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Can We Obtain a Reliable Convergent Chaotic Solution in any Given Finite Interval of Time?

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Generally, it is difficult to obtain convergent chaotic solution in an arbitrarily given finite interval of time. Some researchers even believe that all chaotic responses are simply numerical noise and have nothing to do with solutions of differential equations. However, using 1200 CPUs of the National Supercomputer TH-A1 at Tianjin and a parallel integration algorithm of the so-called "Clean Numerical Simulation" (CNS) based on the 3500th-order Taylor expansion and data in 4180-digit multiple precision, one can obtain reliable convergent chaotic solution of the Lorenz equation in a rather long time interval $[0, 10\,000]$. This supports Lorenz's optimistic viewpoint [Lorenz, 2008] that "numerical approximations can converge to a chaotic true solution throughout any finite range of time". It also supports Tucker's proof [Tucker, 1999, 2002] for the famous Smale's 14th problem that the strange attractor of the Lorenz equation indeed exists.

Keywords: Chaotic solution; Lorenz equation; clean numerical simulation; convergent.

Using a digit computer, Lorenz [1963] found the famous "butterfly effect" of "deterministic nonperiodic" solution of three-coupled ordinary differential equations, called the Lorenz equation today: the so-called chaotic solutions are extremely sensitive to initial conditions. This work was a milestone in the field of nonlinear dynamics. Tucker [1999, 2002] further proved that the Lorenz equation supports a strange attractor. Tucker's work is a great breakthrough that provides a positive answer to the Smale's 14th problem [Smale, 1998].

However, it was *practically* difficult to obtain a reliable numerical simulation of a chaotic dynamic system in an arbitrarily given finite range of time, because Lorenz [1989, 2006] further found that chaotic solutions are sensitive not only to initial conditions but also to numerical algorithms: different numerical algorithms with different time-steps may lead to completely different numerical results of chaos. For example, chaotic numerical simulations of the Lorenz equation given by different traditional procedures were often repeatable only in a interval of time less than 30 Lorenz time unit (LTU). So, "computed" dynamic behaviors observed for a finite time-step in some nonlinear discrete-time difference equations sometimes have nothing to do with the "exact" solution of the original continuous-time differential equations at all, as confirmed by some other researchers [Li *et al.*, 2001; Teixeira *et al.*, 2007]. This numerical phenomenon leads to intense arguments [Yao & Hughes, 2008; Lorenz, 2008] about the reliability of numerical simulations of chaotic dynamic systems. Some believed that "all chaotic responses are simply numerical noise and have nothing to do with the solutions of differential equations" [Yao & Hughes, 2008]. On the other side, using double precision data and a few examples based on the 15th-order Taylor-series procedure [Corliss & Chang, 1982; Barrio *et al.*, 2005] with decreasing time-step, Lorenz [2008] was optimistic and believed that "numerical approximations can converge to a chaotic true solution throughout any finite range of time, although, if the range is large, confirming the convergence can be utterly impractical."

Currently, using the arbitrary-order Taylor series method (TSM) [Corliss & Chang, 1982; Barrio *et al.*, 2005] and arbitrary-precision data [Oyanarte, 1990] together with a validation and verification check, Liao [2009, 2013, 2014] proposed the method of the "Clean Numerical Simulation" (CNS) to obtain reliable convergent chaotic solutions in a long but finite time interval $[0, T]$. Let $s(M, N)$ denote a numerical simulation of a nonlinear dynamic system given by the CNS, where M denotes the order of the TSM and N the number of digit precision of data, respectively. Here, "convergence" means that, for a given interval $[0, T]$ with a properly chosen time-step Δt, there exist a critical order M^* of the TSM and a critical integer N^* for digit precision such that all numerical simulations $s(M, N)$ given by the CNS are the same, i.e. with negligible differences, as long as $M > M^*$ and $N > N^*$. This is mainly because truncation error and round-off error can be reduced to a required, rather small level as long as M and N are large enough. Unlike traditional numerical algorithms for chaotic systems, the CNS searches for the critical order M^* of the TSM and the critical N^* of digit-precision for a given time interval $[0, T]$ and a chosen time-step Δt.

In 2009, using the CNS with the 400th-order Taylor series method (TSM) and data in 800-digit precision (by means of the computer algebra Mathematica with the time-step $\Delta t = 0.01$), Liao [2009] gained, for the first time, a reliable chaotic solution of the Lorenz equation in a long time interval [0, 1000]. As reported by Liao [2009], for a given interval $[0, T]$, one can obtain reliable convergent chaotic simulations of the Lorenz equation by means of the CNS with the Mth-order TMS and data in N-digit precision, where $M > M^* \approx T/3$ and $N > N^* \approx 2T/5$ in the case of $\Delta t = 0.01$. Using the multiple precision (MP) library [Oyanarte, 1990] and parallel computation, Wang *et al.* [2011] confirmed the reliability of Liao's chaotic solution in [0,1000] and gained a reliable chaotic result of the Lorenz equation in [0, 2500] by means of the CNS based on the 1000th-order TSM and data in 2100-digit precision (with $\Delta t = 0.01$). Recently, using 1200 CPUs of the National Supercomputer TH-A1 and a parallel integration algorithm of the CNS based on the 3500th-order Taylor expansion and data in the 4180-digit multiple precision, Liao and Wang [2014] obtained a reliable convergent chaotic solution of the Lorenz equation in a rather long interval $0 \le t \le 10\,000$: its reliability and convergence were further confirmed by means of the CNS using the 3600th-order TSM and the data in 4515-digit multiple precision. To the best of my knowledge, this kind of reliable convergent chaotic solution of the Lorenz equation in such a long time interval has never been reported before, which provides us a numerical benchmark of reliable chaotic solutions of dynamic systems.

Recently, Kehlet and Logg [2013] gained a reliable convergent chaotic solution of the Lorenz equation on the time interval [0, 1000] using the 200th-order *finite element method* (FEM) and data in 400-digits precision. Its reliability was confirmed by means of the CNS with the 400th-order TSM and data in 800-digit precision. Note that, unlike the FEM approach of Kehlet and Logg [2013], the CNS is a kind of *finite difference method* (FDM). Therefore, reliable convergent chaotic results of the Lorenz equation in a long time interval [0, 1000] can indeed be obtained by the two completely *different* numerical approaches! All of these support Lorenz's optimistic viewpoints: "numerical approximations can converge to a chaotic true solution throughout any finite range of time".

Thus, "Smale's 14th Problem" has a perfect answer: the strange attractor of the Lorenz equation not only exists [Tucker, 1999, 2002], but also can be calculated accurately in practice [Liao & Wang, 2014].

Finally, it should be emphasized that the CNS can provide results with so small numerical noises that it can be used to investigate the propagation of micro-level physical uncertainty of some chaotic dynamic systems [Liao, 2012, 2014]. Some very accurate CNS results of a chaotic three-body system suggest that the micro-level physical uncertainty might be the origin of the macroscopic randomness of the three-body system [Liao, 2014], say, the macroscopic uncertainty of a chaotic three-body system might be excited by their *inherent* micro-level physical uncertainty, *without* any other *external* "disturbances" at all.

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