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HIGH-ORDER BEM FORMULATIONS FOR STRONGLY NON-LINEAR PROBLEMS GOVERNED BY QUITE GENERAL NON-LINEAR DIFFERENTIAL OPERATORS

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SUMMARY

In this paper the basic idea of homotopy in topology is applied to give a kind of high-order BEM formulation for general non-linear problems governed by non-linear differential operators which need *not* contain any linear operators at all. As a result, the traditional BEM for non-linear problems is just a special case of the proposed method. Three simple examples are used to show the effectiveness of the proposed quite general BEM for non-linear problems.

KEY WORDS: general BEM; general non-linear differential operator; homotopy

1. INTRODUCTION

1.1. BEM for linear problems

Boundary element method was developed initially for solving equations governed by linear differential operators as follows:

$$\hat{L}(u) = f(\vec{r}). \tag{1}$$

Selecting a weighting function ω, the weighted residual expression of (1) is

$$\int_\Omega [\hat{L}(u) - f(\vec{r})]\omega \mathrm{d}\Omega = 0. \tag{2}$$

Integrating this expression by the divergence theorem, we obtain its inverse form

$$\int_\Omega u\tilde{L}(\omega)\mathrm{d}\Omega = \int_\Gamma [u\hat{B}(\omega) - \omega\hat{B}(u)]\mathrm{d}\Gamma + \int_\Omega f\omega \mathrm{d}\Omega, \tag{3}$$

where the operator $\tilde{L}$ is adjoint operator of $\hat{L}$, the operator $\hat{B}$ is the corresponding boundary operator for the operator $\hat{L}$ and Γ is the boundary of the domain Ω. If we choose the weighting function ω in (3) as the fundamental solution satisfying the equation

$$\tilde{L}(\omega) = \delta(\vec{r} - \vec{r}_0), \tag{4}$$

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we can get the following boundary integral equation for the source point $\vec{r}$ on the boundary Γ:

$$c(\vec{r})u(\vec{r}) = \int_\Gamma [u\hat{B}(\omega) - \omega\hat{B}(u)]\mathrm{d}\Gamma + \int_\Omega f\omega \mathrm{d}\Omega. \tag{5}$$

1.2. Traditional BEM for non-linear problems and its restrictions

Although the BEM is in principle based on the linear superposition of fundamental solutions,[1–5] many researchers[6–11] have applied it to solve non-linear boundary value problems governed by equations of the non-linear differential operator

$$A(u) = f(\vec{r}). \tag{6}$$

If the above non-linear differential operator A can be divided into two parts $\hat{L}$ and $\hat{N}$, i.e. $A = \hat{L} + \hat{N}$, where $\hat{L}$ is a linear operator and $\hat{N}$ is a non-linear one, then equation (6) can be rewritten as

$$\hat{L}(u) = f(\vec{r}) - \hat{N}(u). \tag{7}$$

Similarly, we can obtain from (5) an integral operator equation

$$c(\vec{r})u(\vec{r}) = \int_\Gamma [u\hat{B}(\hat{\omega}) - \hat{\omega}\hat{B}(u)]\mathrm{d}\Gamma + \int_\Omega [f - \hat{N}(u)]\hat{\omega}\mathrm{d}\Omega, \tag{8}$$

where $\hat{\omega}$ is the fundamental solution of the adjoint operator of the linear differential operator $\hat{L}$. Note that the domain integral of the above equation contains the unknown function $u(\vec{r})$, so iterations are necessary.

The basic idea of the above traditional BEM for non-linear problems is to move all non-linear terms to the right side of the equation and then find the fundamental solution $\hat{\omega}$ of the linear operator $\hat{L}$ remaining on the left side, although it is not always easy to find the fundamental solution $\hat{\omega}$ if the linear operator $\hat{L}$ is unfamiliar. This means that the fundamental solution and the corresponding linear operator are very important and absolutely necessary for the traditional BEM. However, there obviously exists a possibility that there is *nothing* left after moving all non-linear terms to the right side of the equation! In this special case the traditional BEM mentioned above does not work at all.

We can give a practical example to explain this point. Consider the general 2D steady state heat transfer equation

$$\frac{\partial}{\partial x}\left(k_1(T)\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_2(T)\frac{\partial T}{\partial y}\right) = 0. \tag{9}$$

If the thermal conductivity coefficients $k_1(T)$ and $k_2(T)$ are not only dependent on temperature T but also differ in directions x and y, i.e.

$$k_1(T) = \mu_1 \mathrm{e}^{\varkappa_1 T}, \qquad k_1(T) = \mu_2 \mathrm{e}^{\varkappa_2 T}, \tag{10}$$

then the above equation does not contain any linear terms on the left side, so the traditional BEM is powerless and useless.

Thus the traditional BEM described above for non-linear problems has the following limitations and restrictions.

1. Many non-linear differential operators A do *not* contain any linear operators at all, so $A = \hat{L} + \hat{N}$ does *not* hold. In this case the traditional BEM for non-linear problems is certainly of no use.
2. Even if the non-linear differential operator A contains a linear operator $\hat{L}$, the operator $\hat{L}$ may be too simple for all boundary conditions to be satisfied or it may be so complex that the corresponding fundamental solution $\hat{\omega}$ is unknown or unfamiliar. The traditional BEM is useless in the former case and is difficult to apply in the latter case.

Thus from both theoretical and practical viewpoints it seems necessary to develop a new kind of BEM for quite general non-lineaer problems

(a) which can be applied to solve equations goverened by a quite general non-linear differential operator A that may *not* contain any linear operators $\hat{L}$ at all
(b) which can give us great freedom to select a proper linear operator L that is familiar to us
(c) which contains the traditional BEM, i.e. the traditional BEM for non-linear problems mentioned above should be only a special case of the proposed BEM.

As a well-known non-linear problem, the Navier–Stokes equations are usually very difficult to solve. A boundary element method of solving the steady state Navier–Stokes equations in streamfunction–vorticity formulation was presented in Reference 9, based on a set of fundamental solutions providing complete coupling between the streamfunction and vorticity equations so that iteration is unnecessary in the case $Re = 0$. In Reference 9 the non-linear terms are considered in the traditional way as inhomogeneities and treated by simple direct iteration, but this numerical scheme is unstable for 2D viscous flow in a square cavity for Reynolds numbers greater than 300.

The author has been trying to develop a new kind of non-linear technique, the homotopy analysis method,[6,12,13] by means of the basic ideas of the homotopy technique in topology.[14] The advantage of the homotopy analysis method is that it does not depend on small parameters, so some restrictions and limitations of widely applied perturbation techniques can be overcome. As one example of the applications of the homotopy analysis method, the author gave a kind of high-order streamfunction–vorticity BEM formulation for the 2D steady state Navier–Stokes equation.[6] The corresponding first-order formulae are the same as those given in Reference 9 and are also unstable for viscous flow in a square cavity for Reynolds numbers greater than 300 (which implies that the traditional BEM is only a special case of the proposed BEM), but the higher-order formulae are still stable for cavity flow with $Re = 2000$. The current research[7] of the author shows that these high-order BEM formulae are still stable for viscous cavity flow even in the case $Re = 10^4$, which corresponds to a very strong non-linearity.

This paper is a continuation of the author's work described above. The basic ideas described in Reference 6 are greatly generalized and a new, quite general BEM which satisfies the three demands (a)–(c) listed above is proposed. Finally, three simple examples are used to show the effectiveness of this *general boundary element method.*

2. BASIC IDEAS OF THE PROPOSED BEM

2.1. *Treatment of governing equations*

Let A be a quite general non-linear differential operator, no matter whether or not it can be divided into a sum of a linear operator and a non-linear one. We consider again equation (6), but remember that A is now a quite general differential operator.

Selecting a proper, familiar linear operator L, we can construct a homotopy $v(\vec{r}, p) : \Omega \times [0, 1] \rightarrow \mathbb{R}$ which satisfies

$$L(v) = (1 - p)L(u_0) + p[L(v) - A(v) + f(\vec{r})], \quad p \in [0, 1], \quad \vec{r} \in \Omega, \tag{11}$$

where $u_0(\vec{r})$ is an initial solution which can be selected with great freedom, $p \in [0, 1]$ is an imbedding parameter and $v(\vec{r}, p)$ is now a function of both $\vec{r} \in \Omega$ and $p \in [0, 1]$. We call equation (11) the *zero-order deformation equation.*

Obviously, from equation (11) the two expressions

$$v(\vec{r}, 0) = u_0(\vec{r}), \tag{12}$$

$$v(\vec{r}, 1) = u(\vec{r}) \tag{13}$$

hold, where $u(\vec{r})$ is the solution of equation (6). This means that $u_0(\vec{r})$ and $u(\vec{r})$ are homotopic.

Assume that the 'continuous deformation' $v(\vec{r}, p)$ is smooth enough about p so that

$$v^{[m]}(\vec{r}, p) = \frac{\partial^m v(\vec{r}, p)}{\partial p^m}, \quad m = 1, 2, 3, \cdots, \tag{14}$$

called *mth-order deformation derivatives*, exist. Then, according to Taylor's formula, we have from (12) and (14) that

$$v(\vec{r}, p) = v(\vec{r}, 0) + \sum_{m=1}^{\infty} \left.\frac{\partial^m v(\vec{r}, p)}{\partial p^m}\right|_{p=0} \left(\frac{p^m}{m!}\right) = u_0(\vec{r}) + \sum_{m=1}^{\infty} \left(\frac{p^m}{m!}\right) v_0^{[m]}(\vec{r}), \tag{15}$$

where $v_0^{[m]}(\vec{r})$ is the value of the mth-order deformation derivatives $v^{[m]}(\vec{r}, p)$ at $p = 0$, which can be obtained in the way described later.

The value of the convergence radius ρ of the Taylor series (15) is generally finite. When $\rho \leqslant 1$, we obtain

$$v(r, \lambda) = u_0(\vec{r}) + \sum_{m=1}^{\infty} \left(\frac{v_0^{[m]}(\vec{r})}{m!}\right) \lambda^m, \tag{16}$$

where $0 < \lambda < \rho \leqslant 1$. Note that $v(r, \lambda)$ given by the above expression is mostly a better approximation than the initial solution $u_0(\vec{r})$. In fact, expression (16) gives a family of high-order iterative formulae

$$u_{k+1}(r) = u_k(\vec{r}) + \sum_{m=1}^{M} \left(\frac{v_0^{[m]}(\vec{r})}{m!}\right) \lambda^m, \tag{17}$$

where $v_0^{[m]}(\vec{r})$ $(m = 1, 2, 3, \ldots)$ are dependent on $u_k(\vec{r})$, which can be determined in the way described later, and M denotes the order of the iterative formula.

Differentiating the zero-order deformation equation (11) with respect to the imbedding parameter p, we obtain the *first-order deformation equation*

$$L(v^{[1]}) = L(v - u_0) - A(v) + f(\vec{r}) + p\left(L(v^{[1]}) - \frac{\partial[A(v)]}{\partial v} v^{[1]}\right). \tag{18}$$

Similarly, we can obtain the *second-order deformation equation* by differentiating the first-order deformation equation (18) with respect to p:

$$L(v^{[2]}) = 2\left(L(v^{[1]}) - \frac{\partial[A(v)]}{\partial v} v^{[1]}\right) + p\left(L(v^{[2]}) - \frac{\partial[A(v)]}{\partial v} v^{[2]} - \frac{\partial^2[A(v)]}{\partial v^2} (v^{[1]})^2\right). \tag{19}$$

At $p = 0$ we have respectively the first- and second-order deformation equations

$$L(v_0^{[1]}) = f(\vec{r}) - A(u_0), \tag{20}$$

$$L(v_0^{[2]}) = 2\left.\left(L(v_0^{[1]}) - \frac{\partial[A(v)]}{\partial v} v_0^{[1]}\right)\right|_{p=0}. \tag{21}$$

Generally the *mth-order deformation equation* at $p = 0$ is in the form

$$L(v_0^{[m]}) = f_m(\vec{r}), \quad m = 1, 2, 3, \ldots, \tag{22}$$

where

$$f_1(\vec{r}) = f(\vec{r}) - A(u_0), \tag{23}$$

$$f_m(\vec{r}) = m\left(L(v_0^{[m-1]}) - \left.\frac{\mathrm{d}^{m-1}[A(v)]}{\mathrm{d}p^{m-1}}\right|_{p=0} \right), \quad m > 1. \tag{24}$$

It is very interesting that $f_1(\vec{r})$ is the negative residual of the original equation (6) for *any* freely selected linear operator L.

According to expression (5), the linear boundary value problem (22) can be easily solved by the boundary integral equation

$$c(\vec{r})v_0^{[m]}(\vec{r}) = \int_\Gamma [v_0^{[m]}B(\omega) - \omega B(v_0^{[m]})]\mathrm{d}\Gamma + \int_\Omega f_m \omega \mathrm{d}\Omega, \tag{25}$$

where B is the corresponding boundary operator for the freely selected linear operator L.

After selecting the initial solution $u_0(\vec{r})$, the function $f_m(\vec{r})$ described by (23) or (24) for the domain integral is known for each m. Note that we now have very great freedom to select the corresponding linear operator L, i.e. we can now select a familiar, proper linear operator L even if the considered non-linear operator A does *not* contain any linear operators at all! In particular, when $A = \hat{L} + \hat{N}$ and we select $L = \hat{L}$ as the linear operator, formula (25) in the case of $m = 1$, combined with formula (17), gives the same expression as equation (8). Therefore the three demands (a)–(c) listed in Section 1.2 are well satisfied.

2.2. *Treatment of boundary conditions*

The boundary condition can be treated in a similar way. For simplicity, consider the following boundary condition on Γ:

$$H(u, u') = 0, \tag{26}$$

where u' denotes the first-order derivatives w.r.t. co-ordinates. We construct the same homotopy $v(r, p)$ which satisfies not only equation (11) but also the boundary condition

$$H(v, v') = (1 - p)H(u_0, u_0'), \quad p \in [0, 1], \quad r \in \Gamma. \tag{27}$$

Differentiating the above equation m times w.r.t. the embedding parameter p and then setting $p = 0$, we obtain the corresponding *linear* boundary conditions for $v_0^{[m]}(\vec{r})$ as

$$\left.\frac{\partial H}{\partial u}\right|_{p=0} v_0^{[m]} + \left.\frac{\partial H}{\partial u'}\right|_{p=0} (v_0^{[m]})' = h_m(r), \quad m = 1, 2, 3, \ldots, \tag{28}$$

where

$$h_1(\vec{r}) = -H(u_0, u_0'), \tag{29}$$

$$h_2(\vec{r}) = -\left.\left(\frac{\partial^2 H}{\partial u^2}\left(v_0^{[1]}\right)^2 + 2\frac{\partial^2 H}{\partial u u'} v_0^{[1]}\left(v_0^{[1]}\right)' + \frac{\partial^2 H}{\partial (u')^2}\left[\left(v_0^{[1]}\right)'\right]^2 \right)\right|_{p=0}, \tag{30}$$

Certainly the *linear* governing equation (22) with the *linear* boundary condition (28) can be easily solved by the BEM. Note that we do not give any restrictions on the boundary conditions $H(u, u')$.

This means that the boundary condition $H(u, u')$ also may not contain any linear terms. Therefore the proposed BEM can even be used to solve those non-linear problems whose governing equations and boundary conditions do not contain any linear terms at all!

3. NUMERICAL EXAMPLES

3.1. Example 1

As the first example, let us consider the following non-linear boundary value problem described in Reference 10:

$$u_{xx} = \beta^2 u^2, \quad x \in [0, 1], \quad \beta > 0, \tag{31}$$

with the boundary conditions

$$u(0) = 1 \quad \text{and} \quad u(1) = 0{\cdot}25. \tag{32}$$

Selecting a familiar linear operator $L(u) = u_{xx} - \beta^2 u$, we can construct a homotopy $v(x; p) : [0, 1] \times [0, 1] \to R$ as

$$v_{xx} - \beta^2 v = p\beta^2(v^2 - v) + (1 - p)[(u_0)_{xx} - \beta^2 u_0], \quad x \in [0, 1], \quad p \in [0, 1], \tag{33}$$

with the corresponding boundary conditions

$$v(0, p) = 1 \quad \text{and} \quad v(1, p) = 0{\cdot}25, \quad p \in [0, 1], \tag{34}$$

where $u_0(x)$ is an initial solution which satisfies the boundary conditions (32). For simplicity we select here

$$u_0(x) = (1 + x)^{-2}, \tag{35}$$

which is the solution of (31), (32) in the case $\beta = \sqrt{6}$, as the initial solution for each β.

Owing to (17), the corresponding family of high-order iterative formulae is

$$u_{k+1}(x) = u_k(x) + \sum_{m=1}^{M} \left(\frac{v_0^{[m]}(x)}{m!} \right) \lambda^m, \quad x \in [0, 1], \quad k \geqslant 0, \tag{36}$$

where $v_0^{[m]}(x)$ satisfies the linear differential equation

$$[v_0^{[m]}(x)]_{xx} - \beta^2 v_0^{[m]}(x) = f_m(x), \quad x \in [0, 1], \quad m \geqslant 1, \tag{37}$$

with the boundary conditions

$$v_0^{[m]}(0) = v_0^{[m]}(1) = 0, \quad m \geqslant 1. \tag{38}$$

Here, according to (23) and (24), we have

$$f_1(x) = \beta^2 u_0^2 - (u_0)_{xx}, \tag{39}$$

$$f_2(x) = 2\beta^2(2u_0 - 1)v_0^{[1]}, \tag{40}$$

$$f_3(x) = 3\beta^2[2(v_0^{[1]})^2 + (2u_0 - 1)v_0^{[2]}], \tag{41}$$

The fundamental solution for the one-dimensional modified Helmholtz operator $L(u) = u_{xx} - \beta^2 u$ has been described in Reference 10 as

$$\omega(x, y) = -\frac{e^{-\beta|x-y|}}{2\beta}. \tag{42}$$

Owing to (25), we obtain the boundary integral equation

$$v_0^{[m]}(y) = [v_0^{[m]}\omega_x - (v_0^{[m]})_x\omega]\Big|_{x=0}^{x=1} + \int_0^1 f_m(x)\omega(x, y)\mathrm{d}x. \tag{43}$$

Substituting (38) and (42) into the above expression, we have

$$v_0^{[m]}(y) = -\frac{1}{2\beta}\left(a_m e^{-\beta y} - b_m e^{-\beta(1-y)} + \int_0^1 f_m(x)e^{-\beta|x-y|}\mathrm{d}x\right), \tag{44}$$

where the unknowns a_m and b_m are the values of $(v_0^{[m]})_x$ at $x = 0$ and $x = 1$ respectively, which can be easily determined by the boundary condition (38) as follows:

$$\begin{bmatrix} e^{-\beta} & -1 \\ 1 & -e^{-\beta} \end{bmatrix} \begin{bmatrix} b_m \\ a_m \end{bmatrix} = \begin{bmatrix} \int_0^1 f_m(x)e^{-\beta x}\,\mathrm{d}x \\ \int_0^1 f_m(x)e^{-\beta(1-x)}\mathrm{d}x \end{bmatrix}. \tag{45}$$

For the sake of the numerical domain integral we divide the domain [0, 1] into N equal subdomains, where $N = 1000$. The values of λ used for the different β^2 and the corresponding iteration times for the first-, second- and third-order formulae are given in Table I. The convergent results for $\beta^2 = 10, 10^2, 10^3, 10^4, 10^5, 10^6, 10^7$ and even 10^8 are shown in Figure 1. All these results agree very well with those given in Reference 10. It is interesting that the higher the order of the iterative formula, the shorter the iteration time. This implies that a higher-order iterative formulation will give a faster convergent iterative procedure. Note also that now we just need to solve a set of two linear algebraic equations in the two unknowns a_m and b_m. The boundary integral method for non-linear problems described in Reference 10 must solve a set of linear algebraic equations containing $2N$ unknowns, which certainly requires much more CPU time for our case of $N = 1000$. Note also that we successfully obtain the convergent results for each β^2 ranging from 10 to 10^8 by using the same initial solution $(1 + x)^{-2}$, which is a very bad guess for large β^2.

Table I. Numerical parameters and iteration times for Example 1

β^2	λ	Iteration times		
		First-order	Second-order	Third-order
10	1·000	3	2	2
10^2	1·000	14	8	6
10^3	1·000	46	28	20
10^4	0·900	85	62	51
10^5	0·750	104	81	71
10^6	0·500	127	105	98
10^7	0·250	144	124	118
10^8	0·125	119	96	88

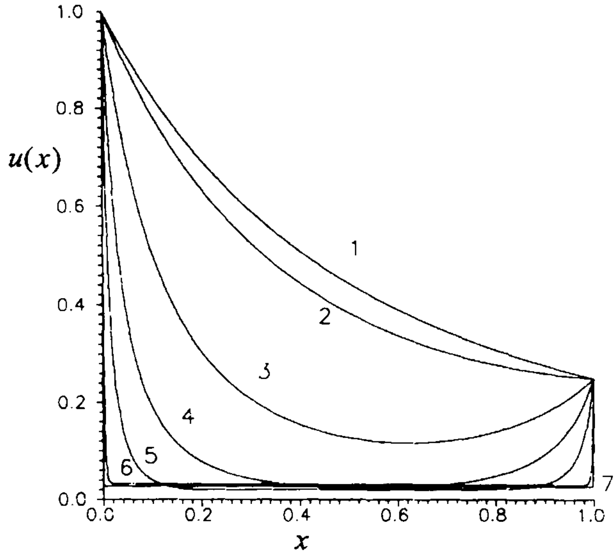

Figure 1. Initial solution and convergent results of Example 1: 1, $\beta^2 = 6$; 2, $\beta^2 = 10$; 3, $\beta^2 = 10^2$; 4, $\beta^2 = 10^3$; 5, $\beta^2 = 10^4$; 6, $\beta^2 = 10^6$; 7, $\beta^2 = 10^8$

3.2. *Example 2*

As the second example, let us consider the non-linear differential equation

$$u_{xx}^2 - \beta^2 \cos(u^2) = 0, \quad x \in [0, 1], \tag{46}$$

which has the same boundary conditions as (32).

Note that equation (46) does *not* contain any linear operators at all. However, we can still choose the familiar linear operator $L(u) = u_{xx} - \alpha^2 u$ which has been used in Example 1 and then construct a homotopy.

$$v_{xx} - \alpha^2 v = (1-p)[(u_0)_{xx} - \alpha^2 u_0] + p[\beta^2 \cos(v^2) - v_{xx}^2 + v_{xx} - \alpha^2 v], \quad x \in [0, 1], \quad p \in [0, 1], \tag{47}$$

with the boundary conditions

$$v(0, p) = 1 \quad \text{and} \quad v(1, p) = 0{\cdot}25. \tag{48}$$

Similarly, we can obtain the same high-order iterative formula (36) and the corresponding $v_0^{[m]}(x)$ satisfies the linear equations

$$[v_0^{[m]}(x)]_{xx} - \alpha^2 v_0^{[m]}(x) = f_m(x) \tag{49}$$

and has the same boundary conditions as (38), where, owing to (23) and (24), we have

$$f_1(x) = \beta^2 \cos(u_0^2) - [(u_0)_{xx}]^2, \tag{50}$$

$$f_2(x) = 2\{[1 - 2(u_0)_{xx}](v_0^{[1]})_{xx} - [\alpha^2 + 2\beta^2 u_0 \sin(u_0^2)]v_0^{[1]}\}, \tag{51}$$

$$f_3(x) = 3\{[1 - 2(u_0)_{xx}](v_0^{[2]})_{xx} - [\alpha^2 + 2\beta^2 u_0 \sin(u_0^2)]v_0^{[2]} - 2[(v_0^{[1]})_{xx}]^2 - 2\beta^2[\sin(u_0^2) + 2u_0^2 \cos(u_0^2)](v_0^{[1]})^2\}, \tag{52}$$

We choose $|\alpha| = \sqrt{\beta}$ and use the following two iterative procedures:

(A) first using $u_0(x) = 1 - 0{\cdot}75x$ as the initial solution to obtain the convergent result for $\beta^2 = 10$, then using this new result to obtain the solution for a larger value of β^2 and so on
(B) using $u_0(x) = 1 - 0{\cdot}75x$ as the initial solution for each β^2.

For the sake of the numerical domain integral we also divide the domain [0, 1] into N equal subdomains. The iteration times and corresponding numerical parameters λ for the first-order iterative formula in the case of $N = 1000$ for different values of β^2 ranging from 10 to 10^8 are given in Table II. The corresponding convergent results are shown in Figure 2. For the same value of β^2 we use the same value of λ for both iterative procedures (A) and (B) described above. Each iterative procedure successfully gives the convergent results, although procedure (A) needs a little less iteration time than procedure (B), because a better initial solution is used in procedure (A) in each case where $\beta^2 > 10$. It seems that the value of λ is most important for the convergence of the iterative procedure; mostly, the iterative procedure converges if λ is selected small enough. It is interesting that even the first-order iterative formula can give convergent results for equation (46) governed by a non-linear operator that does *not* contain any linear operators at all! This seems to be a good example to show the effectiveness of the proposed new BEM for quite general non-linear differential operators.

3.3. *Example 3*

As the third example, let us consider the equation

$$2u_{xx}\cos(u_{xx}) + \gamma(u^2 + u_x^2) = \gamma - \sin(x)\cos[\sin(x)], \quad x \in [0, 2\pi], \quad \gamma > 0, \tag{53}$$

with the boundary conditions

$$\sin(u) + \cos(uu_x) = \gamma/(1+\gamma), \quad x = 0, \tag{54}$$

$$\sin(u) + \cos(uu_x) = \gamma/(1+\gamma), \quad x = 2\pi, \tag{55}$$

Note that the governing equation (53) and the boundary conditions (54) and (55) do not contain any linear terms at all. Defining

$$\tilde{A}(u) = 2u_{xx}\cos(u_{xx}) + \gamma(u^2 + u_x^2), \tag{56}$$

$$\tilde{f}(x) = \gamma - \sin(x)\cos[sin(x)], \tag{57}$$

$$\tilde{H}(u, u_x) = \sin(u) + \cos(uu_x) - \gamma/(1+\gamma), \tag{58}$$

Table II. Numerical parameters and iteration times for Example 2

β^2	λ	Iteration times	
		Procedure (A)	Procedure (B)
10	0·25000	14	14
10^2	0·06250	14	15
10^3	0·02500	16	18
10^4	0·00500	36	39
10^5	0·00100	46	51
10^6	0·00050	30	33
10^7	0·00010	37	44
10^8	0·00005	26	29

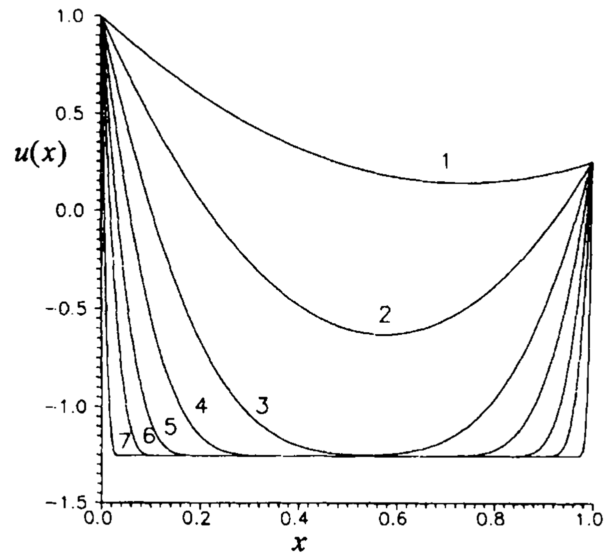

Figure 2. Initial solution and convergent results of Example 2:
1, $\beta^2 = 10$; 2, $\beta^2 = 10^2$; 3, $\beta^2 = 10^3$; 4, $\beta^2 = 10^4$; 5, $\beta^2 = 10^5$; 6, $\beta^2 = 10^6$; 7, $\beta^2 = 10^8$

we can construct a homotopy $v(x,p) : [0, 2\pi] \times [0, 1] \to \mathbb{R}$ as

$$L(v) = p[L(v) - \tilde{A}(v) + \tilde{f}(x)] + (1-p)L(u_0), \quad x \in [0, 2\pi], \quad p \in [0, 1], \tag{59}$$

with the boundary conditions

$$\tilde{H}(v, v_x) = (1-p)\tilde{H}(u_0, u_0'), \quad p \in [0, 1], \quad x = 0 \vee x = 2\pi, \tag{60}$$

where $L(u) = u_{xx} - \beta^2 u$ and the initial solution $u_0(x)$ can be *freely* selected.

Certainly we can use the same iterative formula as (36). For simplicity, only the first-order iterative formula

$$u_{k+1}(x) = u_k(x) + v_0^{[1]}(x)\lambda, \quad x \in [0, 2\pi], \quad k = 0, 1, 2, 3, \ldots \tag{61}$$

is used, where $v_0^{[1]}(x)$ is determined by the *first-order deformation equation*

$$[v_0^{[1]}(x)]_{xx} - \beta^2 v_0^{[1]}(x) = \tilde{f}(x) - \tilde{A}[u_k(x)], \quad x \in [0, 2\pi], \quad k = 0, 1, 2, 3, \ldots \tag{62}$$

with the boundary conditions

$$\{\alpha_k v_0^{[1]}(x) + \beta_k [v_0^{[1]}(x)]'\}|_{x=0} = -\tilde{H}[u_k(x), u_k'(x)]|_{x=0} \quad \text{at} \quad x = 0, \tag{63}$$

$$\{\alpha_k v_0^{[1]}(x) + \beta_k [v_0^{[1]}(x)]'\}|_{x=2\pi} = -\tilde{H}[u_k(x), u_k'(x)]|_{x=2\pi} \quad \text{at} \quad x = 2\pi, \tag{64}$$

where

$$\alpha_k = \cos[u_k(x)] - u_k'(x)\sin[u_k(x)u_k'(x)], \quad k = 0, 1, 2, 3, \ldots, \tag{65}$$

$$\beta_k = -u_k(x)\sin[u_k(x)u_k'(x)], \quad k = 0, 1, 2, 3, \ldots. \tag{66}$$

Note that the governing equation (62) and the two boundary conditions (63) and (64) are linear and can be easily solved by the BEM in a similar way as used in Examples 1 and 2. The only difference is that now there exist four unknowns $u(0)$, $u_x(0)$, $u(2\pi)$ and $u_x(2\pi)$, so a set of four linear algebraic equations has to be solved.

We select $u_0(x) = 0$ and $\beta^2 = 0{\cdot}1$ for each γ in Example 3. For the numerical domain integral we also divide $[0, 2\pi]$ into N equal subdomains, where $N = 1000$. The numerical parameters λ and corresponding iteration times for different values of γ are given in Table III. The numerical results are shown in Figure 3. It should be emphasized that the solution in the case of $\gamma = 100$ is very close to the function

$$g(x) = \begin{cases} -\sin(x), & x \in [0, \pi/2], \\ -1, & x \in [\pi/2, 3\pi/2], \\ -\sin(2\pi - x), & x \in [3\pi/2/2\pi], \end{cases} \tag{67}$$

which is one of the solutions of equations (53)–(55) in the case where γ tends to infinity:

$$u^2 + u_x^2 = 1, \quad x \in [0, 2\pi], \tag{68}$$

$$\sin(u) + \cos(uu_x) = 1, \quad x = 0, \tag{69}$$

$$\sin(u) + \cos(uu_x) = 1, \quad x = 2\pi. \tag{70}$$

Note that equation (68) is only a first-order non-linear differential equation, which is in principle very different from the original non-linear second-order differential equation (53). Thus this example seems to be a very good one to indicate the effectiveness of the proposed BEM in solving quite strongly non-linear problems.

Finally, it should be pointed out that there are many different ways to construct a homotopy w.r.t. boundary conditions. For example, it is very easy to find an initial solution satisying the two boundary conditions $u(0) = 1$ and $u(1) = 0{\cdot}25$ for Examples 1 and 2, so we use homotopies (34) and (48) respectively, which imply that $u_0(0) = 1$ and $u_0(1) = 0{\cdot}25$ must hold. However, it is obviously very difficult to find an initial solution satisfying the boundary conditions (54) and (55) of Example 3, so a much more general form of homotopy such as expression (60) is applied, which allows us to freely select an initial solution $u_0(x)$, such as $u_0(x) = 0$ used in Example 3. Therefore we have great freedom to select an initial solution even if the boundary conditions are very complex. This kind of freedom is very important, especially for 2D and 3D problems. This is also an advantage of the proposed BEM.

In each example described above, we use

$$\mathrm{RMS} = \sqrt{\left(\frac{\sum_{i=1}^{N} |\delta u_i|^2}{N}\right)} < \Delta \tag{71}$$

as the convergence criterion, where δu_i denotes the difference of u at x_i in iterations, and in this paper we use $\Delta = 10^{-4}$ for the first example and $\Delta = 10^{-6}$ for the second and third examples. Using

Table III. Numerical parameters and iteration times for Example 3

γ	λ	Iteration times
0·00	0·50	22
0·25	0·25	43
0·50	0·20	52
0·75	0·15	67
1·00	0·10	97
5·00	0·025	295
100	0·001	1406

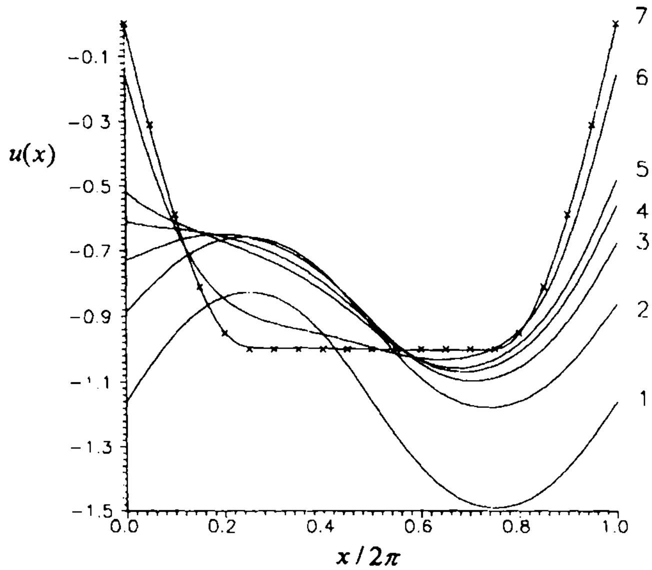

Figure 3. Initial solution and convergent results of Example 3: 1, $\gamma = 0{\cdot}00$; 2, $\gamma = 0{\cdot}25$; 3, $\gamma = 0{\cdot}50$; 4, $\gamma = 0{\cdot}75$; 5, $\gamma = 1{\cdot}00$; 6, $\gamma = 5{\cdot}00$; 7, $\gamma = 100$ (symbols, $g(x)$ defined as (67))

$N = 500$, we also obtain convergent results for each example which agree very well with those for $N = 1000$.

4. DISCUSSION AND CONCLUSIONS

In this paper we apply the basic ideas of the homotopy analysis method[12,13] to propose a kind of general BEM for non-linear problems governed by a quite general non-linear differential operator A, where A need *not* contain any linear operators at all.

As mentioned in Section 1.2, a proper linear operator and its corresponding fundamental solution are very important and absolutely necessary for the traditional BEM. However, for the proposed BEM it is not important at all whether or not there exists such a linear operator, because we now have very great freedom to *select* a proper, familar linear operator even if the considered non-linear problem does not contain any linear terms at all. Therefore the proposed BEM seems to be quite general and can be applied to solve a large number of complex non-linear problems, even including those whose governing equations and boundary conditions do not contain any linear terms at all.

We have also mentioned in Section 1.2 that it is not always easy to find the fundamental solution to an unfamilar linear operator. However, the proposed BEM gives us great freedom to select a familar linear operator even if we do not know the fundamental solutions of the unfamiliar linear operator of the considered problem.

The third advantage of the proposed BEM is that a quite general computer programme can be written for a large number of different sorts of non-linear problems. For example, a quite general computer programme has been written by the author for the three quite different non-linear problems given in this paper. Thus the proposed BEM provides the possibility of developing a kind of quite general software for solving non-linear problems.

The fourth advantage of the proposed BEM is that it contains logically the traditional BEM, because the traditional BEM is only a special case of the proposed BEM. This kind of continuation in logic has been proved again and again to be very important in many mathematical fields.

Note that the domain integral exists when the proposed BEM is applied to solve non-linear problems. However, another kind of powerful BEM technique, the dual reciprocity method,[15] which can take the domain integral to the boundary, has been developing quite quickly. If the dual reciprocity method is really powerful even for very complex problems, than a combination of the proposed BEM and the dual reciprocity method would perhaps give a kind of very efficient numerical technique for solving non-linear problems.

Note also that this paper is only an application of the homotopy anlaysis method[12,13] which has been proposed by the author to overcome some limitations and restrictions of the widely applied perturbation techniques. The successful application of the homotopy analysis method in numerical methods indicates that it seems to be really a powerful tool for solving strongly non-linear problems.

Although three examples are used to show the effectiveness of the proposed BEM, they seem to be too simple. Certainly the proposed BEM must be applied to a large number of complex 2D and 3D practical non-linear problems in order to improve and develop it. On the other hand, deeper mathematical research should be done. For example, it is true that we now have really much greater freedom to select a proper linear operator. However, how can we use this kind of freedom? Certainly all properly selected linear operators which can give a convergent iterative procedure construct a mathematical space **S**. It seems that there should exist the 'best' one in this mathematical space **S** which should give the fastest convergent iterative procedure. However, how can we find the best linear operator?

It is certain that deeper mathematical research and more practical applications are necessary in order to improve and develop the proposed BEM.

REFERENCES

1. C. A. Brebbia, *The Boundary Element for Engineers*, Pentech, London, 1980.
2. C. A. Brebbia and J. J. Connor, *Advances in Boundary Elements 1: Computations and Fundamentals*, Computational Mechanics Publications, Southampton, 1989.
3. C. A. Brebbia, *Boundary Elements X*, Vol. 1, *Mathematical and Computational Aspects*, Computational Mechanics Publications, Southampton, 1988.
4. I. Herrera, *Boundary Methods—An Algebraic Theory*, Pitman, Boston, MA, 1984.
5. S. -S. Shu, *Boundary Value Problems of Linear Partial Differential Equations for Engineers and Scientists*, World Scientific, Hong Kong, 1987.
6. S. J. Liao and J. M. Zhu, 'Higher-order streamfunction–vorticity formulation of 2D steady-state Navier–Stokes equations', *Int. j. numer. methods fluids*, **15**, 595–612 (1992).
7. S. J. Liao, 'A short note on the high-order streamfunction–vorticity formulations of 2D steady-state Navier–Stokes equations', *Int. j. numer. methods fluids*, **22**, 1–9 (1996).
8. S. J. Liao, 'The quite general BEM for strongly non-linear problems', in C. A. Brebbia *et al.* (eds), *Boundary Elements XVIII*, Computational Mechanics Publications, Southampton, 1995, pp. 67–74.
9. H. A. Rodriguez-Prada, F. F. Pironti and A. E. Asez, 'Fundamental solutions of the streamfunction–vorticity formulation of the Navier–Stokes equation', *Int. j. numer. methods fluids*, **10**, 1990.
10. N. Tosaka and K. Kakuda, 'The generalized BEM for non-linear problems', in C. A. Brebbia (ed.), *Boundary Elements X*, Vol. 1, *Mathematical and Computational Aspects*, Computational Mechanics Publications, Southampton 1988, pp. 1–17.
11. N. Tosaka and S. Miyake, 'Integral equation analysis for geometrically non-linear-problems of elastic bodies', in N. Tasaka and Q. Du (eds.), *Theory and Applications of Boundary Element Methods*, Pergamon, Oxford, 1987, pp. 251–260.
12. S. J. Liao, 'A kind of linearly-invariance under homotopy and some simple applications of it in mechanics', *Report 520*, Institute of Shipbuilding, University of Hamburg, 1992.
13. S. J. Liao, 'An approximate solution technique not depending on small parameters: a special example', *Int. j. non-lin. mech.*, **30**, 371–380 (1995).
14. R. Brown, *Topology: A General Account of General Topology, Homotopy Types and the Fundamental Groupoid*, Wiley, New York, 1988.
15. P. W. Partridge, 'New developments in the dual reciprocity method', in C. A. Brebbia, Kim, Osswald and Power (eds), *Boundary Element XVII*, Computational Mechanics Publications, Southampton, 1995, pp. 11–18.