains small or large parameters is not important at all for the validity of the HAM, because the only assumption made on Eq. (7) is that $F(u)$ should be an odd function of u. We also emphasize that, the initial approximation $u_0(t)$ of $u(t)$, together with its corresponding frequency ω_0, can be selected with large freedom that provides us with a great potential to construct better or even the best zeroth-order deformation equations. This kind of freedom is rather important. It would produce much better approximations.

Suppose that $U(\tau; p)$ and $\Lambda(p)$ have derivatives with respect to the imbedding variable p evaluated at $p = 0$:

$$u_0^{[k]}(\tau;p) = \left.\frac{\partial^k U(\tau;p)}{\partial p^k}\right|_{p=0} \quad (k \ge 1), \tag{16}$$

$$\lambda_0^{[k]}(p) = \left.\frac{\partial^k \Lambda(p)}{\partial p^k}\right|_{p=0} \quad (k \ge 1), \tag{17}$$

which are called the kth-order deformation derivatives. Then, by Taylor's formula, we have

$$U(\tau;p) = u_0(\tau) + \sum_{k=1}^{\infty} \frac{u_0^{[k]}(\tau)}{k!} p^k, \tag{18}$$

$$\Lambda(p) = \lambda_0 + \sum_{k=1}^{\infty} \frac{\lambda_0^{[k]}}{k!} p^k. \tag{19}$$

Setting $p = 1$ and noting that $U(\tau; 1) = u(\tau)$ and $\Lambda(1) = \lambda$, we obtain

$$u(\tau) = u_0(\tau) + \sum_{k=1}^{\infty} \frac{u_0^{[k]}(\tau)}{k!} \tag{20}$$

and

$$\lambda = \lambda_0 + \sum_{k=1}^{\infty} \frac{\lambda_0^{[k]}}{k!}, \tag{21}$$

provided that the radii of convergence of series (18) and (19) are not less than 1. Note that (20) gives a relation between the initial approximation $u_0(\tau)$ and solution $u(\tau)$; meanwhile, (21) provides a link between the initial approximation $\lambda_0 = 1/\omega_0$ and the reciprocal of the frequency $\lambda = 1/\omega$. Now, the key to the problem becomes how to solve these kth-order deformation derivatives $u_0^{[k]}(\tau)$ and $\lambda_0^{[k]}(k \ge 1)$. For this purpose, we must first of all give equations governing $u_0^{[k]}(\tau)$ and $\lambda_0^{[k]}(k \ge 1)$.

Differentiating Eqs. (14) and (15) with respect to p and then setting $p = 0$, we have

$$\frac{d^2u_0^{[1]}(\tau)}{d\tau^2} + \lambda_0^2\left(\frac{dF}{du}\bigg|_{u=u_0(\tau)}\right)u_0^{[1]}(\tau) = -\left\{\frac{d^2u_0(\tau)}{d\tau^2} + \sigma F[u_0(\tau)]\right\} \quad (22)$$

with two initial conditions

$$u_0^{[1]}(\tau) = 0, \quad \frac{du_0^{[1]}(\tau)}{d\tau} = 0, \quad \text{for} \quad \tau = 0, \quad (23)$$

where

$$\sigma = \lambda_0^2 + 2\lambda_0\lambda_0^{[1]}. \quad (24)$$

We call Eqs. (22) and (23) the first-order deformation equations. In the same way, we can obtain all of the kth-order deformation equations governing $u_0^{[k]}(\tau)$ and $\lambda_0^{[k]}(k \geq 2)$, which are similar in form to Eqs. (22) and (23) except the inhomogeneous terms. However, in this paper, we would like to show that even the first-order HAM approximation

$$\lambda \approx \lambda_0 + \lambda_0^{[1]}$$

can give satisfactory period formulas uniformly valid for any possible amplitudes of oscillation. Let us emphasize that the first-order deformation Eqs. (22) and (23) are *linear* with respect to the first-order deformation derivative $u_0^{[1]}(\tau)$. In fact, every kth-order deformation equation is *linear* with respect to the corresponding kth-order deformation derivative ($k \geq 1$), as proved by Liao (1992a) in a rather general case. It means that every term $u_0^{[k]}(\tau)(k \geq 1)$ in (20) is governed by a *linear* equation. Therefore, by (20), the original *nonlinear* problem governed by Eqs. (7) and (8) can be transformed to an infinite number of *linear* subproblems about the kth-order deformation derivatives $u_0^{[k]}(\tau)(k \geq 1)$. In the Introduction of this paper, we point out that the perturbation method uses the small parameter assumption to transform a nonlinear problem into an infinite number of linear subproblems. By means of the HAM, we also accomplish this kind of transformation but *without* using the small parameter assumption. Thus, the HAM is in principle different from the perturbation method.

For conservative oscillations with an odd restoring force $F(u)$, there exists a periodic motion around the equilibrium point $u = 0$ with frequency ω and amplitude β. Therefore, a reasonable and the simplest initial approximation of $u(\tau)$ is

$$u_0(\tau) = \beta \cos(\tau). \quad (25)$$

Moreover, $F'(u) = dF/du$ is clearly an even function of u. Thus, $F(\beta \cos \tau)$ and $F'(\beta \cos \tau)$ can be expressed by

$$F(\beta \cos \tau) = \sum_{k=0}^{\infty} A_{2k+1} \cos[(2k+1)\tau] \quad (26)$$

and

$$\frac{dF}{du}\bigg|_{u=\beta\cos\tau} = \frac{B_0}{2} + \sum_{k=1}^{\infty} B_{2k} \cos(2k\tau), \quad (27)$$

respectively, where the coefficients A_{2k+1} and B_{2k} ($k = 0, 1, 2, 3, \ldots$) are determined by

$$A_{2k+1} = \frac{1}{\pi}\int_0^{2\pi} F(\beta \cos \tau) \cos[(2k+1)\tau]d\tau \quad (k = 0, 1, 2, 3, \ldots), \quad (28)$$

$$B_{2k} = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{dF}{du}\bigg|_{u=\beta\cos\tau}\right)\cos(2k\tau)d\tau \quad (k = 0, 1, 2, 3, \ldots). \quad (29)$$

Clearly, the solution $u_0^{[1]}(\tau)$ of Eq. (22) can be expressed by

$$u_0^{[1]}(\tau) = \sum_{k=0}^{\infty} C_{2k+1} \cos[(2k+1)\tau]. \quad (30)$$

Substituting (26), (27), and (30) into (22) and then equating the coefficients, we obtain a set of linear algebraic equations for unknown coefficients C_{2k+1} ($k = 0, 1, 2, 3, \ldots$),

$$\mathbf{Ec} = \frac{2}{\lambda_0^2}\left(\beta\begin{bmatrix}1\\0\\0\\\vdots\end{bmatrix} - \sigma\begin{bmatrix}A_1\\A_3\\A_5\\\vdots\end{bmatrix}\right) = \frac{2}{\lambda_0^2}(\beta\mathbf{b} - \sigma\mathbf{a}), \quad (31)$$

where $\mathbf{a} = [A_1\ A_3\ A_5 \cdots]^T$, $\mathbf{b} = [1\ 0\ 0\ 0 \cdots]^T$, $\mathbf{c} = [C_1\ C_3\ C_5 \cdots]^T$ are vectors and $\mathbf{E}$ is a matrix with elements e_{ij} defined by

$$e_{11} = B_0 + B_2 - 2\lambda_0^{-2},$$
$$e_{12} = B_2 + B_4,$$
$$e_{13} = B_4 + B_6,$$
$$e_{21} = B_2 + B_4,$$
$$e_{22} = B_0 + B_6 - 18\lambda_0^{-2},$$
$$e_{23} = B_2 + B_8,$$
$$e_{31} = B_4 + B_6,$$
$$e_{32} = B_2 + B_8,$$
$$e_{33} = B_0 + B_{10} - 50\lambda_0^{-2},$$
$$\cdots$$

and so on. According to Cramer's theory, the solution of Eq. (31) can be expressed by

$$C_{2k-1} = \frac{2}{\lambda_0^2 \det(\mathbf{E})}[\beta \det(\mathbf{S}_k) - \sigma \det(\mathbf{R}_k)] \quad (k = 1, 2, 3, \ldots), \quad (32)$$

where both $\mathbf{S}_k = \{s_{ij}^k\}$ and $\mathbf{R}_k = \{r_{ij}^k\}$ are matrices with elements

$$s_{ij}^k = \begin{cases} e_{ij} & \text{when} \quad j \neq k, \\ b_i & \text{when} \quad j = k, \end{cases} \quad (33)$$

and

$$r_{ij}^k = \begin{cases} e_{ij} & \text{when} \quad j \neq k, \\ a_i & \text{when} \quad j = k. \end{cases} \quad (34)$$

Hence, according to (30), we have

$$u_0^{[1]}(\tau) = \frac{2}{\lambda_0^2 \det(\mathbf{E})}\sum_{k=1}^{\infty}[\beta \det(\mathbf{S}_k) - \sigma \det(\mathbf{R}_k)] \cos[(2k-1)\tau]. \quad (35)$$

From initial conditions (23), we obtain

$$\sigma = \left[\frac{\sum_{k=1}^{\infty} \det(\mathbf{S}_k)}{\sum_{k=1}^{\infty} \det(\mathbf{R}_k)}\right]\beta. \quad (36)$$

According to (24) and (36), σ is a function of both β and λ_0. If only n terms are used in (30), we have

$$\sigma(\beta, \lambda_0) \approx \left[\frac{W_n(\beta, \lambda_0)}{H_n(\beta, \lambda_0)}\right]\beta, \tag{37}$$

where

$$W_n(\beta, \lambda_0) = \sum_{k=1}^{n} \det(\mathbf{S}_k), \tag{38}$$

$$H_n(\beta, \lambda_0) = \sum_{k=1}^{n} \det(\mathbf{R}_k). \tag{39}$$

Then, by Eqs. (21) and (24), we obtain the first-order approximation of λ as

$$\lambda_1 = \lambda_0 + \lambda_0^{[1]} = \frac{1}{2}\left[\lambda_0 + \frac{\sigma(\beta, \lambda_0)}{\lambda_0}\right]. \tag{40}$$

Note that in all of the above derivations, we do *not* set a definite value to λ_0. As we have emphasized, the HAM provides us with large freedom to select the value of $\lambda_0 = 1/\omega_0$. Obviously, if we replace λ_0 by λ_1 in (40), we will obtain a new approximation λ_2 which should be even better than λ_1. Therefore, expression (40) provides, in fact, such an iterative process

$$\lambda_{k+1} = \frac{1}{2}\left[\lambda_k + \frac{\sigma(\beta, \lambda_k)}{\lambda_k}\right]. \tag{41}$$

In the limit as k tends to infinity and $\lambda = \lim_{k\to+\infty} \lambda_k$, we have

$$\lambda^2 = \sigma(\beta, \lambda). \tag{42}$$

Substituting (37) into (42) gives

$$\lambda^2 = \left[\frac{W_n(\beta, \lambda)}{H_n(\beta, \lambda)}\right]\beta, \tag{43}$$

which is an algebraic equation about λ^2. If only one term in Eq. (30) is used ($n = 1$), we have

$$W_1(\beta, \lambda) = b_1 = 1, \tag{44}$$

$$H_1(\beta, \lambda) = a_1 = A_1, \tag{45}$$

so that

$$\lambda = \sqrt{\frac{\beta}{A_1}}. \tag{46}$$

Therefore, we obtain the first approximate period formula

$$T_1 = 2\pi\sqrt{\frac{\beta}{A_1}}. \tag{47}$$

If two terms are used in (30), we have

$$W_2(\beta, \lambda) = \begin{vmatrix} 1 & e_{12} \\ 0 & e_{22} \end{vmatrix} + \begin{vmatrix} e_{11} & 1 \\ e_{21} & 0 \end{vmatrix}$$

$$= B_0 - B_2 - B_4 + B_6 - 18/\lambda^2 \tag{48}$$

and

$$H_2(\beta, \lambda) = \begin{vmatrix} A_1 & e_{12} \\ A_3 & e_{22} \end{vmatrix} + \begin{vmatrix} e_{11} & A_1 \\ e_{21} & A_3 \end{vmatrix}$$

$$= A_1(B_0 - B_2 - B_4 + B_6 - 18/\lambda^2) + A_3(B_0 - B_4 - 2/\lambda^2). \tag{49}$$

Substituting (48) and (49) into (43) we obtain

$$\lambda^2 = \frac{\beta(B_0 - B_2 - B_4 + B_6 - 18/\lambda^2)}{[A_1(B_0 - B_2 - B_4 + B_6 - 18/\lambda^2) + A_3(B_0 - B_4 - 2/\lambda^2)]}, \tag{50}$$

which gives the second approximate period formula

$$T_2 = \frac{2\pi}{\sqrt{f(\beta) \pm \sqrt{f^2(\beta) - h(\beta)}}}, \tag{51}$$

where

$$f(\beta) = \frac{\beta(B_0 - B_2 - B_4 + B_6) + 18A_1 + 2A_3}{36\beta}, \tag{52}$$

$$h(\beta) = \frac{A_1(B_0 - B_2 - B_4 + B_6) + A_3(B_0 - B_4)}{18\beta}. \tag{53}$$

The sign before $\sqrt{f^2(\beta) - h(\beta)}$ should be determined by the condition $\lim_{\beta\to 0}(T_1/T_2) = 1$.

3 Some Applications

3.1 Example 1: Duffing Equation. Let us first consider the well-known Duffing equation

$$u'' + u + \epsilon u^3 = 0 \tag{54}$$

with initial conditions $u(0) = \beta$ and $u'(0) = 0$, where ϵ is a known parameter and $\beta > 0$ is the amplitude of oscillation. Note that both ϵ and β are not necessarily small. When $\epsilon < 0$, $u = 0$ is a center and $u = \pm(-\epsilon)^{-1/2}$ are saddle points so that there exist periodic oscillations around $u = 0$ only for $\sqrt{|\epsilon|}\beta < 1$. However, when $\epsilon > 0$, there always exist oscillations around the stable equilibrium point $u = 0$ for *any* possible values of $\sqrt{\epsilon}\beta$ so that the corresponding amplitude of oscillation, β, may be very large ($0 \le \beta < +\infty$). The exact period T of the Duffing Eq. (54) is

$$T = \frac{4}{\sqrt{1 + \epsilon\beta^2}} K(\mu), \tag{55}$$

where $\mu = \sqrt{\frac{1}{2}\epsilon\beta^2/(1 + \epsilon\beta^2)}$ and $K(\mu)$ is the complete elliptic integral of the first kind. Its first-order perturbation approximation is

$$T_{\text{pert}} = \frac{2\pi}{\left(1 + \frac{3}{8}\epsilon\beta^2\right)}. \tag{56}$$

Since the restoring force is $F(u) = u + \epsilon u^3$, we have

$$F(\beta\cos\tau) = \beta(1 + \tfrac{3}{4}\epsilon\beta^2)\cos(\tau) + \tfrac{1}{4}\epsilon\beta^3\cos(3\tau), \tag{57}$$

$$\left.\frac{dF}{du}\right|_{u=\beta\cos(\tau)} = \left(1 + \frac{3}{2}\epsilon\beta^2\right) + \frac{3}{2}\epsilon\beta^2\cos(2\tau). \tag{58}$$

According to (28) and (29), we have

$$A_1 = \beta(1 + \tfrac{3}{4}\epsilon\beta^2), \tag{59}$$

$$A_3 = \tfrac{1}{4}\epsilon\beta^3, \tag{60}$$

$$B_0 = 2 + 3\epsilon\beta^2, \tag{61}$$

$$B_2 = \tfrac{3}{2}\epsilon\beta^2, \tag{62}$$

$$A_{2k+1} = B_{2k} = 0 \quad (k \ge 2). \tag{63}$$

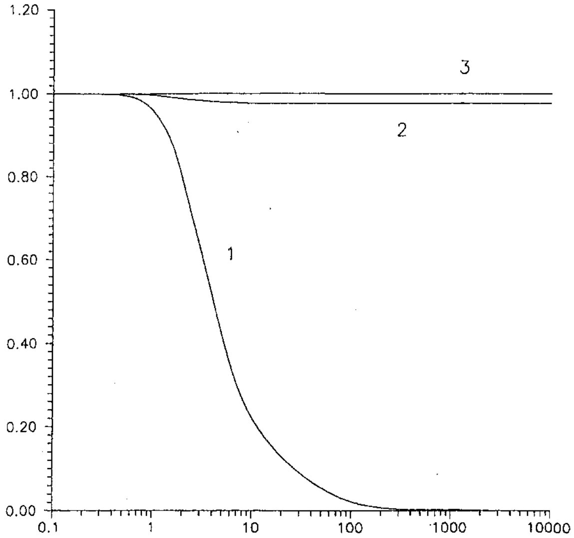

Fig. 1 Comparison of the approximate periods with the exact one (example 1: $\epsilon > 0$); Horizontal axis: $\sqrt{|\epsilon|}\beta$; Vertical axis: ratio of approximate periods to the exact; Curve 1: T_{pert}/T_{exact}; Curve 2: T_1/T_{exact}; Curve 3: T_2/T_{exact}

Substituting (59) into (47) gives

$$T_1 = \frac{2\pi}{\sqrt{1 + \frac{3}{4}\epsilon\beta^2}}. \tag{64}$$

Moreover, according to (51), we have

$$T_2 = \frac{24\pi}{\sqrt{80 + 62\epsilon\beta^2 + \sqrt{4096 + 5888\epsilon\beta^2 + 1684\epsilon^2\beta^4}}}. \tag{65}$$

Here, we select the plus sign before the second square root to ensure $\lim_{\epsilon\beta^2\to 0}(T_1/T_2) = 1$.

Comparisons of the approximate periods (56), (64), and (65) with the exact period (55) are shown in Fig. 1 for $\epsilon > 0$ and Fig. 2 for $\epsilon < 0$, respectively. Clearly, formula (64) and especially (65) are more accurate than (56) and can give good approximations of period T for both small and large values of $\sqrt{|\epsilon|}\beta$. Note that, for $\epsilon > 0$, (64) and (65) give maximum relative errors less than 2.2 percent and 0.2 percent, respectively. In fact,

$$\lim_{\sqrt{|\epsilon|}\beta\to\infty} \frac{T_1}{T} = \left(\frac{\pi}{\sqrt{3}}\right)\frac{1}{K(1/\sqrt{2})} \approx 0.9783 \tag{66}$$

and

$$\lim_{\sqrt{|\epsilon|}\beta\to\infty} \frac{T_2}{T} = \left(\frac{6\pi}{\sqrt{62 + \sqrt{1684}}}\right)\frac{1}{K(1/\sqrt{2})} \approx 1.0016, \tag{67}$$

where K denotes the complete elliptic integral of the first kind. Note that the accuracy of (64) and (65) are *not* strongly dependent upon the values of $\sqrt{|\epsilon|}\beta$ because they are uniformly valid for *any* possible values of $\sqrt{|\epsilon|}\beta$. Therefore, both (64) and (65) are valid for strongly nonlinear oscillations with large amplitudes. Finally, let us point out that the first approximations given by other alternative techniques such as the harmonic balance method may give the same result as (64). However, when we apply the harmonic balance method to obtain a second-order approximation, we have to solve a cubic algebraic equation. Therefore, Eq. (65) is relatively simple and easy to use, because it involves only square roots.

3.2 Example 2: Simple Pendulum Attached to a Rotating Rigid Frame.

As the second example we consider a simple pendulum attached to a rotating rigid frame, governed by

$$\theta'' + \omega_0^2(1 - \kappa\cos\theta)\sin\theta = 0 \tag{68}$$

with two initial conditions $\theta(0) = \beta$ and $\theta'(0) = 0$, where $0 < \kappa < 1$ is a constant and $\beta(0 \le \beta < \pi)$ denotes the amplitude of oscillation. The exact period T of Eq. (68) is

$$T = \frac{4}{\omega_0}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \mu^2\sin^2\phi}\sqrt{1 - \kappa + \kappa\mu^2(1 + \sin^2\phi)}}, \tag{69}$$

where $\mu = \sin(\beta/2)$. Its first-order perturbation approximation is

$$T_{pert} = \left(\frac{2\pi}{\omega_0}\right)\frac{16\sqrt{1-\kappa}}{[16(1-\kappa) - (1 - 4\kappa)\beta^2]}. \tag{70}$$

Now, the restoring force is $F(\theta) = \omega_0^2(1 - \kappa\cos\theta)\sin\theta$. Therefore, we have

$$F(\beta\cos\tau) = \omega_0^2\sum_{k=0}^{\infty}(-1)^k\{2J_{2k+1}(\beta) - \kappa J_{2k+1}(2\beta)\}\cos[(2k+1)\tau], \tag{71}$$

$$F'(\beta\cos\tau) = \omega_0^2[J_0(\beta) - \kappa J_0(2\beta)] + 2\omega_0^2\sum_{k=1}^{\infty}(-1)^k[J_{2k}(\beta) - \kappa J_{2k}(2\beta)]\cos(2k\tau), \tag{72}$$

where $J_k(x)$ is the Bessel function of the first kind. Then, by (26), (27), (47), and (51) we obtain

$$T_1 = \frac{2\pi}{\omega_0}\sqrt{\frac{\beta}{2J_1(\beta) - \kappa J_1(2\beta)}}, \tag{73}$$

$$T_2 = \frac{2\pi}{\omega_0}\frac{1}{\sqrt{Q(\beta) + \sqrt{Q^2(\beta) - P(\beta)}}}, \tag{74}$$

where

$$Q(\beta) = \frac{1}{18\beta}\{\beta[J_0(\beta) + J_2(\beta) - J_4(\beta) - J_6(\beta)] - \kappa\beta[J_0(2\beta) + J_2(2\beta) - J_4(2\beta) - J_6(2\beta)]$$

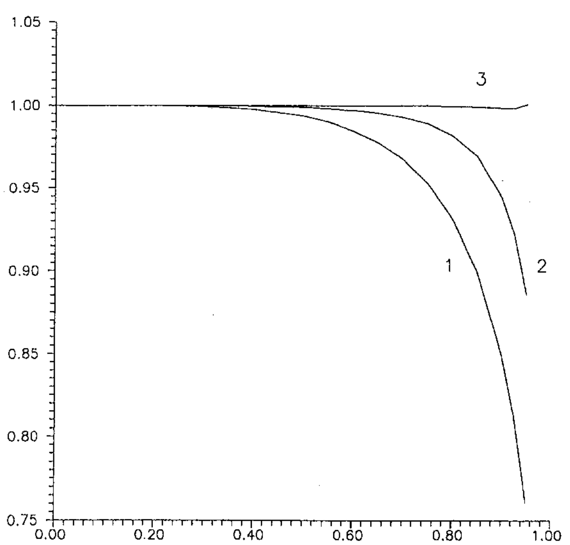

Fig. 2 Comparison of the approximate periods with the exact one (example 1: $\epsilon < 0$); Horizontal axis: $\sqrt{|\epsilon|}\beta$; Vertical axis: ratio of approximate periods to the exact; Curve 1: T_{pert}/T_{exact}; Curve 2: T_1/T_{exact}; Curve 3: T_2/T_{exact}

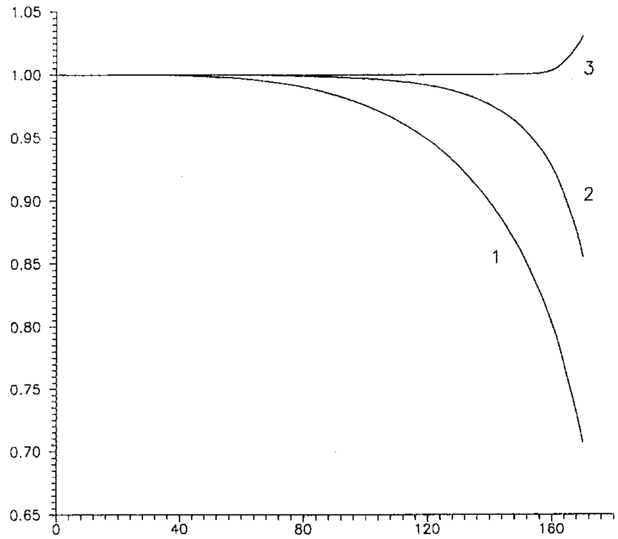

Fig. 3 Comparison of the approximate periods with the exact one (example 2: $\kappa = 0.1$); Horizontal axis: β (degree); Vertical axis: ratio of approximate periods to the exact; Curve 1: T_{pert}/T_{exact}; Curve 2: T_1/T_{exact}; Curve 3: T_2/T_{exact}

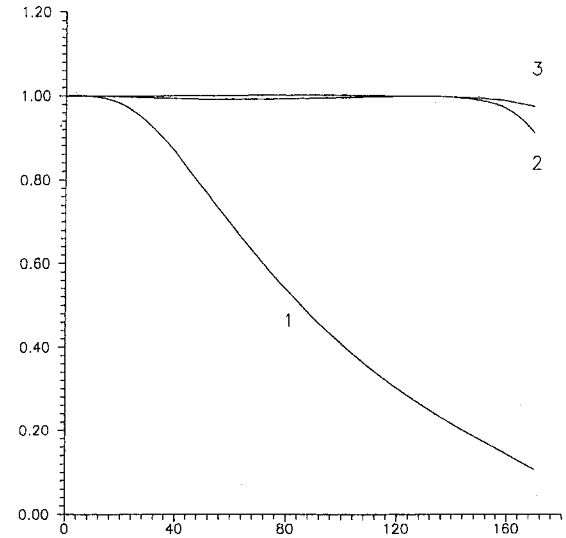

Fig. 5 Comparison of the approximate periods with the exact one (example 2: $\kappa = 0.9$); Horizontal axis: β (degree); Vertical axis: ratio of approximate periods to the exact; Curve 1: T_{pert}/T_{exact}; Curve 2: T_1/T_{exact}; Curve 3: T_2/T_{exact}

$$+ [18J_1(\beta) - 2J_3(\beta)] - \kappa[9J_1(2\beta) - J_3(2\beta)]\}, \quad (75)$$

$$P(\beta) = \frac{1}{9\beta}\{[2J_1(\beta) - \kappa J_1(2\beta)][J_0(\beta) + J_2(\beta) - J_4(\beta) - J_6(\beta)] - \kappa[2J_1(\beta) - \kappa J_1(2\beta)][J_0(2\beta) + J_2(2\beta) - J_4(2\beta) - J_6(2\beta)] - [2J_3(\beta) - \kappa J_3(2\beta)]([J_0(\beta) - J_4(\beta)] - \kappa[J_0(2\beta) - J_4(2\beta)])\}. \quad (76)$$

Comparisons of approximations (70), (73), and (74) with the exact period (69) are shown in Fig. 3 for $\kappa = 0.1$, Fig. 4 for $\kappa = 0.5$, and Fig. 5 for $\kappa = 0.9$, respectively. Note that the perturbation result (70) gives good approximations only when both κ and β are small. However, in contrast to the perturbation approximation, period formula (73) and especially (74) can give good approximations even for large values of β and κ. Therefore, the two period formulas (73) and (74) are valid not only for strong nonlinearity (large κ) but also for large-amplitude oscillations (large β). Note that our results are not satisfactory if β is very close to π. This may be due to the fact that $\beta = \pi$ is an unstable equilibrium point, called the saddle point, where the corresponding motion is a circle.

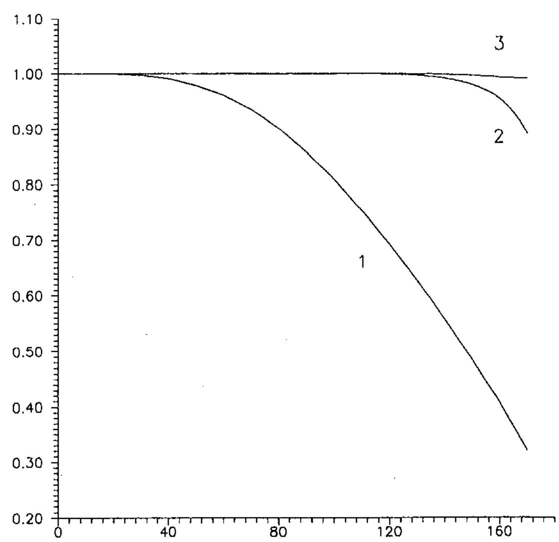

Fig. 4 Comparison of the approximate periods with the exact one (example 2: $\kappa = 0.5$); Horizontal axis: β (degree); Vertical axis: ratio of approximate periods to the exact; Curve 1: T_{pert}/T_{exact}; Curve 2: T_1/T_{exact}; Curve 3: T_2/T_{exact}

Note that alternative approaches, such as the harmonic balance method, the KBM method, and the weighted linearization method, can give the same first approximation as (73). However, when applying the harmonic balance method to get the second or even higher approximations, we have to solve a set of transcendental equations. It seems difficult to obtain (74) by alternative approaches.

3.3 Example 3: Oscillations With Restoring Force $F(u) = u/(1 + u^2)$.

As the third example, let us consider the following oscillation governed by

$$u'' + \frac{u}{1 + u^2} = 0 \quad (77)$$

with two initial conditions $u(0) = \beta$ and $u'(0) = 0$. The exact period T of Eq. (77) is

$$T = 4\int_0^\beta \frac{dt}{\sqrt{\ln(1 + \beta^2) - \ln(1 + t^2)}}. \quad (78)$$

Its first-order perturbation approximation is

$$T_{pert} = \frac{2\pi}{(1 - \frac{3}{8}\beta^2)}. \quad (79)$$

Note that the restoring force is $F(u) = u/(1 + u^2)$. By (28), (29), (47), and (51) we obtain

$$T_1 = \frac{2\pi\beta}{\sqrt{2\left(1 - \frac{1}{\sqrt{1 + \beta^2}}\right)}} \quad (80)$$

and

$$T_2 = \frac{2\pi}{\sqrt{G + \sqrt{G^2 - H}}}, \tag{81}$$

where

$$G = \tfrac{4}{3}\gamma^3 + \tfrac{2}{9}\alpha^2(2 + 3\gamma) - \tfrac{4}{9}\alpha^4(16 - 29\gamma + 7\gamma^3) - \tfrac{16}{9}\alpha^6(5 - 6\gamma + \gamma^3), \tag{82}$$

$$H = -\tfrac{8}{9}\alpha^2\{-4\gamma^3 + \alpha^2(1 - 3\gamma + 8\gamma^3 - 8\gamma^4) + 4\alpha^4(5 - 12\gamma + 6\gamma^2 + 3\gamma^3 - 2\gamma^4) + 16\alpha^6(1 - 2\gamma + \gamma^2)\}, \tag{83}$$

and

$$\alpha = \frac{1}{\beta}, \quad \gamma = \frac{1}{\sqrt{1 + \beta^2}}. \tag{84}$$

Comparisons of the perturbation approximation (79) and the HAM approximations (80) and (81) with the exact period T are shown in Fig. 6. Clearly, the perturbation approximation (79) is valid only for small values of β. It breaks down for large amplitudes of oscillation. However, the HAM approximations (80) and (81) are valid for any possible amplitudes of oscillation.

From (80) and (81) we have

$$\lim_{\beta\to 0} T_1 = \lim_{\beta\to 0} T_2 = 2\pi \tag{85}$$

and

$$\lim_{\beta\to\infty} T_1 \approx \sqrt{2}\pi\beta, \tag{86}$$

$$\lim_{\beta\to\infty} T_2 \approx \left(\frac{3\sqrt{2}}{\sqrt{2 + \sqrt{22}}}\right)\beta. \tag{87}$$

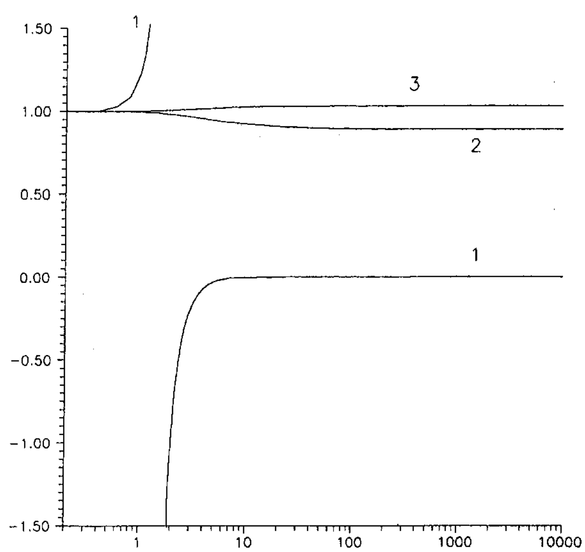

Fig. 6 Comparison of the approximate periods with the exact one (example 3); Horizontal axis: β (degree); Vertical axis: ratio of approximate periods to the exact; Curve 1: T_{pert}/T_{exact}; Curve 2: T_1/T_{exact}; Curve 3: T_2/T_{exact}

It means that period T of Eq. (77) is equal to 2π for small-amplitude β. However, for large-amplitude oscillations, T is approximately directly proportional to the amplitude of oscillation β. It seems difficult to obtain this qualitative conclusion directly from the exact expression (78).

3.4 Example 4: Oscillations With Restoring Force $F(u) = \alpha u + \lambda u^n$ ($n = 3, 5, 7, \ldots$).

As the last example, let us consider a family of nonlinear differential equations

$$u'' + \alpha u + \gamma u^n = 0, \quad (\alpha \ge 0, \gamma > 0, n = 3, 5, 7, \ldots), \tag{88}$$

with two initial conditions

$$u(0) = \beta, \quad u'(0) = 0. \tag{89}$$

The corresponding exact period T is

$$T = 4\int_0^{\pi/2} \frac{d\phi}{\sqrt{\alpha + \left(\frac{2\gamma}{n+1}\right)\beta^{n-1}(1 + \sin^2\phi + \sin^4\phi + \ldots + \sin^{n-1}\phi)}}. \tag{90}$$

From (28), (29), (47), and (51) we obtain two approximations of period T,

$$T_1 = \frac{2\pi}{\sqrt{\alpha + \left(\frac{\gamma}{2}\right)\left(\frac{\beta}{2}\right)^{n-1}\Theta(n+1)}}, \tag{91}$$

$$T_2 = \frac{12\pi\sqrt{\beta}}{\sqrt{\Psi + \sqrt{\Psi^2 - \Pi}}}, \tag{92}$$

where

$$\Psi = 20\alpha\beta + \gamma\left(\frac{\beta}{2}\right)^n[(12 + 24n)\Theta(n+1) + (2 - 14n)\Theta(n+3) + 2n\Theta(n+5)], \tag{93}$$

$$\Pi = 144\alpha^2\beta^2 + 144\gamma\left(\frac{\beta}{2}\right)^n\left\{\alpha\beta[(12n - 2)\Theta(n+1) + (1 - 7n)\Theta(n+3) + n\Theta(n+5)] + n\gamma\left(\frac{\beta}{2}\right)^n[\Theta(n+1)\Theta(n+5) - \Theta^2(n+3)]\right\}, \tag{94}$$

and

$$\Theta(n) = \frac{\Gamma(n+1)}{\Gamma^2\left(\frac{n}{2} + 1\right)} = \frac{n!}{\left[\left(\frac{n}{2}\right)!\right]^2}. \tag{95}$$

Formulas (91) and (92) are valid for any possible amplitudes and give the maximum errors as the dimensionless amplitude $\gamma\beta^{n-1}$ tends to infinity, as shown in Fig. 7, where $3 \le n \le 50$. Note that, even for $n = 19$, the maximum error given by (92) is less than ten percent for amplitude $\beta \in [0, \infty)$. All of these

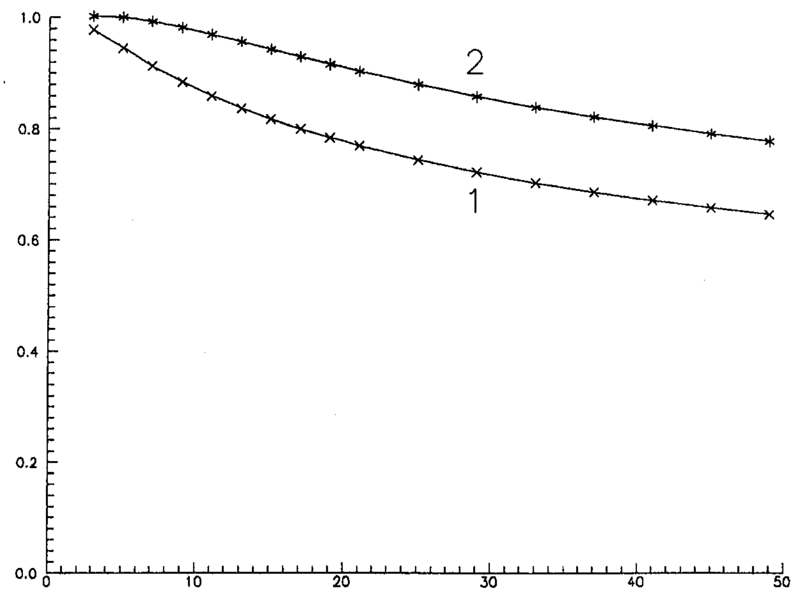

Fig. 7 Maximum errors of example 4 at different values of n as β tends to infinity; Horizontal axis: n; Vertical axis: ratio of approximate periods to the exact; Curve 1: minimum value of T_1/T_{exact}; Curve 2: minimum value of T_2/T_{exact}

illustrate that the validity of the HAM period formulas (47) and (51) for nonlinear oscillations does not strongly depend upon the form of the restoring forces.

In this section we have applied the HAM period formulas (47) and (51) to four typical conservative single-degree-of-freedom oscillation systems. The restoring forces of these four oscillations are rather different. These examples illustrate that, unlike perturbation approximations, the HAM period formulas (47) and (51) are uniformly valid for *any* possible amplitudes of oscillation and can give satisfactory period approximations for oscillations with strong nonlinearity and large amplitudes. These four examples give us reasons to believe that the HAM period formulas (47) and (51) should be valid for many other nonlinear conservative oscillation systems with odd restoring forces.

4 Discussion

In this paper we have applied a new analytical technique, called the Homotopy Analysis Method (HAM), to derive two period formulas (47) and (51) for conservative single-degree-of-freedom oscillation systems with odd nonlinearity. We have further applied these two general formulas to four examples to illustrate that, unlike perturbation approximations, these two formulas are *uniformly* valid for oscillations with *any* possible amplitudes and strong nonlinearity. In other words, the validity of the two period formulas (47) and (51) is independent of small parameters. This is because the HAM is in principle based on homotopy in topology without the assumption of small parameters. Therefore, the validity of the HAM itself does not depend on the presence of small parameters. This point is very important, because the small-parameter assumption brings a lot of restrictions to the perturbation method. Therefore, the HAM can be applied to solve more nonlinear problems than the perturbation method, especially those which do not contain any small parameters.

The basic ideas of the HAM are simple. Some homotopies are first constructed by introducing an imbedding parameter $p \in [0, 1]$ so that unknown functions, together with their periods in case of periodic motions, become also dependent upon this imbedding parameter. These homotopies, or deformations, governed by the zeroth-order deformation equations, can be expressed as Maclaurin series in p. The most important point is that every term in this Maclaurin series, called the kth-order ($k \geq 1$) deformation derivatives at $p = 0$, is governed by a linear equation which can be solved easily. In this way, a nonlinear problem is transformed into an infinite number of linear subproblems. As pointed out by Liao (1992a), this kind of transformation is the key to solve nonlinear problems. Note that the perturbation method relies on the small-parameter assumption to accomplish this kind of transformation. Thus, the validity of the perturbation method itself and its approximate results are naturally dependent upon small parameters. However, unlike the perturbation method, the validity of the HAM and its approximations are independent of small parameters, as shown in this paper. Moreover, the HAM provides us with great freedom to select initial approximations, and through iteration and limiting processes these initial approximations lead to uniformly valid satisfactory results.

Note that the two period formulas (47) and (51) are uniformly valid for the four different examples. We have also applied (47) and (51) to many other nonlinear conservative oscillations with odd nonlinearity and have found that in all the cases they can give satisfactory period approximations in significant parameter ranges. These successful applications of (47) and (51) give us confidence in believing that they should be valid for many other nonlinear conservative oscillations with odd nonlinear restoring forces. As a new attempt to develop a nonlinear analytical technique free from the small-parameter assumption, the HAM seems promising, although it needs further improvements and more applications.

There are other techniques to obtain first approximations for the period of a conservative oscillator, such as the harmonic balance method, the KBM method, and the weighted linearization method. In most cases these alternative techniques give the same first approximations as (47). However, it is difficult to obtain the second or even higher-order approximations by means of these alternatives. For example, the harmonic balance method requires the solution of a set of nonlinear algebraic or even transcendental equations which usually have no exact analytical solutions and must be solved numerically. In comparison with these alternative approaches to obtain the second approximations, the new period formula (51) given by the HAM is both simple and general. It involves only square roots and is valid for any odd restoring forces $F(u)$.

Finally, we would like to point out that the perturbation method is still a useful tool, especially in cases where a proper small parameter is present. What we attempt to develop is a new nonlinear analytical technique in the absence of small parameters. This paper shows one step in this attempt and we still have a long way to go.

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