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ABSTRACT

In this paper, we investigate the nonlinear interaction of two primary progressive waves traveling in the same/opposite direction. Without loss of generality, two cases are considered: waves traveling in the same direction and waves traveling in the opposite direction. There exist an infinite number of resonant wave components in each case, corresponding to an infinite number of singularities in mathematical terms. Resonant wave systems with an infinite number of singularities are rather difficult to solve by means of traditional analytic approaches such as perturbation methods. However, this mathematical obstacle is easily cleared by means of the homotopy analysis method (HAM): the infinite number of singularities can be completely avoided by choosing an appropriate auxiliary linear operator in the frame of the HAM. In this way, we successfully gain steady-state systems with an infinite number of resonant components, consisting of the nonlinear interaction of the two primary waves traveling in the same/opposite direction. In physics, this indicates the general existence of so-called steady-state resonant waves, even in the case of an infinite number of resonant components. In mathematics, it illustrates the validity and potential of the HAM to be applied to rather complicated nonlinear problems that may have an infinite number of singularities.

NOMENCLATURE

c_0	convergence-control parameter
f	expression defined by (13)
g	acceleration due to gravity
H_s	wave steepness
$\boldsymbol{k}_0$	wavenumber of the resonant wave
$\boldsymbol{k}_{0,\iota}$	wavenumber of the ιth resonant wave
$\boldsymbol{k}_1$	wavenumber of the first primary wave
$\boldsymbol{k}_2$	wavenumber of the second primary wave
k_d	wavenumber corresponding to the dominant frequency
$\bar{\boldsymbol{k}}_i$	wavenumber of the unperturbed wave
$\boldsymbol{k}_i'$	wavenumber of the perturbed wave
(k_p, k_q)	wavevector of perturbation
k_{JK}	wavenumber depending on J and K
κ_{JK}	expression in (70)
q	embedding parameter
R	difference in the extreme values of wave elevation of linear and nonlinear wave groups
S_m	term in high-order deformation equation (26)
$\bar{S}_m$	term in high-order deformation equation (26)
t	time
ξ_i	variable in the coordinate system (ξ_1, ξ_2, z)
φ	velocity potential
φ_0	initial guess of φ
φ_m	m-order solution of φ
φ_1^*	special solution of φ_1
Ψ_{m_1,m_2}	expression in φ
$\bar{\varphi}$	velocity potential of the unperturbed wave
φ'	velocity potential of the perturbed wave
Φ	φ in the coordinate system $(\xi_1, \xi_2, z; q)$
η	free-surface elevation
η_0	initial guess of η
η_m	m-order solution of η
$\eta_{\max}$	maximum elevation of the nonlinear waves
$\eta_{\min}$	minimum elevation of the nonlinear waves
$\eta_{\max}'$	maximum elevation of the linear waves
$\bar{\eta}$	free-surface elevation of the unperturbed wave

η'	free-surface elevation of the perturbed wave
ζ	η in the coordinate system $(\xi_1, \xi_2; q)$
a_{m_1,m_2}	coefficient in the solution expression of φ
b_{m_1,m_2}	coefficient in the solution expression of η
a^*	coefficient of $\Psi_{1,0}$ in φ_1
b^*	coefficient of $\Psi_{0,1}$ in φ_1
$\bar{b}_{m_1,m_2}$	coefficient in (59)
$\bar{d}_{m_1,m_2}$	coefficient in (64)
ω_0	linear frequency of the resonant wave
$\omega_{0,\iota}$	linear frequency of the ιth resonant wave
ω_1	linear frequency of the first primary wave
ω_2	linear frequency of the second primary wave
$\mathrm{d}\omega_\iota$	frequency mismatch of the ιth resonant wave
$\bar{\omega}$	linear frequency of the unperturbed wave
ω'	linear frequency of the perturbed wave
σ	frequency of the perturbation
σ_i	actual frequency of the primary wave
Λ_i	σ_i with q
$\sigma_{i,m}$	m-order solution of σ_i
ϵ_i	dimensionless frequency of the primary wave
$\mathcal{N}_1$	nonlinear operator defined by (10)
$\mathcal{N}_2$	nonlinear operator defined by (11)
$\mathcal{L}$	auxiliary linear operator
$\mathcal{L}_0$	linear operator corresponding to the linear parts of (10)
Δ_m^φ	term in high-order deformation equation (26)
Δ_m^η	term in high-order deformation equation (28)
χ_m	$\chi_1 = 0$ and $\chi_m = 1$ for $m > 1$
γ_m	unknown introduced in φ_m
Π	sum of the squared amplitude of all components
Π_0	sum of the squared amplitude of two primary components
μ_i	piecewise function in $\mathcal{L}$
$\lambda^i_{m_1,m_2}$	eigenvalue associated with $\mathcal{L}_0$ or $\mathcal{L}$

I. INTRODUCTION

When studying the nonlinear interaction of water waves by means of the traditional perturbation method, singularities may be encountered in the calculation process, and these are difficult to deal with mathematically. For example, when the criterion for wave resonance,

$$2\boldsymbol{k}_1 - \boldsymbol{k}_2 = \boldsymbol{k}_0, \tag{1a}$$

$$2\omega_1 - \omega_2 = \omega_0, \tag{1b}$$

is exactly satisfied, where $\boldsymbol{k}_i$ is the wavenumber and ω_i is the related linear wave frequency, the nonlinear interaction of four gravity waves in deep water contains one singularity that corresponds to the resonant mode at the third order.[1–3] For the so-called steady-state resonant waves whose wave spectrum does not change with time, perturbation methods fail because of singularities in the transfer function, as pointed out by Madsen and Fuhrman.[4] For short-crested waves, there are two difficulties in obtaining their profile, as identified by Okamura.[5] The first is that the radius of convergence is much smaller than the maximum wave steepness; the second is associated with the division by zero due to harmonic resonance. The nonlinear interaction of double cnoidal waves governed by the Korteweg–de Vries (KdV) equation contains an infinite number of singularities when using the perturbation method.[6,7] In dealing with these singularities, one must add terms with to-be-determined coefficients for each singularity, which makes the calculation process cumbersome and complicated. Additionally, as the amplitude increases, the accuracy of the Stokes series deteriorates to become effectively useless, even with Padé improvement, as noted by Boyd.[8,9] Moreover, resonance phenomena such as Bragg resonance, capillary-gravity wave resonance, and acoustic-gravity wave resonance all contain singularity problems.[10–13]

The above-mentioned difficulties/restrictions can be overcome by means of the homotopy analysis method (HAM),[14–17] an analytic technique for highly nonlinear problems that has been successfully applied in areas such as boundary-layer flows,[18,19] American put options in finance,[20–22] nonlinear water waves,[23–29] and von Kármán circular plates.[30,31] The HAM has some obvious advantages. First, unlike perturbation methods, the HAM does not depend upon any small/large physical parameters. Different from all other analytic approximation methods, the HAM guarantees the convergence of the series solution by choosing a proper value of the so-called convergence-control parameter c_0. In addition, the HAM provides great freedom to select an auxiliary linear operator and the initial condition. Using this freedom, the secular terms corresponding to the singularities can be easily avoided by choosing an appropriate auxiliary linear operator. For example, Liao[23] reconsidered the nonlinear interactions between pairs of intersecting wave trains in deep water that exactly satisfy the resonance criterion (1a) and (1b), and gained, for the first time, a convergent solution of the steady-state resonant waves by means of the HAM, which was not possible using perturbation methods. Xu *et al.*[24] and Liu and Liao[25] extended the work of Liao[23] from a single quartet in deep water to more complicated cases in infinite and finite water depths. Liao *et al.*[26] further applied the HAM to solve a gravity wave problem in deep water with a small divisor caused by a single near-resonant term, and this was extended by Liu *et al.*[27] to steady-state wave groups with multiple near-resonances. The near-resonance criteria are

$$m_\iota \boldsymbol{k}_1 + n_\iota \boldsymbol{k}_2 = \boldsymbol{k}_{0,\iota}, \tag{2a}$$

$$m_\iota \omega_1 + n_\iota \omega_2 = \omega_{0,\iota} + \mathrm{d}\omega_\iota, \tag{2b}$$

where $\iota = 1, 2, \ldots, l$, $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are the wavenumbers of the two primary components, and $\mathrm{d}\omega_\iota$ is a small real number that represents the angular frequency mismatch of the ιth resonant component. The l nearly resonant components correspond to l small divisors in the calculation process. For the nonlinear interaction of double cnoidal waves governed by the KdV equation, Xu *et al.*[32] overcame the difficulties of singularities in a convenient way by choosing an auxiliary linear operator which contains an irrational number in the framework of the HAM.

For the nonlinear interaction of collinear waves, in addition to the research of Haupt and Boyd[6,7] on the nonlinear interaction of double cnoidal waves governed by the KdV equation, Sharma and Dean[33] derived a second-order solution for bichromatic bidirectional water waves. Zhang and Chen[34] provided a third-order analytical solution for the interactions among three collinear deep-water wave components. Madsen and Fuhrman[35] then derived a third-order solution for dichromatic bidirectional water waves in finite depths. Fully nonlinear bichromatic waves traveling in the same direction in deep water were investigated by Lin *et al.*,[36] who used the HAM to obtain convergent solutions. Liu *et al.*[37] extended this work from infinite water to finite depths. Recently, Liu *et al.*[38] obtained convergent high-order solutions for nonlinear bichromatic waves with finite wave amplitudes in deep water. Multiple steady-state near-resonances corresponding to multiple small divisors were considered in the wave system. For two primary waves traveling in opposite directions, the most researched problem concerns nonlinear standing waves. Standing waves of the finite amplitude in infinite water depths were first investigated by Rayleigh,[39] who calculated a perturbation series based on the wave amplitude to third order. After Rayleigh's work, Penney and Price[40] derived a fifth-order approximation for the wave amplitude by means of a successive expansion method. Since then, Aoki,[41] Tabjbakhsh and Keller,[42] Schwartz and Whitney,[43] and Okamura,[5] among others, have made considerable progress on nonlinear standing waves, albeit mostly using numerical methods.

The objective of this paper is to investigate nonlinear steady-state wave groups in deep water with an infinite number of resonances that satisfy the resonance criteria when l tends to infinity under the near-resonance criteria (2a) and (2b). In the present paper, we study the nonlinear interaction of two collinear traveling waves. An infinite number of singularities associated with an infinite number of resonant components are successfully handled by means of the HAM. Standing waves are simply special cases of two progressive waves traveling in the opposite direction with the same wavelength. We will illustrate that, by choosing an appropriate auxiliary linear operator in the HAM framework, the infinite number of singularities can be conveniently avoided. Additionally, as the nonlinearity increases, one more wave component is added to the initial nontrivial component, as the HAM provides us with this freedom. In this way, we can successfully obtain convergent solutions for steady-state collinear wave groups, even with rather high wave steepness.

The structure of the paper is as follows. The mathematical formulas in the HAM framework are described in Sec. II. Convergent results for steady-state resonant collinear waves with different values of wave steepness and their energy distributions and wave profiles are presented in Sec. III. The conclusions to this study and a discussion of our results are given in Sec. IV.

II. MATHEMATICAL FORMULAS

A. Governing equations

Let us consider the nonlinear interaction of two trains of progressive gravity waves traveling on the same line in water of infinite depth. A Cartesian coordinate system is adopted, with the x-axis and y-axis located on the mean water plane and the z-axis pointing vertically upward. Under the assumption of an inviscid and incompressible fluid, and without surface tension, the governing equations read

$$\nabla^2 \varphi = 0, \ -\infty \le z \le \eta(x,y,t), \qquad (x,y) \in \mathbb{R}^2, \tag{3}$$

$$\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial z} + \frac{\partial|\nabla\varphi|^2}{\partial t} + \nabla\varphi\cdot\nabla\left(\frac{1}{2}|\nabla\varphi|^2\right) = 0, \ \text{on} \ z = \eta(x,y,t), \tag{4}$$

$$g\eta + \frac{\partial\varphi}{\partial t} + \frac{1}{2}|\nabla\varphi|^2 = 0, \qquad \text{on} \ z = \eta(x,y,t), \tag{5}$$

$$\lim_{z\to-\infty}\frac{\partial\varphi}{\partial z} = 0, \tag{6}$$

where φ denotes the velocity potential, η is the free-surface elevation, g is the acceleration due to gravity, and t is the time, respectively.

Consider a steady-state wave system in deep water consisting of two progressive waves, with $\boldsymbol{k}_i$ denoting the wavenumber and σ_i the actual wave frequency. In the present paper, we assume that $\sigma_i > 0$. Due to the nonlinearity, the actual wave frequency σ_i is different from the linear frequency $\omega_i = \sqrt{g|\boldsymbol{k}_i|}$ and depends upon the wave amplitudes. Write

$$\epsilon_i = \frac{\sigma_i}{\omega_i}, \quad i = 1, 2, \tag{7}$$

where the value of ϵ_i is slightly different from 1. In this paper, the actual wave frequency σ_i is unknown and needs to be determined. Then, we define the variables

$$\xi_i = \boldsymbol{k}_i\cdot\boldsymbol{r} - \sigma_i\cdot t, \quad i = 1, 2, \tag{8}$$

where $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$. For a steady-state wave system, all wave amplitudes a_i, wavenumbers $\boldsymbol{k}_i$, and actual wave frequencies σ_i, where $i = 1, 2$, are independent of time. Using the new variables ξ_i, the original initial/boundary-value problem governed by (3)–(6) can be transformed into a boundary-value problem. Therefore, in the coordinate system (ξ_1, ξ_2, z), the governing equation (3) becomes

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\boldsymbol{k}_i\cdot\boldsymbol{k}_j\frac{\partial^2\varphi}{\partial\xi_i\xi_j} + \frac{\partial^2\varphi}{\partial z^2} = 0, -\infty \le z \le \eta(\xi_1,\xi_2), \tag{9}$$

subject to the following two boundary conditions on the unknown free surface $z = \eta(\xi_1, \xi_2)$:

$$\begin{aligned}\mathcal{N}_1[\varphi] = &\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\frac{\partial^2\varphi}{\partial\xi_i\xi_j} + g\frac{\partial\varphi}{\partial z} - 2\sum_{i=1}^{2}\sigma_i\frac{\partial f}{\partial\xi_i}\\ &+ \sum_{i=1}^{2}\sum_{j=1}^{2}\boldsymbol{k}_i\cdot\boldsymbol{k}_j\frac{\partial\varphi}{\partial\xi_i}\frac{\partial f}{\partial\xi_j} + \frac{\partial\varphi}{\partial z}\frac{\partial f}{\partial z} = 0,\end{aligned} \tag{10}$$

$$\mathcal{N}_2[\varphi,\eta] = \eta - \frac{1}{g}\left(\sum_{i=1}^{2}\sigma_i\frac{\partial\varphi}{\partial\xi_i} - f\right) = 0, \tag{11}$$

and the impermeability condition at the bottom,

$$\lim_{z\to-\infty}\frac{\partial\varphi}{\partial z} = 0, \tag{12}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are the two nonlinear operators defined above and

$$f = \frac{1}{2}\left[\sum_{i=1}^{2}\sum_{j=1}^{2}\boldsymbol{k}_i\cdot\boldsymbol{k}_j\frac{\partial\varphi}{\partial\xi_i}\frac{\partial\varphi}{\partial\xi_j} + \left(\frac{\partial\varphi}{\partial z}\right)^2\right]. \tag{13}$$

The velocity potential $\varphi(\xi_1, \xi_2, z)$ can be expressed by

$$\varphi(\xi_1,\xi_2,z) = \sum_{m_1=0}^{+\infty}\sum_{m_2=-\infty}^{+\infty}a_{m_1,m_2}\Psi_{m_1,m_2}(\xi_1,\xi_2,z), \tag{14}$$

with the definition

$$\Psi_{m_1,m_2}(\xi_1,\xi_2,z) = \sin(m_1\xi_1 + m_2\xi_2)\cdot\exp(|m_1\boldsymbol{k}_1 + m_2\boldsymbol{k}_2|z), \tag{15}$$

where a_{m_1,m_2} are constants to be determined later. Note that the velocity potential (14) automatically satisfies the linear governing equation (9) and the bottom boundary condition (12). Therefore, the wave elevation $\eta(\xi_1, \xi_2)$ can be expressed by

$$\eta(\xi_1,\xi_2) = \sum_{m_1=0}^{+\infty}\sum_{m_2=-\infty}^{+\infty} b_{m_1,m_2}\cos(m_1\xi_1 + m_2\xi_2), \tag{16}$$

where b_{m_1,m_2} are constants to be determined later. Note that the unknown coefficients a_{m_1,m_2} and b_{m_1,m_2} are determined by the two nonlinear boundary conditions (10) and (11).

B. Solution procedure

The above-mentioned nonlinear partial differential equations (PDEs) related to the steady-state resonant waves have been successfully solved using the HAM in many different cases. Liao[23] used the HAM to obtain, for the first time, solutions of the steady-state gravity waves of resonant quartets in deep water. Thereafter, Xu *et al.*,[24] Liu and Liao,[25] Liao *et al.*,[26] and Liu *et al.*[27] extended the study of steady-state gravity waves to more complicated cases, including both exactly- and nearly resonant waves. Recently, steady-state acoustic-gravity waves were obtained by Yang *et al.*[29] using the HAM. For the sake of simplicity, we briefly describe some key mathematical processes here; detailed mathematical derivations can be found in the above-mentioned articles. Let $q \in [0, 1]$ denote the embedding parameter. In the HAM framework, we first construct a family of solutions $\Phi(\xi_1, \xi_2, z; q)$, $\zeta(\xi_1, \xi_2; q)$, $\Lambda_1(q)$, and $\Lambda_2(q)$ in $q \in [0, 1]$ by means of the so-called zeroth-order deformation equations,

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\frac{\partial^2\Phi}{\partial\xi_i\xi_j} + g\frac{\partial\Phi}{\partial z} = 0,\ -\infty \le z \le \zeta(\xi_1,\xi_2;q), \tag{17}$$

subject to the two boundary conditions on the unknown elevation $z = \eta(\xi_1, \xi_2; q)$,

$$(1-q)\mathcal{L}[\Phi(\xi_1,\xi_2,z;q) - \varphi_0(\xi_1,\xi_2,z)] = c_0\, q\, \mathcal{N}_1[\Phi(\xi_1,\xi_2,z;q),\Lambda_1(q),\Lambda_2(q)], \tag{18}$$

$$(1-q)\zeta(\xi_1,\xi_2;q) = c_0\, q\, \mathcal{N}_2\big[\Phi(\xi_1,\xi_2,z;q),\zeta(\xi_1,\xi_2;q),\Lambda_1(q),\Lambda_2(q)\big], \tag{19}$$

and the impermeability condition at the bottom,

$$\lim_{z\to-\infty}\frac{\partial\Phi}{\partial z} = 0, \tag{20}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are the two nonlinear operators defined by (10) and (11), $\mathcal{L}$ is an auxiliary linear operator that can be almost freely chosen, $\varphi_0(\xi_1, \xi_2, z)$ is the initial guess of $\varphi(\xi_1, \xi_2, z)$, and $c_0 \neq 0$ is the so-called "convergence-control parameter" (which has no physical meaning). Here, Φ, ζ, Λ_1, and Λ_2 correspond to the unknown φ, η, σ_1, and σ_2, respectively. Note that we have great freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $\varphi_0(\xi_1, \xi_2, z)$. Obviously, when $q = 0$, we have the solution

$$\Phi(\xi_1,\xi_2,z;0) = \varphi_0(\xi_1,\xi_2,z),\quad \zeta(\xi_1,\xi_2;0) = 0. \tag{21}$$

When $q = 1$, Eqs. (17)–(20) are equivalent to the original equations (9)–(12), respectively, and so we have

$$\Phi(\xi_1,\xi_2,z;1) = \varphi(\xi_1,\xi_2,z),\quad \zeta(\xi_1,\xi_2;1) = \eta(\xi_1,\xi_2),\quad \Lambda_1(1) = \sigma_1,\quad \Lambda_2(1) = \sigma_2. \tag{22}$$

Thus, as q increases from 0 to 1, $\Phi(\xi_1, \xi_2, z; q)$ deforms continuously from the initial guess $\varphi_0(\xi_1, \xi_2, z)$ to the unknown potential function $\varphi(\xi_1, \xi_2, z)$, as does $\zeta(\xi_1, \xi_2; q)$ from 0 to the unknown wave profile $\eta(\xi_1, \xi_2)$ and $\Lambda_1(q)$, $\Lambda_2(q)$ from the initial guesses $\sigma_{1,0}$, $\sigma_{2,0}$ to the unknown frequencies σ_1, σ_2, respectively.

Assuming that the convergence-control parameter c_0 is chosen so that the Maclaurin series of $\Phi(\xi_1, \xi_2, z; q)$, $\zeta(\xi_1, \xi_2; q)$, $\Lambda_1(q)$, and $\Lambda_2(q)$ with respect to the embedding parameter q, i.e.,

$$\Phi(\xi_1,\xi_2,z;q) = \sum_{m=0}^{+\infty}\varphi_m(\xi_1,\xi_2,z)q^m, \tag{23a}$$

$$\zeta(\xi_1,\xi_2;q) = \sum_{m=0}^{+\infty}\eta_m(\xi_1,\xi_2)q^m, \tag{23b}$$

$$\Lambda_1(q) = \sum_{m=0}^{+\infty}\sigma_{1,m}q^m, \tag{23c}$$

$$\Lambda_2(q) = \sum_{m=0}^{+\infty}\sigma_{2,m}q^m \tag{23d}$$

exist and converge at $q = 1$, we have the so-called homotopy-series solution,

$$\varphi(\xi_1,\xi_2,z) = \sum_{m=0}^{+\infty}\varphi_m(\xi_1,\xi_2,z), \tag{24a}$$

$$\eta(\xi_1,\xi_2) = \sum_{m=0}^{+\infty}\eta_m(\xi_1,\xi_2), \tag{24b}$$

$$\sigma_1 = \sum_{m=0}^{+\infty}\sigma_{1,m}, \tag{24c}$$

$$\sigma_2 = \sum_{m=0}^{+\infty}\sigma_{2,m}, \tag{24d}$$

respectively.

Substituting the Maclaurin series (23) into the zeroth-order deformation equations (17)–(20) and then equating powers of q, we obtain the so-called high-order deformation equations,

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\frac{\partial^2\varphi_m}{\partial\xi_i\xi_j} + g\frac{\partial\varphi_m}{\partial z} = 0,\quad -\infty \le z \le 0, \tag{25}$$

subject to the linear boundary condition,

$$\mathcal{L}[\varphi_m] = c_0\Delta^{\varphi}_{m-1} + \chi_m S_{m-1} - \overline{S}_m,\quad \text{on}\ \ z = 0, \tag{26}$$

and the bottom condition,

$$\frac{\partial\varphi_m}{\partial z} = 0,\ \ \text{as}\ \ z \to -\infty, \tag{27}$$

together with

$$\eta_m = c_0\Delta^{\eta}_{m-1} + \chi_m\eta_{m-1},\quad \text{on}\ \ z = 0, \tag{28}$$

where $\chi_1 = 0$ and $\chi_m = 1$ for $m > 1$. Both $\overline{S}_m$ and S_m are dependent upon the auxiliary linear operator and will thus be defined later. The definitions of Δ^{φ}_{m-1} and Δ^{η}_{m-1} are the same as those given by Yang *et al.*[29] Note that all of the Δ^{φ}_{m-1}, Δ^{η}_{m-1}, $\overline{S}_m$, and S_m on the right-hand side of (26) and (28) are determined by the known previous approximations η_j and φ_j ($j = 0, 1, 2, \ldots, m-1$) and can thus be regarded as known terms. Some highlights and distinguishing features for the considered problem are given below.

C. Singularities of two steady-state progressive waves traveling on the same line

Consider a wave system with resonant components of the wavenumber $\boldsymbol{k}_{0,\iota}$, generated by two primary components with wavenumbers $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ traveling on the same line. The resonance criteria

$$m_\iota \boldsymbol{k}_1 + n_\iota \boldsymbol{k}_2 = \boldsymbol{k}_{0,\iota}, \quad m_\iota\omega_1 + n_\iota\omega_2 = \omega_{0,\iota} \tag{29}$$

hold, where $\iota = 1, 2, \ldots$, m_ι and n_ι are integers, and $\omega_i = \sqrt{g|\boldsymbol{k}_i|}$ and $\omega_{0,\iota} = \sqrt{g|\boldsymbol{k}_{0,\iota}|}$ are the linear frequencies. Such collinear wave systems often have an infinite number of resonances, i.e., singularities. For example, let us consider two cases. The first is the wave system consisting of two primary waves traveling in the same direction:

$$\boldsymbol{k}_1 = \left(\frac{9}{4}, 0\right)\ \mathrm{m}^{-1}, \boldsymbol{k}_2 = (1, 0)\ \mathrm{m}^{-1}. \tag{30}$$

The second is the wave system consisting of two primary waves traveling in opposite directions:

$$\boldsymbol{k}_1 = \left(\frac{9}{4}, 0\right)\ \mathrm{m}^{-1}, \boldsymbol{k}_2 = (-1, 0)\ \mathrm{m}^{-1}. \tag{31}$$

The corresponding linear frequencies of the primary waves are $\omega_1 = 3\sqrt{g}/2$ and $\omega_2 = \sqrt{g}$, respectively.

Let

$$\mathcal{L}_0[\varphi] = \omega_1^2\frac{\partial^2\varphi}{\partial\xi_1^2} + 2\omega_1\omega_2\frac{\partial^2\varphi}{\partial\xi_1\partial\xi_2} + \omega_2^2\frac{\partial^2\varphi}{\partial\xi_2^2} + g\frac{\partial\varphi}{\partial z} \tag{32}$$

denote a linear operator corresponding to the linear parts of the boundary condition (10). This has the property

$$\mathcal{L}_0[\Psi_{m_1,m_2}(\xi_1,\xi_2,z)] = \lambda^a_{m_1,m_2}\Psi_{m_1,m_2}(\xi_1,\xi_2,z), \tag{33}$$

where

$$\lambda^a_{m_1,m_2} = g|m_1\boldsymbol{k}_1 + m_2\boldsymbol{k}_2| - (m_1\omega_1 + m_2\omega_2)^2. \tag{34}$$

Therefore, its inverse operator $\mathcal{L}_0^{-1}$ is given by

$$\mathcal{L}_0^{-1}[\Psi_{m_1,m_2}(\xi_1,\xi_2,z)] = \frac{\Psi_{m_1,m_2}(\xi_1,\xi_2,z)}{\lambda^a_{m_1,m_2}}. \tag{35}$$

Obviously, $\lambda^a_{m_1,m_2} = 0$ corresponds to the wave resonance and leads to the singularity.

For two wave trains traveling in the same direction, $\lambda^a_{m_1,m_2} = 0$ in the case of (30) gives an algebraic equation in the integers m_1 and m_2,

$$\left|\frac{9}{4}m_1 + m_2\right| - \left(\frac{3}{2}m_1 + m_2\right)^2 = 0, \tag{36}$$

which has an infinite number of solutions (some of which are listed in Table I). Therefore, there exist an infinite number of zero-valued denominators in (35), and these are associated with an infinite number of resonances. Similarly, when the two wave trains are traveling in opposite directions, $\lambda^a_{m_1,m_2} = 0$ in the case of (31) gives an algebraic equation in the integers m_1 and m_2,

$$\left|\frac{9}{4}m_1 - m_2\right| - \left(\frac{3}{2}m_1 + m_2\right)^2 = 0, \tag{37}$$

which also has an infinite number of solutions (some of which are listed in Table II), and these are also associated with an infinite number of resonances. It is rather difficult to handle such nonlinear problems with an infinite number of singularities by means of traditional analytic approximation methods, such as perturbation techniques.

TABLE I. Values of (m_1, m_2) corresponding to $\lambda^a_{m_1,m_2} = 0$ in the case of $\boldsymbol{k}_1 = (9/4, 0)\ \mathrm{m}^{-1}$ and $\boldsymbol{k}_2 = (1, 0)\ \mathrm{m}^{-1}$.

(m_1, m_2)	(m_1, m_2)
(1, 0)	(8, −9)
(0, 1)	(8, −14)
(1, −2)	(16, −20)
(5, −5)	(16, −27)
(5, −9)	…

TABLE II. Values of (m_1, m_2) corresponding to $\lambda^a_{m_1,m_2} = 0$ in the case of $\boldsymbol{k}_1 = (9/4, 0)\ \mathrm{m}^{-1}$ and $\boldsymbol{k}_2 = (-1, 0)\ \mathrm{m}^{-1}$.

(m_1, m_2)	(m_1, m_2)
(1, 0)	(13, −13)
(0, 1)	(13, −27)
(1, −4)	(17, −18)
(8, −7)	(17, −34)
(8, −18)	…

D. Choice of auxiliary linear operator and initial potential

The above-mentioned mathematical difficulty in the problem considered in this paper can be easily solved by means of the HAM, mainly because, unlike perturbation methods, the HAM provides great freedom to choose the so-called auxiliary linear operator and the initial values of unknown functions.

1. Avoidance of an infinite number of singularities

Different from perturbation methods, the HAM provides great freedom to choose the auxiliary linear operator. Therefore, we can choose such an auxiliary linear operator as

$$\mathcal{L}[\varphi] = \omega_1^2\frac{\partial^2\varphi}{\partial\xi_1^2} + 2\omega_1\omega_2\frac{\partial^2\varphi}{\partial\xi_1\partial\xi_2} + \omega_2^2\frac{\partial^2\varphi}{\partial\xi_2^2} + \mu_1 g\frac{\partial\varphi}{\partial z}, \tag{38}$$

where

$$\mu_1 = \begin{cases} 1, & m_1 = 1,\ m_2 = 0;\ m_1 = 0,\ m_2 = 1, \\ \pi/3, & else. \end{cases} \tag{39}$$

The auxiliary linear operator (38) has the property

$$\mathcal{L}[\Psi_{m_1,m_2}(\xi_1,\xi_2,z)] = \lambda^b_{m_1,m_2}\Psi_{m_1,m_2}(\xi_1,\xi_2,z), \quad (40)$$

where

$$\lambda^b_{m_1,m_2} = \mu_1 g|m_1\boldsymbol{k}_1 + m_2\boldsymbol{k}_2| - (m_1\omega_1 + m_2\omega_2)^2. \quad (41)$$

Thus, its inverse operator $\mathcal{L}^{-1}$ is

$$\mathcal{L}^{-1}[\Psi_{m_1,m_2}(\xi_1,\xi_2,z)] = \frac{\Psi_{m_1,m_2}(\xi_1,\xi_2,z)}{\lambda^b_{m_1,m_2}}. \quad (42)$$

Obviously, $\lambda^b_{m_1,m_2} = 0$ corresponds to a singularity.

In the case of (30), $\lambda^b_{m_1,m_2} = 0$ leads to the algebraic equation

$$\mu_1\left|\frac{9}{4}m_1 + m_2\right| - \left(\frac{3}{2}m_1 + m_2\right)^2 = 0, \quad (43)$$

where μ_1 is defined by (39) and m_1, m_2 should be integers. When $\mu_1 = 1$, the above equation has two integer solutions, $m_1 = 1$, $m_2 = 0$ and $m_1 = 0$, $m_2 = 1$, i.e.,

$$\lambda^b_{1,0} = \lambda^b_{0,1} = 0. \quad (44)$$

However, when $\mu_1 = \pi/3$ and unless $m_1 = 0$ and $m_2 = 0$, the first term $\mu_1\left|\frac{9}{4}m_1 + m_2\right|$ is an irrational number, but the second term $\left(\frac{3}{2}m_1 + m_2\right)^2$ is rational so that

$$\mu_1\left|\frac{9}{4}m_1 + m_2\right| - \left(\frac{3}{2}m_1 + m_2\right)^2 \neq 0$$

always holds, i.e., $\lambda^b_{m_1,m_2} \neq 0$. Thus, in the case of (30), the auxiliary linear operator (38) has only *two* singularities, although the original linear operator (32) has an *infinite* number of singularities. In this way, an infinite number of zero-valued denominators associated with an infinite number of resonant components of the nonlinear interaction of two wave trains traveling in the same/opposite direction are avoided by choosing an appropriate auxiliary linear operator (38). One can even choose different irrational values of μ_1, such as $\mu_1 = \sqrt{2}$ or $\mu_1 = \pi/4$; all of them work quite well and give convergent results. Note that it is the HAM that provides us with the freedom to choose such an auxiliary linear operator (38).

Similarly, in the case of (31), the auxiliary linear operator (38) has only two singularities, i.e., $\lambda^b_{1,0} = \lambda^b_{0,1} = 0$.

When the auxiliary linear operator is defined by (38), the terms $\bar{S}_m$ and S_m in the high-order deformation equations (26) read

$$\bar{S}_m = \sum_{n=1}^{m-1}\left(\omega_1^2\beta_{2,0}^{m-n,n} + 2\omega_1\omega_2\beta_{1,1}^{m-n,n} + \omega_2^2\beta_{0,2}^{m-n,n} + \mu_1 g\gamma_{0,0}^{m-n,n}\right), \quad (45)$$

$$S_m = \omega_1^2\beta_{2,0}^{m,0} + 2\omega_1\omega_2\beta_{1,1}^{m,0} + \omega_2^2\beta_{0,2}^{m,0} + \mu_1 g\gamma_{0,0}^{m,0} + \bar{S}_m, \quad (46)$$

where $\beta_{i,j}^{n,m}$ and $\gamma_{i,j}^{n,m}$ have the same definitions as those given by Liao.[23]

For the auxiliary linear operator (38), we choose the initial potential

$$\varphi_0 = a\sqrt{g/k_1}\,\Psi_{1,0} + b\sqrt{g/k_2}\,\Psi_{0,1}, \quad (47)$$

where the given constants a and b corresponding to the two primary waves $\Psi_{1,0}$ and $\Psi_{0,1}$ are defined by (15).

2. Acceleration of convergence

As mentioned above, the HAM provides significant freedom to choose an appropriate auxiliary linear operator (38) that avoids the problem of infinite singularities. This kind of freedom is so great that even different irrational values of μ_1 in the auxiliary linear operator (38) give the same results. Note that this freedom can be further used to accelerate the convergence in cases of high nonlinearity, as described below.

Using (38) as the auxiliary linear operator and (47) as the initial guess for the velocity potential in the HAM framework, we obtain convergent series solutions for the steady-state system consisting of the nonlinear interaction of two collinear progressive waves in the same/opposite direction with an infinite number of resonant components. The two primary waves often contain most of the wave energy. However, as the nonlinearity increases, i.e., a and b in the initial guess (47) become larger, other wave components such as $(1, -1)$ might contain an increasing amount of wave energy. Mathematically, this will lead to slower convergence of the series solution. Fortunately, in the HAM framework, we can accelerate the convergence by choosing a more suitable auxiliary operator,

$$\mathcal{L}[\varphi] = \omega_1^2\frac{\partial^2\varphi}{\partial\xi_1^2} + \mu_3\omega_1\omega_2\frac{\partial^2\varphi}{\partial\xi_1\partial\xi_2} + \omega_2^2\frac{\partial^2\varphi}{\partial\xi_2^2} + \mu_2 g\frac{\partial\varphi}{\partial z}, \quad (48)$$

where

$$\mu_2 = \begin{cases} 1, & m_1 = 1,\ m_2 = 0;\ m_1 = 0,\ m_2 = 1; \\ & m_1 = 1,\ m_2 = -1, \\ \pi/3, & else \end{cases} \quad (49)$$

and

$$\mu_3 = \begin{cases} \frac{g|m_1\boldsymbol{k}_1 + m_2\boldsymbol{k}_2| - (m_1^2\omega_1^2 + m_2^2\omega_2^2)}{m_1 m_2\omega_1\omega_2}, & m_1 = 1, m_2 = -1, \\ 2, & else. \end{cases} \quad (50)$$

The auxiliary linear operator (48) has the property

$$\mathcal{L}[\Psi_{m_1,m_2}(\xi_1,\xi_2,z)] = \lambda^c_{m_1,m_2}\Psi_{m_1,m_2}(\xi_1,\xi_2,z), \quad (51)$$

where

$$\lambda^c_{m_1,m_2} = \mu_2 g|m_1\boldsymbol{k}_1 + m_2\boldsymbol{k}_2| - (m_1^2\omega_1^2 + \mu_3 m_1 m_2\omega_1\omega_2 + m_2^2\omega_2^2). \quad (52)$$

Its inverse operator $\mathcal{L}^{-1}$ is

$$\mathcal{L}^{-1}[\Psi_{m_1,m_2}(\xi_1,\xi_2,z)] = \frac{\Psi_{m_1,m_2}(\xi_1,\xi_2,z)}{\lambda^c_{m_1,m_2}}. \quad (53)$$

Similarly, it is easily shown that the auxiliary linear operator (48) has only three singularities,

$$\lambda^c_{1,0} = \lambda^c_{0,1} = \lambda^c_{1,-1} = 0. \quad (54)$$

For the auxiliary linear operator (48), the terms $\bar{S}_m$ and S_m in the high-order deformation equations (26) are

$$\bar{S}_m = \sum_{n=1}^{m-1}\left(\omega_1^2\beta_{2,0}^{m-n,n} + \mu_3\omega_1\omega_2\beta_{1,1}^{m-n,n} + \omega_2^2\beta_{0,2}^{m-n,n} + \mu_2 g\gamma_{0,0}^{m-n,n}\right), \quad (55)$$

$$S_m = \omega_1^2\beta_{2,0}^{m,0} + \mu_3\omega_1\omega_2\beta_{1,1}^{m,0} + \omega_2^2\beta_{0,2}^{m,0} + \mu_2 g\gamma_{0,0}^{m,0} + \bar{S}_m. \quad (56)$$

TABLE III. Values of $\epsilon_1(\epsilon_2)$, $|\eta_{\max}|$, and $|\eta_{\min}|$ in the case of $\boldsymbol{k}_1$ = (1, 0) m^{-1} and $\boldsymbol{k}_2$ = (−1, 0) m^{-1} with different values of steepness.

	$a(b)$ = 7/200 m (steepness ≈ 0.07)			$a(b)$ = 1/10 m (steepness ≈ 0.2)			$a(b)$ = 2/10 m (steepness ≈ 0.4)		
	$\epsilon_1(\epsilon_2)$	$\|\eta_{\max}\|$	$\|\eta_{\min}\|$	$\epsilon_1(\epsilon_2)$	$\|\eta_{\max}\|$	$\|\eta_{\min}\|$	$\epsilon_1(\epsilon_2)$	$\|\eta_{\max}\|$	$\|\eta_{\min}\|$
HAM	0.999 386	0.0725 947	0.0676 843	0.994 908	0.223 647	0.182 918	0.978 666	0.515 985	0.341 165
Tadjbakhsh and Keller	0.999 387	0.0726 239	0.0677 194	0.995 014	0.224 167	0.183 873	0.979 602	0.515 616	0.350 775
Schwartz and Whitney	0.999 388	0.0724 802	0.0675 839	0.995 033	0.220 606	0.180 863	0.980 549	0.483 818	0.335 678

In this case, we choose the initial potential to be

$$\varphi_0 = a\sqrt{g/k_1}\,\Psi_{1,0} + b\sqrt{g/k_2}\,\Psi_{0,1} + \gamma_0\Psi_{1,-1}, \tag{57}$$

where the two constants a and b are given, but γ_0 is to be determined. In this way, the convergence of the series solution can be accelerated, as shown below.

III. RESULT ANALYSIS

A. A validation case: Nonlinear standing waves

Nonlinear standing waves are special cases of two progressive waves traveling in the opposite directions with an infinite number of exact resonances. In this section, we first consider the nonlinear standing waves with the wavevectors satisfying the following condition:

$$\boldsymbol{k}_1 = (1,0)\ \mathrm{m}^{-1}, \boldsymbol{k}_2 = (-1,0)\ \mathrm{m}^{-1}. \tag{58}$$

Based on the HAM, we successfully obtained the convergent solutions of nonlinear standing waves. The HAM result agrees well with the 3rd-order perturbation solution given by Tabjbakhsh and Keller[42] and the 25th-order perturbation solution given by Schwartz and Whitney,[43] as presented in Table III for different values of steepness. As mentioned by Schwartz and Whitney,[43] the 25th-order perturbation solution is not uniformly convergent for a steepness greater than 0.3. Therefore, when the steepness ≈0.4, the solution of the HAM is more closer to the result of Tabjbakhsh and Keller.[42] This illustrates the validity of our HAM approach.

B. Convergence of the series solution given by the HAM

To demonstrate the validity of the proposed approach based on the HAM, let us first consider the case of $\boldsymbol{k}_1 = (\frac{9}{4},0)$ m^{-1} and $\boldsymbol{k}_2$ = (1, 0) m^{-1} with $a = b = 2/100$ m by means of the auxiliary linear operator (38) and the initial guess (47). Substituting the initial guess (47) into the so-called first-order deformation equation (26) (m = 1) in the HAM framework, we have

$$\begin{aligned}\mathcal{L}[\varphi_1] &= c_0\Delta_0^{\varphi} - \overline{S}_1\\ &= \overline{b}_{1,0}\sin(\xi_1) + \overline{b}_{0,1}\sin(\xi_2) + \overline{b}_{1,-1}\sin(\xi_1-\xi_2),\\ &\quad + \overline{b}_{2,-1}\sin(2\xi_1-\xi_2) + \overline{b}_{1,-2}\sin(\xi_1-2\xi_2),\end{aligned} \tag{59}$$

where $\overline{b}_{m_1,m_2}$ depends upon the unknown $\sigma_{1,0}$ and $\sigma_{2,0}$. According to the property of the inverse operator (42), the coefficients $\overline{b}_{1,0}$ and $\overline{b}_{0,1}$ of the terms corresponding to the primary components $\sin(\xi_1)$ and $\sin(\xi_2)$ on the right-hand side of (59) must be zero so as to avoid secular terms. This gives us the set of two nonlinear algebraic equations for $\sigma_{1,0}$ and $\sigma_{2,0}$, i.e.,

$$0.924\,379 - 0.041\,761\sigma_{1,0}^2 = 0, \tag{60a}$$

$$0.617\,251 - 0.626\,42\sigma_{2,0}^2 = 0, \tag{60b}$$

and the unique solution with the assumption $\sigma_{i,0} > 0$ is $\sigma_{1,0} = 4.704\,77$ and $\sigma_{2,0} = 3.139\,05$. As long as $\sigma_{1,0}$ and $\sigma_{2,0}$ can be determined, φ_0 is known. With the initial guess $\eta_0 = 0$, it is then straightforward to calculate η_1 directly by means of (28),

$$\begin{aligned}\eta_1 = c_0\Delta_0^{\eta} &= \overline{a}_{0,0} + \overline{a}_{1,0}\cos(\xi_1)\\ &+ \overline{a}_{0,1}\cos(\xi_2) + \overline{a}_{1,-1}\cos(\xi_1-\xi_2),\end{aligned} \tag{61}$$

where all coefficients $\overline{a}_{m_1,m_2}$ are known.

Substituting the known φ_0 into (26) at m = 1, we then obtain the special solution φ_1^* of φ_1, say,

$$\varphi_1^* = \mathcal{L}^{-1}[c_0\Delta_0^{\varphi} - \overline{S}_1]. \tag{62}$$

According to the property of the auxiliary linear operator (42), the general solution of φ_1 is

$$\varphi_1 = \varphi_1^* + a^*\Psi_{1,0} + b^*\Psi_{0,1}. \tag{63}$$

However, as the components $a\sqrt{g/k_1}\,\Psi_{1,0}$ and $b\sqrt{g/k_2}\,\Psi_{0,1}$ of the two primary waves are known and fixed, we have $a^* = b^* = 0$, and therefore,

$$\begin{aligned}\varphi_1 = \varphi_1^* = \mathcal{L}^{-1}[c_0\Delta_0^{\varphi} - \overline{S}_1] &= \overline{d}_{1,1}\Psi_{1,1} + \overline{d}_{1,-1}\Psi_{1,-1}\\ &+ \overline{d}_{2,-1}\Psi_{2,-1} + \overline{d}_{1,-2}\Psi_{1,-2},\end{aligned} \tag{64}$$

where $\overline{d}_{m_1,m_2} = \overline{b}_{m_1,m_2}/\lambda^b_{m_1,m_2}$. Note that φ_1 contains the unknown $\sigma_{1,1}$ and $\sigma_{2,1}$. Thus, further substituting φ_1 into (26) at m = 2 and

TABLE IV. The averaged residual squares of ε_m^{ϕ} and ε_m^{η} in the case of $\boldsymbol{k}_1$ = (9/4, 0) m^{-1} and $\boldsymbol{k}_2$ = (1, 0) m^{-1} with $a = b = 2/100$ m and $c_0 = -0.5$.

m (order of approximation)	ε_m^{ϕ}	ε_m^{η}
2	0.000 054 84	0.000 096 34
4	5.93×10^{-6}	5.78×10^{-6}
6	5.95×10^{-7}	3.67×10^{-7}
8	4.19×10^{-8}	2.57×10^{-8}
10	5.31×10^{-9}	3.49×10^{-9}

TABLE V. mth-order analytical approximations of the dimensionless angular frequencies and the wave amplitude components ($a'_{m_1,m_2} = |a_{m_1,m_2}|/|a_{1,0}|$) in the case of $\boldsymbol{k}_1$ = (9/4, 0) m^{-1} and $\boldsymbol{k}_2$ = (1, 0) m^{-1} with $a = b$ = 2/100 m and c_0 = −0.5.

m	ϵ_1	ϵ_2	$a'_{0,1}$	$a'_{2,0}$	$a'_{1,-1}$	$a'_{1,1}$	$a'_{0,2}$
2	1.0015	1.0015	1.001	0.007	0.024	0.011	0.003
4	1.0016	1.0012	1.001	0.017	0.017	0.024	0.007
6	1.0016	1.0009	1.000	0.020	0.015	0.029	0.010
8	1.0016	1.0009	1.000	0.022	0.014	0.032	0.011
10	1.0016	1.0008	1.000	0.023	0.014	0.033	0.011
12	1.0016	1.0008	1.000	0.023	0.014	0.033	0.011
13	1.0016	1.0008	1.000	0.023	0.014	0.033	0.011

forcing the coefficients of the terms corresponding to the primary components $\sin(\xi_1)$ and $\sin(\xi_2)$ to be zero so as to avoid secular terms, $\sigma_{1,1}$ and $\sigma_{2,1}$ can be determined. Furthermore, we obtain η_2 by substituting the known φ_0, φ_1, η_0, and η_1 into (28) at m = 2, and so on. Similarly, we can determine all of the unknowns step by step.

Note that the so-called "convergence-control parameter" c_0 exists in the series solution. This parameter modifies the convergence of the solution and thus provides a convenient way to guarantee the convergence. This is an advantage of the HAM that distinguishes it from other analytic approximation methods. With a proper "convergence-control parameter" c_0 determined by minimizing the averaged residual squares of the two boundary conditions in a similar way to those described by Liao,[23] Xu *et al.*,[32] and Yang *et al.*,[29] the corresponding angular frequencies and wave amplitudes converge quickly. For the case of $\boldsymbol{k}_1$ = (9/4, 0) m^{-1} and $\boldsymbol{k}_2$ = (1, 0) m^{-1} with $a = b$ = 2/100 m, we choose $c_0 = -0.5$. The corresponding residual error squares of the two boundary conditions decease rather quickly to the level 10^{-9} at the 10th-order, as shown in Table IV. Then, we can obtain the convergent wave frequencies and amplitudes, which are given in Table V. Similarly, for two primary waves traveling in the same direction, we gain the convergent solutions when $a = b$ = 1/100 m and $a = b$ = 3/100 m by means of the auxiliary linear operator (38); several examples are listed in Table VI. Note that, in this paper, we do not give the wave amplitude $|a_{m_1,m_2}|$ when $|a_{m_1,m_2}| < 10^{-4}$.

However, when $a = b$ increases from 1/100 m to 3/100 m, the second-order wave component (1, −1) becomes larger in the velocity potential φ, while the convergence slows down. Hence, when $a \geq$ 4/100 m or $b \geq$ 4/100 m, we use the auxiliary linear operator (48) and the corresponding initial guess (57) so as to accelerate the convergence of the series solution. For the mth-order approximation, we have

$$\varphi_m = \mathcal{L}^{-1}\left[c_0\Delta^{\varphi}_{m-1} - \overline{S}_m\right] + \gamma_m \Psi_{1,-1}, \tag{65}$$

TABLE VI. Dimensionless angular frequencies and dimensionless wave amplitude components ($a'_{m_1,m_2} = |a_{m_1,m_2}|/|a_{1,0}|$) in the case of $\boldsymbol{k}_1$ = (9/4, 0) m^{-1} and $\boldsymbol{k}_2$ = (1, 0) m^{-1} for different values of wave steepness.

a (m)	b (m)	ϵ_1	ϵ_2	$a'_{0,1}$	$a'_{2,0}$	$a'_{1,-1}$	$a'_{1,1}$	$a'_{0,2}$	$a'_{3,0}$	$a'_{2,1}$	$a'_{2,-1}$	$a'_{1,2}$	$a'_{3,-1}$
1/100	1/100	1.0004	1.0002	1.000	0.010	0.010	0.020	0.010	...	...	...	...	...
2/100	2/100	1.0016	1.0008	1.000	0.025	0.015	0.035	0.010	...	...	0.005	...	...
3/100	3/100	1.0037	1.0018	1.000	0.040	0.020	0.050	0.017	...	0.003	0.013	0.003	...
4/100	4/100	1.0067	1.0033	1.000	0.048	0.028	0.068	0.020	0.003	0.008	0.023	0.005	0.003
6/100	6/100	1.0158	1.0078	0.998	0.080	0.043	0.103	0.032	0.010	0.017	0.061	0.010	0.012
8/100	2/100	1.0179	1.0115	0.249	0.097	0.016	0.035	0.002	0.015	0.007	0.035	0.001	0.009
8/100	4/100	1.0206	1.0123	0.500	0.103	0.032	0.071	0.011	0.017	0.015	0.071	0.005	0.019
8/100	6/100	1.0255	1.0138	0.749	0.114	0.050	0.107	0.026	0.022	0.025	0.108	0.011	0.031
8/100	8/100	1.0330	1.0161	0.998	0.133	0.070	0.147	0.047	0.032	0.037	0.150	0.020	0.048

TABLE VII. Dimensionless angular frequencies and dimensionless wave amplitude components ($a'_{m_1,m_2} = |a_{m_1,m_2}|/|a_{1,0}|$) in the case of $\boldsymbol{k}_1$ = (9/4, 0) m^{-1} and $\boldsymbol{k}_2$ = (−1, 0) m^{-1} for different values of wave steepness.

a (m)	b (m)	ϵ_1	ϵ_2	$a'_{0,1}$	$a'_{2,0}$	$a'_{1,-1}$	$a'_{1,1}$	$a'_{0,2}$	$a'_{3,0}$	$a'_{2,1}$	$a'_{2,-1}$	$a'_{1,-2}$	$a'_{3,1}$	$a'_{3,-1}$
2/100	2/100	1.0004	0.9996	1.000	0.025	0.035	0.015	0.010	...	...	...	...	...	...
5/100	5/100	1.0027	0.9975	1.000	0.056	0.082	0.032	0.024	0.004	0.004	0.010	0.006	...	0.002
8/100	8/100	1.0075	0.9933	0.999	0.092	0.130	0.050	0.037	0.012	0.012	0.024	0.014	0.002	0.005
1/10	1/10	1.0126	0.9891	0.998	0.118	0.162	0.064	0.045	0.021	0.019	0.037	0.022	0.006	0.010
12/100	2/100	1.0413	0.9771	0.165	0.155	0.032	0.012	0.002	0.037	0.005	0.009	0.001	0.002	0.003
12/100	5/100	1.0378	0.9782	0.414	0.154	0.081	0.031	0.009	0.036	0.012	0.022	0.005	0.005	0.008
12/100	8/100	1.0320	0.9800	0.662	0.152	0.126	0.050	0.022	0.035	0.019	0.035	0.014	0.007	0.012
12/100	1/10	1.0267	0.9816	0.829	0.148	0.162	0.063	0.035	0.033	0.024	0.045	0.021	0.009	0.015
12/100	12/100	1.0202	0.9836	0.997	0.146	0.195	0.076	0.051	0.031	0.029	0.053	0.031	0.011	0.017

TABLE VIII. Energy distribution (%) of the wave system in the case of $\boldsymbol{k}_1$ = (9/4, 0) m^{-1} and $\boldsymbol{k}_2$ = (1, 0) m^{-1} for different values of wave steepness.

a (m)	b (m)	$a_{1,0}^2/\Pi$	$a_{0,1}^2/\Pi$	$a_{2,0}^2/\Pi$	$a_{1,-1}^2/\Pi$	$a_{1,1}^2/\Pi$	$a_{0,2}^2/\Pi$	$a_{3,0}^2/\Pi$	$a_{2,1}^2/\Pi$	$a_{2,-1}^2/\Pi$	$a_{1,2}^2/\Pi$	$a_{3,-1}^2/\Pi$	Π_0/Π
1/100	1/100	49.99	49.99	…	…	0.01	…	…	…	…	…	…	99.98
2/100	2/100	49.95	49.95	0.03	0.01	0.05	0.01	…	…	…	…	…	99.90
3/100	3/100	49.89	49.89	0.06	0.02	0.12	0.01	…	…	0.01	…	…	99.78
4/100	4/100	49.80	49.78	0.11	0.03	0.22	0.02	…	…	0.03	…	…	99.58
6/100	6/100	49.44	49.36	0.31	0.09	0.52	0.05	0.01	0.01	0.08	…	…	98.80
8/100	2/100	93.05	5.79	0.88	0.02	0.11	…	0.02	…	0.11	…	0.01	98.84
8/100	4/100	78.64	19.57	0.84	0.08	0.39	0.01	0.02	0.02	0.39	…	0.03	98.20
8/100	6/100	62.42	39.94	0.82	0.16	0.27	0.04	0.03	0.04	0.73	0.01	0.06	97.36
8/100	8/100	48.29	48.04	0.85	0.24	1.04	0.11	0.05	0.07	1.09	0.02	0.11	96.34

where γ_m is an additional unknown constant. Note that the auxiliary linear operator (48) has three singularities. According to the property of the inverse operator (53), the coefficients of the three terms corresponding to the wave components $\sin(\xi_1)$, $\sin(\xi_2)$, and $\sin(\xi_1 - \xi_2)$ on the right-hand side of (26) must be zero to avoid secular terms. This provides a set of algebraic equations to determine the three unknowns $\sigma_{1,m}$, $\sigma_{2,m}$, and γ_m. Similarly, we can choose an appropriate value of the convergence-control parameter c_0 to guarantee the convergence of the series solution. In this way, we successfully gain the convergent angular frequencies and wave amplitudes for different values of a and b, as listed in Table VI. For two primary waves traveling in opposite directions, we gain convergent solutions in a similar way; several examples are listed in Table VII. Note that, as the wave amplitude increases, an increasing number of wave components join the whole wave system.

The contributions of Stokes' corrections mean that, for the two primary waves traveling in the same direction, the nonlinear frequencies σ_1 and σ_2 are always greater than the corresponding linear frequencies ω_1 and ω_2. For the two primary waves traveling in opposite directions, the nonlinear frequency σ_1 associated with the higher wavenumber is greater than the corresponding linear frequency ω_1, but the nonlinear frequency σ_2 associated with the small wavenumber is less than the related linear frequency ω_2. Therefore, the angular frequencies of the second primary wave are always slightly less than 1, which is different from the case of two primary waves traveling in the same direction. In addition, a larger difference between the amplitudes of the two primary waves produces a greater deviation of ϵ_i from 1, as can be seen from Table VII.

C. Energy distribution of the wave system

Let Π denote the sum of the squared amplitude of all components and write

$$\Pi_0 = a_{1,0}^2 + a_{0,1}^2. \tag{66}$$

For the collinear steady-state resonant wave system consisting of the nonlinear interaction of two primary waves traveling in the same/opposite direction, the energy distributions of different wave components with $a_{i,j}^2/\Pi \geq 0.01\%$ are presented in Tables VIII and IX.

The two primary components as a whole occupy most of the wave energy in all of the cases considered here. For a fixed a and a variable b in the initial guess (47) or (57), the wave energy is mainly distributed between the two primary waves, as shown in Fig. 1. A similar conclusion was reported by Xu *et al.*[32] for the nonlinear interaction of double cnoidal waves. Note that, when the wave amplitude increases, the wave energy slowly shifts from the primary components to the second-, third-, and even higher-order wave components.

D. Wave profiles

Wave profiles for the steady-state resonant wave system consisting of the interaction of two primary waves traveling in the

TABLE IX. Energy distribution (%) of the wave system in the case of $\boldsymbol{k}_1$ = (9/4, 0) m^{-1} and $\boldsymbol{k}_2$ = (−1, 0) m^{-1} for different values of wave steepness.

a (m)	b (m)	$a_{1,0}^2/\Pi$	$a_{0,1}^2/\Pi$	$a_{2,0}^2/\Pi$	$a_{1,-1}^2/\Pi$	$a_{1,1}^2/\Pi$	$a_{0,2}^2/\Pi$	$a_{3,0}^2/\Pi$	$a_{2,1}^2/\Pi$	$a_{2,-1}^2/\Pi$	$a_{1,-2}^2/\Pi$	$a_{3,1}^2/\Pi$	$a_{3,-1}^2/\Pi$	Π_0/Π
2/100	2/100	49.96	49.95	0.03	0.05	0.01	…	…	…	…	…	…	…	99.91
5/100	5/100	49.73	49.70	0.16	0.33	0.05	0.03	…	…	…	…	…	…	99.43
8/100	8/100	49.30	49.20	0.42	0.83	0.13	0.07	0.01	0.01	0.03	0.01	…	…	98.50
1/10	1/10	48.89	48.71	0.67	1.29	0.20	0.10	0.02	0.02	0.07	0.02	…	…	97.60
12/100	2/100	94.85	2.59	2.29	0.10	0.01	…	0.13	…	0.01	…	…	…	97.44
12/100	5/100	83.03	14.19	1.97	0.54	0.08	0.01	0.11	0.01	0.04	…	…	0.01	97.22
12/100	8/100	67.35	29.54	1.54	1.13	0.17	0.03	0.08	0.02	0.08	0.01	…	0.01	96.90
12/100	1/10	57.29	39.37	1.27	1.51	0.23	0.07	0.06	0.03	0.11	0.03	…	0.01	96.66
12/100	12/100	48.38	48.03	1.03	1.84	0.28	0.13	0.05	0.04	0.14	0.05	0.01	0.01	96.42

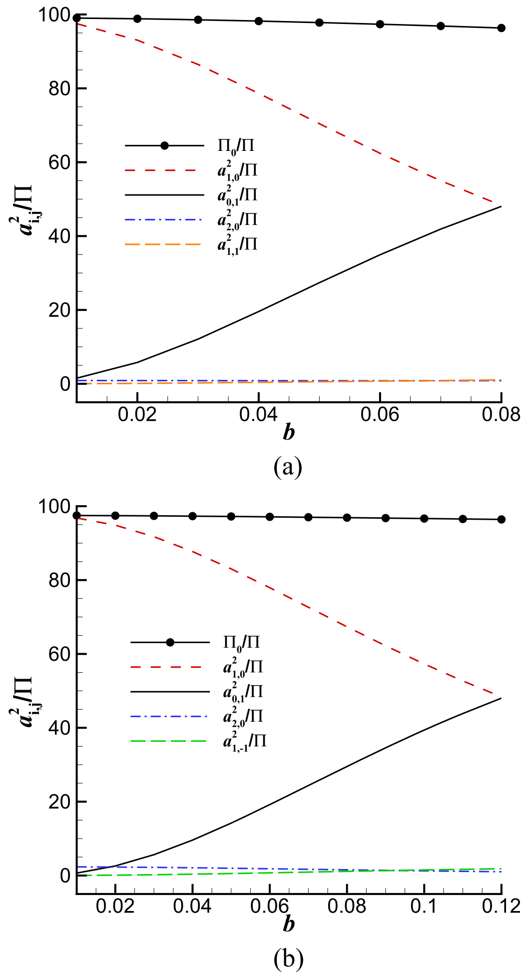

FIG. 1. Wave energy distribution (%) for increased b (m) in the initial guess: (a) $\boldsymbol{k}_1$ = (9/4, 0) m^{-1}, $\boldsymbol{k}_2$ = (1, 0) m^{-1}, and a = 8/100 m, (b) $\boldsymbol{k}_1$ = (9/4, 0) m^{-1}, $\boldsymbol{k}_2$ = (−1, 0) m^{-1}, and a = 12/100 m.

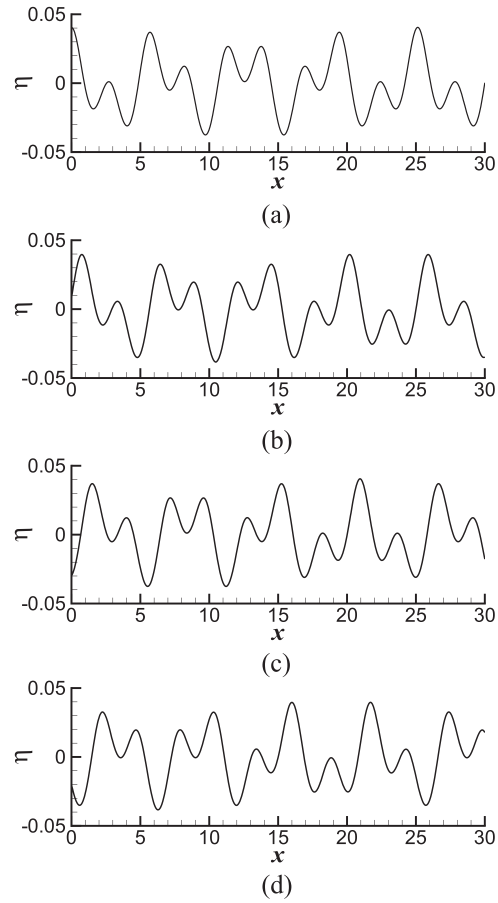

FIG. 2. Wave profiles at different time instants in the case of $\boldsymbol{k}_1$ = (9/4, 0) m^{-1} and $\boldsymbol{k}_2$ = (1, 0) m^{-1} with $a = b$ = 2/100 m: (a) t = 0, (b) $t = 0.25T_1$, (c) $t = 0.5T_1$, and (d) $t = 0.75T_1$, where T_1 is the actual wave period of the primary wave with k_1 = 9/4 m^{-1}.

same/opposite direction in the case of $a = b$ = 2/100 m are shown in Figs. 2 and 3, respectively.

We define the wave steepness as

$$H_s = k_d \frac{\max[\eta(\xi_1,\xi_2)] - \min[\eta(\xi_1,\xi_2)]}{2}, \tag{67}$$

where $\xi_i \in [0, 2\pi]$ and k_d is the wavenumber corresponding to the dominant frequency (k_1 in this paper). Without loss of generality, let us consider the two cases (30) and (31). The wave steepness is dependent upon the values of a and b in the initial guess (47) or (57). The maximum and minimum elevations and the wave steepness for the collinear steady-state resonant waves in the same/opposite direction are given in Tables X and XI, respectively. Larger values of a and b produce a greater wave steepness. For the steady-state resonant system consisting of two primary wave trains traveling in the same direction, the wave steepness H_s reaches 0.4187 in the case of $a = b$ = 8/100 m, corresponding to a finite amplitude steady-state wave group mentioned by Liu *et al.*[27] and Shemer *et al.*[44] For the steady-state resonant system consisting of two primary wave trains traveling in opposite directions, the wave steepness H_s reaches 0.5745 when $a = b$ = 12/100 m, also corresponding to a finite amplitude steady-state wave group. As a comparison, the limit wave steepness of the Stokes wave and the standing wave are given here. The limit wave steepness of the Stokes wave is about 0.44,[45] and the limit wave steepness of the standing wave is about 0.64.[46] It can be seen that the steady-state wave systems with the maximum steepness we obtained in this paper are in the range of finite amplitude steady-state wave groups and the maximum steepness closes to the limit wave steepness.

In linear wave theory, $|\eta'_{\max}| = |\eta'_{\min}| = a + b$, where $\eta'_{\max}$ and $\eta'_{\min}$ represent the maximum and minimum elevations of the linear combined waves, respectively. Define

$$R = \max\left\{\left|\frac{\eta_{\max} - \eta'_{\max}}{\eta'_{\max}}\right|, \left|\frac{\eta_{\min} - \eta'_{\min}}{\eta'_{\min}}\right|\right\}, \tag{68}$$

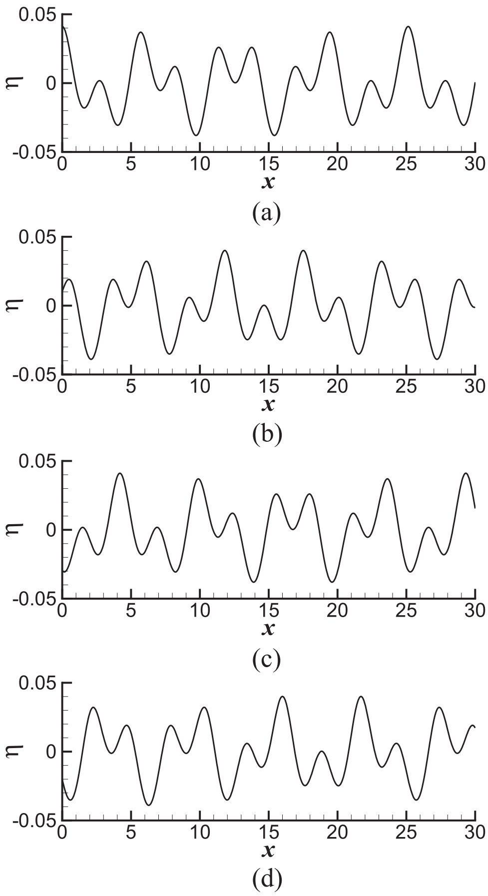

FIG. 3. Wave profiles at different time instants in the case of $\boldsymbol{k}_1$ = (9/4, 0) m^{-1} and $\boldsymbol{k}_2$ = (−1, 0) m^{-1} with $a = b$ = 2/100 m: (a) t = 0, (b) $t = 0.25T_1$, (c) $t = 0.5T_1$, and (d) $t = 0.75T_1$, where T_1 is the actual wave period of the primary wave with k_1 = 9/4 m^{-1}.

which indicates the difference in the extreme values of wave elevation given by the linear and nonlinear wave theories. For the two primary wave trains traveling in the same/opposite direction, as the nonlinearity increases, the difference between $\eta_{\max}$ and $\eta'_{\max}$

TABLE X. Wave steepness of nonlinear wave groups with $\boldsymbol{k}_1$ = (9/4, 0) m^{-1} and $\boldsymbol{k}_2$ = (1, 0) m^{-1} for different values of a and b.

a (m)	b (m)	$\lvert\eta_{\max}\rvert$	$\lvert\eta_{\min}\rvert$	H_s
1/100	1/100	0.0202	0.0197	0.0448
2/100	2/100	0.0405	0.0384	0.0888
3/100	3/100	0.0614	0.0565	0.1327
4/100	4/100	0.0861	0.0771	0.1836
6/100	6/100	0.1379	0.1156	0.2851
8/100	2/100	0.1158	0.0945	0.2366
8/100	4/100	0.1448	0.1160	0.2934
8/100	6/100	0.1765	0.1376	0.3534
8/100	8/100	0.2120	0.1594	0.4178

TABLE XI. Wave steepness of nonlinear wave groups with $\boldsymbol{k}_1$ = (9/4, 0) m^{-1} and $\boldsymbol{k}_2$ = (−1, 0) m^{-1} for different values of a and b.

a (m)	b (m)	$\lvert\eta_{\max}\rvert$	$\lvert\eta_{\min}\rvert$	H_s
2/100	2/100	0.0411	0.0390	0.0901
5/100	5/100	0.1076	0.0943	0.2271
8/100	8/100	0.1817	0.1463	0.3690
1/10	1/10	0.2371	0.1795	0.4687
12/100	2/100	0.1703	0.1244	0.3315
12/100	5/100	0.2067	0.1509	0.4023
12/100	8/100	0.2450	0.1770	0.4748
12/100	1/10	0.2717	0.1943	0.5243
12/100	12/100	0.2993	0.2114	0.5745

becomes more and more obvious, as shown in Figs. 4 and 5, respectively. For example, in the case of $a = b$ = 8/100 m, the linear wave theory gives $\eta'_{\max}$ = 0.1600 m, but the nonlinear wave theory gives $\eta_{\max}$ = 0.2120 m, some 31.9% higher than that in the linear case. Thus, generally speaking, the nonlinearity plays an important role and should not be neglected in the problem considered in this paper. Especially, for the case of $\boldsymbol{k}_1$ = (9/4, 0) m^{-1} and $\boldsymbol{k}_2$ = (−1, 0) m^{-1}, the first primary wave has a large influence on the whole wave system. When a = 12/100 m, which is large enough, as the value of b ($b < a$) changes, the effect of the second primary wave on the nonlinearity of the entire wave system is almost negligible, so the variation of R in Fig. 5(b) is weak.

E. Stability analysis

We consider the linear stability of the two-dimensional collinear steady-state waves with an infinite number of exact resonances to an infinitesimal three-dimensional disturbance. Let

$$\eta(x,y,t) = \overline{\eta}(x,y) + \eta'(x,y,t), \tag{69}$$

$$\varphi(x,y,t) = \overline{\varphi}(x,y) + \varphi'(x,y,t), \tag{70}$$

where ($\overline{\eta}$, $\overline{\varphi}$) and (η', φ') correspond to the unperturbed and infinitesimal perturbative motions, respectively. It is assumed that $\eta' \ll \overline{\eta}$ and $\varphi' \ll \overline{\varphi}$. Substituting (69) and (70) into the original governing equations (3)–(6), we obtain the first-order perturbation equations,

$$\nabla^2\varphi' = 0,\ -\infty \le z \le \overline{\eta}(x,y), \tag{71}$$

$$\varphi'_t + g\eta' + \overline{\varphi_x}\varphi'_x + \overline{\varphi_y}\varphi'_y + \overline{\varphi_z}\varphi'_z + \left(\overline{\varphi}_{tz} + \overline{\varphi}_x\overline{\varphi}_{xz} + \overline{\varphi}_y\overline{\varphi}_{yz} + \overline{\varphi}_z\overline{\varphi}_{zz}\right)\eta' = 0, \quad \text{on } z = \overline{\eta}(x,y), \tag{72}$$

$$\eta'_t + \overline{\eta}_x\varphi'_x + \overline{\varphi}_x\eta'_x + \overline{\eta}_y\varphi'_y + \overline{\varphi}_y\eta'_y - \varphi'_z - \left(\overline{\varphi}_{zz} - \overline{\eta}_x\overline{\varphi}_{xz} - \overline{\eta}_y\overline{\varphi}_{yz}\right)\eta' = 0, \quad \text{on } z = \overline{\eta}(x,y), \tag{73}$$

$$\lim_{z\to-\infty}\varphi'_z = 0. \tag{74}$$

We look for nontrivial solutions of (72) and (73) of the form

$$\eta' = e^{-i\sigma t}e^{i(k_p x + k_q y)}\sum_{J=-\infty}^{+\infty}\sum_{K=-\infty}^{+\infty} a_{JK}e^{i(k_{JKx}x + k_{JKy}y)}, \tag{75}$$

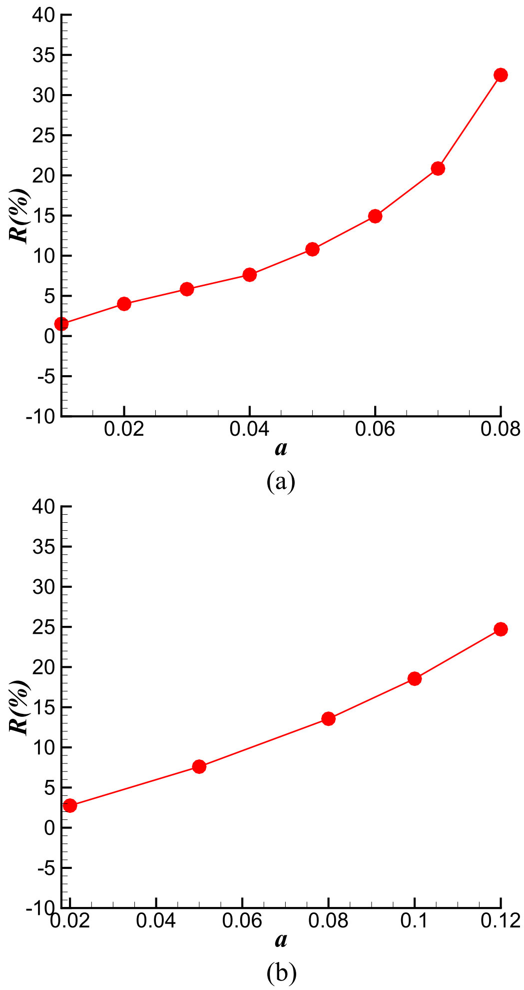

FIG. 4. Values of R with respect to the different values of wave steepness when $a = b$ (m): (a) $\boldsymbol{k}_1 = (9/4, 0)$ m^{-1} and $\boldsymbol{k}_2 = (1, 0)$ m^{-1}, and (b) $\boldsymbol{k}_1 = (9/4, 0)$ m^{-1} and $\boldsymbol{k}_2 = (-1, 0)$ m^{-1}.

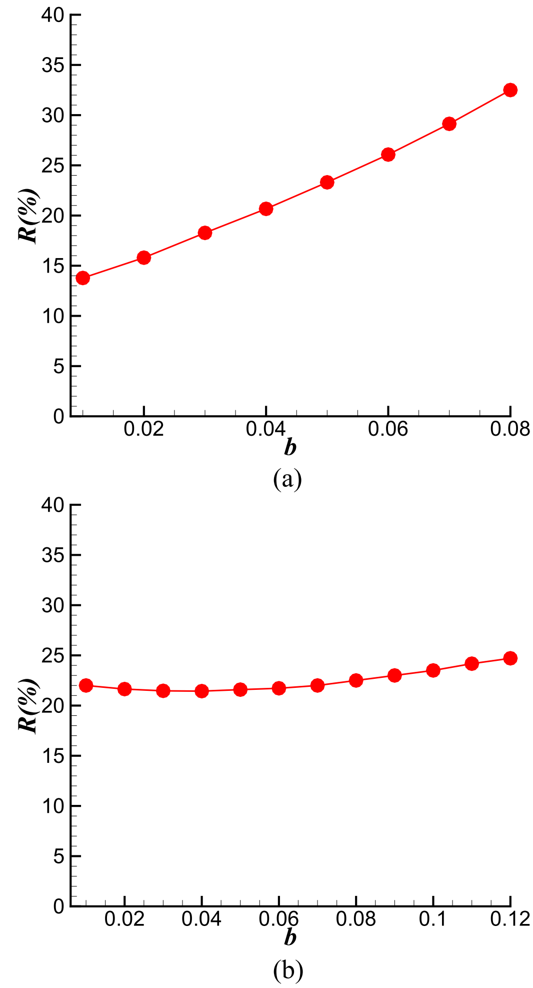

FIG. 5. Values of R with respect to different values of b: (a) $\boldsymbol{k}_1 = (9/4, 0)$ m^{-1}, $\boldsymbol{k}_2 = (1, 0)$ m^{-1}, and $a = 8/100$ m, and (b) $\boldsymbol{k}_1 = (9/4, 0)$ m^{-1}, $\boldsymbol{k}_2 = (-1, 0)$ m^{-1}, and $a = 12/100$ m.

$$\varphi' = \mathrm{e}^{-\mathrm{i}\sigma t}\mathrm{e}^{\mathrm{i}(k_p x + k_q y)}\sum_{J=-\infty}^{+\infty}\sum_{K=-\infty}^{+\infty} b_{JK}\mathrm{e}^{\mathrm{i}(k_{JK,x}x + k_{JK,y}y)}\mathrm{e}^{\kappa_{JK}z}, \tag{76}$$

where $\boldsymbol{k}_{JK} = (Jk_{1x} + Kk_{2x}, Jk_{1y} + Kk_{2y})$, $\kappa_{JK} = \left[(k_p + k_{JK,x})^2 + (k_q + k_{JK,y})^2\right]^{\frac{1}{2}}$, (k_p, k_q) is the wavevector of perturbation for the linear case, k_p and k_q are arbitrary real numbers, σ is the frequency of the perturbation, and a_{JK} and b_{JK} are two coefficients.

Once unperturbed wave appears in the wave field, a change from stability to instability can occur only if the eigenvalues satisfy

$$\sigma^{s_1}_{J_1K_1}(k_p, k_q) = \sigma^{s_2}_{J_2K_2}(k_p, k_q). \tag{77}$$

In a linear approximation, the condition when instability happens may be written in the following form:

$$\kappa^{1/2}_{J_1K_1} + \kappa^{1/2}_{J_2K_2} = (J_1 - J_2)|\boldsymbol{k}_1| + (K_1 - K_2)|\boldsymbol{k}_2|^{1/2}, \tag{78}$$

with $s_1 = -s_2 = 1$. Following McLean,[47] Ioualalen and Kharif,[48] and Liu and Liao,[25] we define two general classes of instabilities from (78),

$$J_1 + K_1 = j,\ J_2 + K_2 = -j \quad \text{for the class I}(j), \tag{79}$$

$$J_1 + K_1 = j,\ J_2 + K_2 = -j - 1 \text{ for the class II}(j), \tag{80}$$

where $j = 1, 2, 3, \ldots$. The present study will be limited to class I for $j = 1$, which corresponds to the dominant instability caused by four-wave interactions. We adopt the case of $\boldsymbol{k}_1 = (9/4, 0)$ m^{-1} and $\boldsymbol{k}_2 = (1, 0)$ m^{-1} to study the instability of the system. Given the symmetry in the basic wave, we consider the following five cases, which belong to classes Ia and Ib, respectively,

$$\text{Class I}a \begin{cases} a1: \ J_1 = 0,\ K_1 = 1,\ J_2 = -1,\ K_2 = 0, \\ a2: \ J_1 = 1,\ K_1 = 0,\ J_2 = -2,\ K_2 = 1, \end{cases} \tag{81}$$

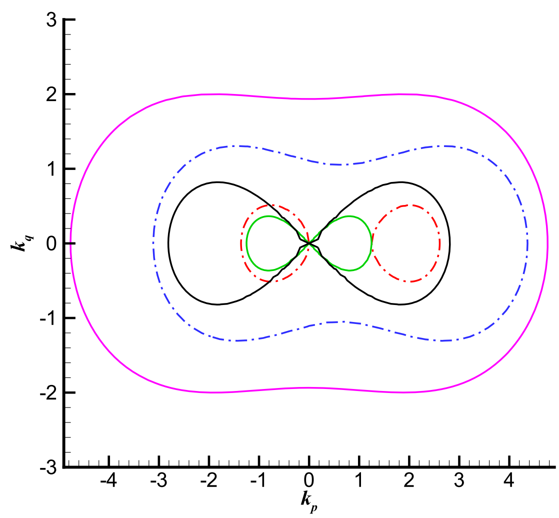

FIG. 6. Resonance curves of class Ia and Ib from the linear dispersion relation when $\boldsymbol{k}_1$ = (9/4, 0) m^{-1} and $\boldsymbol{k}_2$ = (1, 0) m^{-1}. Red: $a1$; blue: $a2$; green: $b1$; black: $b2$; purple: $b3$.

$$\text{Class I}b\begin{cases} b1: \ J_1 = 0,\ K_1 = 1,\ J_2 = 0,\ K_2 = -1, \\ b2: \ J_1 = 1,\ K_1 = 0,\ J_2 = -1,\ K_2 = 0, \\ b3: \ J_1 = 2,\ K_1 = -1,\ J_2 = -2,\ K_2 = 1. \end{cases} \tag{82}$$

When the values of J and K are given, the curves descried by (81) and (82) are plotted in Fig. 6. Class Ib curves are symmetric about the origin, while class Ia curves are symmetric about $k_p = 5/8$ and $k_q = 0$.

The coalescence of the eigenvalues can alternatively be interpreted as a resonance of two infinitesimal modes of wavevectors $\boldsymbol{k}_1'$ and $\boldsymbol{k}_2'$ with the fundamental components of wavevectors $\overline{\boldsymbol{k}}_1$ and $\overline{\boldsymbol{k}}_2$ in the basic wave. The resonance condition is

$$\boldsymbol{k}_1' + \boldsymbol{k}_2' = \overline{\boldsymbol{k}}_1 + \overline{\boldsymbol{k}}_2, \quad \omega_1' + \omega_2' = \overline{\omega}_1 + \overline{\omega}_2, \tag{83}$$

where ω_i' and $\overline{\omega}_i$ denote the frequencies of the corresponding perturbed and unperturbed waves, respectively. For the condition of $b1$,

$$\begin{aligned} \boldsymbol{k}_1' &= (k_p + k_{2x},\ k_q + k_{2y})^t, \\ \boldsymbol{k}_2' &= -(k_p - k_{2x},\ k_q - k_{2y})^t, \\ \overline{\boldsymbol{k}}_1 &= \overline{\boldsymbol{k}}_2 = \boldsymbol{k}_2. \end{aligned}$$

As shown in Fig. 6, the curve of $b1$ is Phillips' Fig. 8, so does the condition of $b2$.

According to our analysis, the collinear steady-state waves with an infinite number of exact resonances are stable as long as the disturbance does not resonate with any components of the basic wave. This conclusion is consistent with that for the Stokes waves and short-crested waves, so it is reasonable.

IV. CONCLUDING REMARKS AND DISCUSSION

We have investigated the nonlinear interaction of two primary progressive waves traveling in the same/opposite direction. In particular, there exist an infinite number of resonant wave components in the considered cases, corresponding to an infinite number of singularities in mathematics. Such resonant wave systems are rather difficult to solve by means of traditional analytic approaches such as perturbation methods. Fortunately, this mathematical obstacle can be easily cleared by means of the homotopy analysis method (HAM); the infinite number of singularities of the considered problem can be completely avoided by choosing an appropriate auxiliary linear operator in the HAM framework. In this way, we successfully gain steady-state systems with an infinite number of resonant components, consisting of the nonlinear interaction of the two primary waves traveling in the same or opposite direction. This indicates the general existence of the so-called steady-state resonant waves, even in the case of an infinite number of resonant components.

Let $\mathcal{N}[u] = 0$ denote a nonlinear equation, where $\mathcal{N}[u] = \mathcal{L}_0[u] + \mathcal{N}_0[u]$ contains a linear part $\mathcal{L}_0[u]$ and a nonlinear part $\mathcal{N}_0[u]$. From the viewpoint of perturbation methods, the linear part $\mathcal{L}_0[u]$ plays a key role and contributes to the bulk of the solution, while the nonlinear part $\mathcal{N}_0[u]$ provides some small modifications. Thus, in the frame of perturbation methods, the linear part $\mathcal{L}_0[u]$ has superiority over the nonlinear part $\mathcal{N}_0[u]$. Unfortunately, for the collinear steady-state resonant wave systems considered in this paper, the original linear operator (32) leads to an infinite number of singularities, which are rather difficult to handle mathematically. However, unlike perturbation methods, the HAM regards the nonlinear equation $\mathcal{L}_0[u] + \mathcal{N}_0[u] = 0$ as a whole and relinquishes the superiority of the linear part $\mathcal{L}_0[u]$ over the nonlinear part $\mathcal{N}_0[u]$. Additionally, the HAM provides great freedom to choose an auxiliary linear operator that might have no obvious relationship with the original $\mathcal{L}_0$. Using this freedom of the HAM, the infinite number of singularities can be avoided, and the mathematical obstacle can be easily overcome. Note that two kinds of auxiliary linear operators, (38) and (48), are successfully used in the HAM framework to obtain convergent series solutions for the considered problems. This strongly suggests that the superiority of the linear part $\mathcal{L}_0[u]$ over the nonlinear part $\mathcal{N}_0[u]$ in the frame of perturbation methods is wrong and unreasonable. Certainly, the steady-state collinear wave systems with an infinite number of resonances considered in this study illustrate the validity, flexibility, and potential of the HAM to be applied to complicated problems with a high degree of nonlinearity.

Our results indicate that the two primary components often occupy most of the wave energy, although this slowly shifts from the primary components to the second-, third-, and even higher-order wave components as the wave amplitude increases. Nonlinearity plays an important role and should therefore not be neglected in the types of problems considered in this paper. The results of this study deepen our understanding and enrich our knowledge about the resonance of gravity waves, which plays an important role in the growth and propagation of waves.

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