# Collisionless periodic orbits in the free-fall three-body problem

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ARTICLE INFO

Keywords:
Three-body system
Periodic orbits
Celestial mechanics
Chaos

ABSTRACT

Although the free-fall triple system has been numerically studied for more than one century, however, only four collisionless periodic orbits have been found. In this paper, using a supercomputer and a new strategy for chaotic dynamic systems, called "clean numerical simulation" (CNS), we successfully gained 316 collisionless periodic orbits of the free-fall triple system with a few chosen values of mass ratios, including three collisionless periodic orbits which were found before. Especially, 313 collisionless free-fall periodic orbits are entirely new. What's more, we can gain periodic free-fall three-body orbits in a random ratio of mass. Thus, this is a good example to prove that there exist an infinite number of periodic solutions for the triple system. In addition, it is found that, for a given ratio of mass, there exists a generalized Kepler's third law for the periodic three-body system. All of these would enrich our knowledge and deepen our understanding about the famous three-body problem as a whole.

## 1. Introduction

The three-body problem was first studied by Newton (1687), and attracted many famous mathematicians and physicists such as Euler (1767), Lagrange (1772) and so on. Poincaré (1890) found that solutions of three-body problem are rather sensitive to initial conditions. His discovery of the so-called "sensitivity dependance on initial conditions" (SDIC) laid the foundation for chaos theory. It well explains why in three hundred years only three families of periodic solutions of three-body system were discovered. These three families of periodic orbits are the Euler–Lagrange family (Euler, 1767; Lagrange, 1772), the Broucke–Hadjidemetriou–Hénon family of periodic orbits (Broucke, 1975; Hadjidemetriou, 1975; Hadjidemetriou and Christides, 1975; Hénon, 1976; 1977) and the figure-eight orbit (Moore, 1993; Chenciner and Montgomery, 2000). In Šuvakov and Dmitrašinović (2013) found 13 new periodic three-body orbits by means of numerical methods. In recent years, periodic three-body problem have been paid much attention (Iasko and Orlov, 2014; Šuvakov, 2014; Li and Liao, 2017; Li et al., 2018; Šindik et al., 2018). Especially, Li and Liao (2017) presented more than six hundred new periodic solutions of triple system with equal mass, and Li et al. (2018) further reported more than 1000 new periodic solutions of triple system in case of unequal mass, mainly because they can accurately simulate orbits of chaotic triple systems by using a new numerical strategy, i.e. clean numerical simulation (CNS)

(Liao, 2009; 2013; 2014; Liao and Wang, 2014; Liao and Li, 2015; Lin et al., 2017). The detailed results of these orbits are given on the website http://numericaltank.sjtu.edu.cn/three-body/three-body.htm.

The initial conditions of the newly-found periodic orbits (Šuvakov and Dmitrašinović, 2013; Iasko and Orlov, 2014; Šuvakov, 2014; Li and Liao, 2017; Li et al., 2018) are isosceles collinear configurations. However, these is another configuration, called the free-fall three-body problem with the zero initial velocities and arbitrary ratios of masses, which have been numerically investigated for more than one century. The free-fall triple system is also called Pythagorean problem, which was first numerically studied by Burrau (1913). Its first periodic orbit with a binary collision was found by Szebehely and Peters (1967). Its first collisionless periodic orbit was found by Standish (1970). In 1990s Tanikawa and his colleagues (Tanikawa et al., 1995; Tanikawa, 2000) reported some collision solutions of free-fall triple system. Moeckel et al. (2012) proved the existence of periodic brake three-body orbits (i.e., periodic free-fall three-body orbits) with collision in the isosceles configuration. Tanikawa and Mikkola (2015) gained the periods and initial conditions of some periodic free-fall three-body orbits with collision. Yasko and Orlov (2015) found three collisionless periodic orbits while searching periodic free-fall three-body orbits. Orlov et al. (2016) investigated periodic solutions of the free-fall three-body system with equal mass. In summary, so far, only four collisionless periodic solutions for free-fall three-body system have been found.

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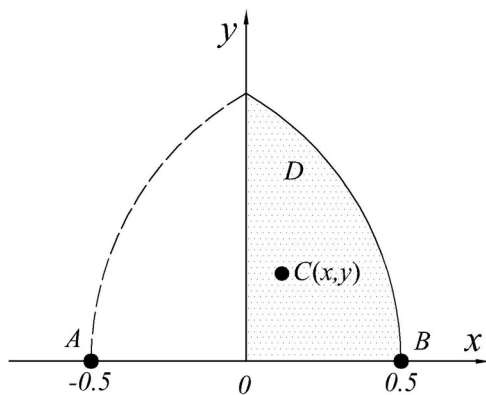  E-mail address: sjliao@sjtu.edu.cn (S. Liao).

**Fig. 1.** The initial position of three bodies in the configuration space. The initial positions of body-1 and body-2 are located at points A ($-0.5$, 0) and B (0.5,0), respectively. And the initial position of body-3 is replaced at the point C ($x$, $y$) in the region $D$.

In this paper, we gain 316 collisionless periodic orbits for free-fall three-body system with different mass ratios by means of the clean numerical simulation (CNS) (Liao, 2009; 2013; 2014; Liao and Wang, 2014; Liao and Li, 2015; Lin et al., 2017), including the periodic orbit found by Standish (1970) and two periodic orbits found by Yasko and Orlov (2015). Especially, 313 collisionless free-fall periodic orbits are entirely new.

## 2. Numerical approach of searching for periodic orbits

In this paper, we consider the planar Newtonian three-body problem. The differential equations of the three-body problem are

$$\ddot{r}_i = \sum_{j=1, j \neq i}^{3} \frac{Gm_j(r_j - r_i)}{|r_i - r_j|^3}, \, i = 1, 2, 3 \tag{1}$$

where $r_i$ and $m_i$ are the position and mass of the $i$th body ($i = 1, 2, 3$), $G$ is the Newtonian constant of gravitation. Without loss of generality, we assume $G = 1$ here.

Let us consider planar three-body problem with zero initial velocities and arbitrary ratios of masses. As shown in Fig. 1, the initial locations of body-1 and body-2 are at points A ($-0.5$, 0) and B (0.5,0), respectively, and the initial position of body-3 is at the point C ($x$, $y$) in the region $D$ which is surrounded by the $x$ and $y$ axes and a circular segment of unit radius with the point A (-0.5,0) as the centre. Agekyan and Anosova demonstrated that all possible initial condition configurations for planar free-fall triple system are included in the region $D$. Their excellent work greatly simplifies the considered problem. Without loss of generality, let the Newtonian gravitation constant be equal to one.

Because the initial positions of body-1 and body-2 are fixed at point ($-0.5$, 0) and (0.5,0), respectively, and the initial position ($x$, $y$) of body-3 determines the orbit of free-fall triple system. Here we denote $y$ $(t) = (r_1(t), r_2(t), r_3(t), \dot{r}_1(t), \dot{r}_2(t), \dot{r}_3(t))$, where $r_i$ and $\dot{r}_i(t)$ are the positions and velocities of the body-$i$ ($i=1,2,3$), respectively. The periodic orbits will return to initial condition $y(T) = y(0)$ at the time $t = T$, where $T$ is the period. We locate the periodic free-fall three-body orbits by finding zeros of the function

$$\|y(t) - y(0)\| = \sqrt{\sum_{i=1}^{3} [r_i(t) - r_i(0)]^2 + \sum_{i=1}^{3} [\dot{r}_i(t) - \dot{r}_i(0)]^2} \tag{2}$$

Here we used the same strategy as that in our previous discovery of more than 2000 new periodic orbits of three-body system (Li and Liao,

2017; Li et al., 2018). First, we scan the initial positions ($x$, $y$) of body-3 in region $D$ with steps of $\Delta x = \Delta y = 0.0001$. We employ the ODE solver dop853 (Hairer et al., 1993) to numerically solve the differential equations of the free-fall triple system. The approximate initial conditions and periods are selected when the function (2) is below 0.05. Second, we use the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002; Abad et al., 2011) to modify these approximate initial conditions ($x$, $y$) and the periods $T$. However, since some periodic three-body solutions might be lost by conventional numerical method with double precision, we now solve the differential equations of the three-body problem by using "clean numerical simulation" (CNS) (Liao, 2009; 2013; 2014; Liao and Wang, 2014; Liao and Li, 2015; Lin et al., 2017). The CNS is based on the *arbitrary* order of Taylor series method (Barton et al., 1971; Corliss and Chang, 1982; Barrio et al., 2005) in *arbitrary* precision (Oyanarte, 1990; Viswanath, 2004), and a convergence verification by means of another simulation with even smaller numerical noises. A periodic free-fall three-body orbit is obtained when the function (2) is less than $10^{-6}$. The detail of numerical strategy is shown in ref. Li and Liao (2017) and Li et al. (2018).

## 3. Results

We consider the periodic orbits are collisionless if the minimum distance between the bodies is greater than $10^{-6}$ in the whole period. We focused our numerical search on collisionless periodic orbits for free-fall triple system with period less than 200 and minimum distance between the bodies greater than $10^{-6}$. Similarly as we did before (Li and Liao, 2017; Li et al., 2018), we use a topological method (Montgomery, 1998; Šuvakov and Dmitrašinović, 2014) to classify the periodic orbits.

There are two three-body relative coordinate vectors $\rho = \frac{1}{\sqrt{2}}(r_1 - r_2)$ and $\lambda = \frac{1}{\sqrt{6}}(r_1 + r_2 - 2r_3)$. With these two vectors, the positions of three-body system can be mapped to a point on a unit space sphere, and the Cartesian components of the point are

$$n = (n_x, \, n_y, \, n_z) = \left( \frac{2\rho \cdot \lambda}{R^2}, \, \frac{\lambda^2 - \rho^2}{R^2}, \, \frac{2(\rho \times \lambda) \cdot e_z}{R^2} \right), \tag{3}$$

where $R = \sqrt{\rho^2 + \lambda^2}$. There are three singular points (punctures) in the space sphere, corresponding to the three binary collision in the real space. A collisionless periodic orbit gives a closed curve around three punctures on the shape sphere. With one puncture as the north pole, the points of the sphere can be mapped to the plane with the other two punctures by a stereographic projection. Then a close curve on the shape sphere can be mapped to a plane with two punctures. The free group words $a$ and $b$ represent a clockwise around the right-hand-side puncture and counter-clockwise turn around the left-hand-side puncture, respectively. The letters $A$ and $B$ denote the opposite direction of $a$ and $b$, respectively. The so-called "free group elements" of periodic orbits $w$ can be written as $w = w_1 w_2 w_3...$, where $w_i$ is any one of $a$, $b$, $A$ and $B$. Two periodic orbits on the shape sphere are shown in Fig. 2. The free group elements of $F_1(1, 0.8, 0.8)$ and $F_3(1, 0.8, 0.2)$ is $BaBbAb$ and $BabBAb$, respectively.

We found 30 collisionless periodic orbits in the free-fall triple system with equal mass ($m_1 = m_2 = m_3 = 1$). Table S I in Supplementary contains the periods and initial conditions of these periodic orbits. Their free group elements are given in Table S VIII in Supplementary.

In addition, we also gained collisionless periodic free-fall solutions of triple system in some cases of unequal masses, as shown in Table 1. In summary, we totally found 316 collisionless periodic orbits in the free-fall triple system with different mass ratios. The periods and initial conditions of the periodic solutions are displayed in Tables S I-VII in Supplementary, and their free group elements are given in Tables S VIII-
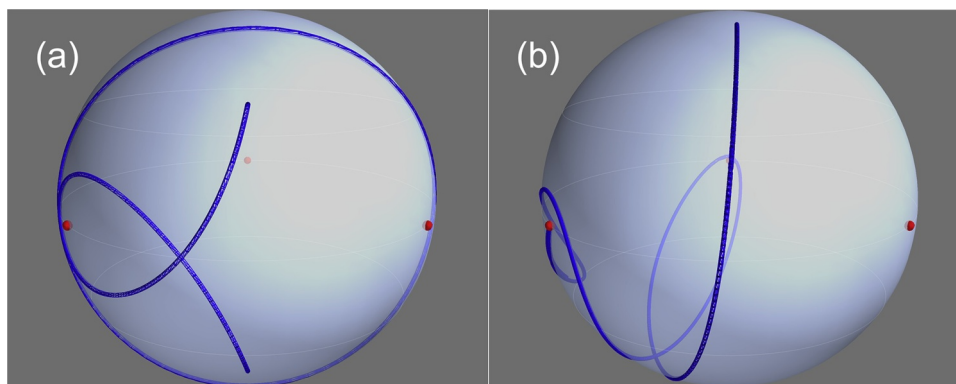
**Fig. 2.** The shape sphere of two periodic orbits: (a) $F_1(1, 0.8, 0.8)$ and (b) $F_3(1, 0.8, 0.2)$.

**Table 1**
The number of periodic orbits with different masses.

| $m_1$ | $m_2$ | $m_3$ | Number of periodic orbits |
|---|---|---|---|
| 1 | 1 | 1 | 30 |
| 1 | 0.8 | 0.8 | 29 |
| 1 | 0.8 | 0.6 | 69 |
| 1 | 0.8 | 0.4 | 44 |
| 1 | 0.8 | 0.2 | 32 |
| 0.6 | 0.8 | 1 | 30 |
| 1 | 0.412 | 0.267 | 15 |
| 0.393 | 0.288 | 0.662 | 19 |
| 1 | 0.1 | 0.01 | 48 |

XIII in Supplementary. The animation of these periodic orbits are provided on the website: http://numericaltank.sjtu.edu.cn/free-fall-3b/free-fall-3b.htm. The periodic orbit $F_1(0.6, 0.8, 1)$ is the one found by Standish (1970). The periodic orbits $F_1(1, 1, 1)$ and $F_2(1, 1, 1)$ are the ones named Oribt 16 and 17 in Yasko and Orlov (2015), respectively. Except for these three periodic orbits, 313 collisionless periodic free-fall three-body orbits are newly-found. Note that the periodic orbit named Orbit 19 in ref. Yasko and Orlov (2015) is not found in this paper, because the initial condition of that orbit is not in the region-*D* considered in this paper.

A few cases of mass ratio listed in Table 2 are considered here. Note that we find 19 collisionless periodic orbits with masses $m_1 = 0.393$, $m_2 = 0.288$ and $m_3 = 0.662$ which are randomly generated between 0 and 1. Therefore, theoretically speaking, we can gain collisionless periodic solutions of the free-fall three-body system in **arbitrary** ratio of mass. What's more, we obtain 48 collisionless periodic orbits even with large mass ratio (i.e. $m_1 = 1$, $m_2 = 0.1$ and $m_3 = 0.01$). So, theoretically speaking, the number of periodic solutions of the free-fall three-

**Table 2**
The initial conditions and periods $T$ of the new collisionless periodic free-fall orbits in the case of $\boldsymbol{r}_1(0) = (-0.5, 0)$, $\boldsymbol{r}_2(0) = (0.5, 0)$, $\boldsymbol{r}_3(0) = (x, y)$, $\dot{\boldsymbol{r}}_1(0) = \dot{\boldsymbol{r}}_2(0) = \dot{\boldsymbol{r}}_3(0) = (0, 0)$ and the Newtonian constant of gravitation $G = 1$.

| $F_i(m_1, m_2, m_3)$ | $x$ | $y$ | $T$ |
|---|---|---|---|
| $F_{10}(1, 1, 1)$ | 0.3089693008 | 0.4236727692 | 4.8914942162 |
| $F_1(1, 0.8, 0.8)$ | 0.0009114239 | 0.3019805958 | 1.8286248401 |
| $F_{24}(1, 0.8, 0.8)$ | 0.1527845023 | 0.068494294 | 7.5001089956 |
| $F_5(1, 0.8, 0.6)$ | 0.129088109 | 0.4010761427 | 2.500764871 |
| $F_{27}(1, 0.8, 0.4)$ | 0.083924021 | 0.3307729197 | 5.3174336486 |
| $F_6(1, 0.8, 0.2)$ | 0.1310631652 | 0.3036588095 | 2.9464698551 |

body system is infinite.

All bodies of these periodic orbits have zero velocities at time $t = T/2$, where $T$ is the period. After $t = T/2$, the three bodies will go back to the initial positions along the original trajectories. For example, the periods and initial conditions of six newly-found collisionless periodic free-fall three-body orbits are presented in Table 2, and their trajectories are shown in Fig. 3. So, it seems that all periodic trajectories of the free-fall three-body system are *not* closed. This is quite different from the periodic solutions of triple system with nonzero initial velocities.

Li and Liao (2017) found that the periodic solutions with equal mass and initial conditions with isosceles collinear configurations have scale-invariant average period $\bar{T}^* = (T/L_f) |E|^{3/2} \approx 2.433$, where $T$, $L_f$ and $E$ are the period, the number of letter of free group word, total energy of the system, respectively. For the collisionless free-fall periodic orbits with equal mass ($m_1 = m_2 = m_3 = 1$) found in this paper, the scale-invariant average period $\bar{T}^* \approx 2.363$, is close to 2.433 with error of 3%. The results imply that the scale-invariant average period of these periodic orbits with equal mass ($m_1 = m_2 = m_3 = 1$) is approximate to a universal constant for different configurations, which are considered as generalized Kepler's third law for the periodic three-body system (Dmitrašinović and Šuvakov, 2015; Li and Liao, 2017; Li et al., 2018). For periodic orbits with unequal mass and initial conditions with isosceles collinear configurations, Li et al. (2018) found that the scale-invariant average period grows linearly with $m_3$ in case of $m_1 = m_2 = 1$. In this paper, for periodic free-fall orbits with $m_1 = 1$ and $m_2 = 0.8$, scale-invariant average period is 1.663, 1.191, 0.776 and 0.527 for $m_3 = 0.8, 0.6, 0.4$ and $0.2$, respectively. It is also found that the scale-invariant average period grows linearly with $m_3$, as shown in Fig. 4. This confirms again that these should exist a generalized Kepler's third law for the **periodic** three-body system.

## 4. Conclusions

Although the free-fall triple system have been numerically studied for more than one century, however, only four collisionless periodic orbits have been found. In this paper, we report 316 collisionless periodic solutions of the free-fall triple system with a few mass ratios, including the periodic orbit found by Standish (1970) and two periodic orbits found by Yasko and Orlov (2015). Especially, 313 periodic orbits are entirely new. It should be emphasized that we can gain periodic solutions of the free-fall three-body system in *random* ratio of mass so that there should exist an infinite number of periodic orbits of the free-fall triple system. In addition, it is found that, in case of a randomly given ratio of mass, there always exists a generalized Kepler's third law
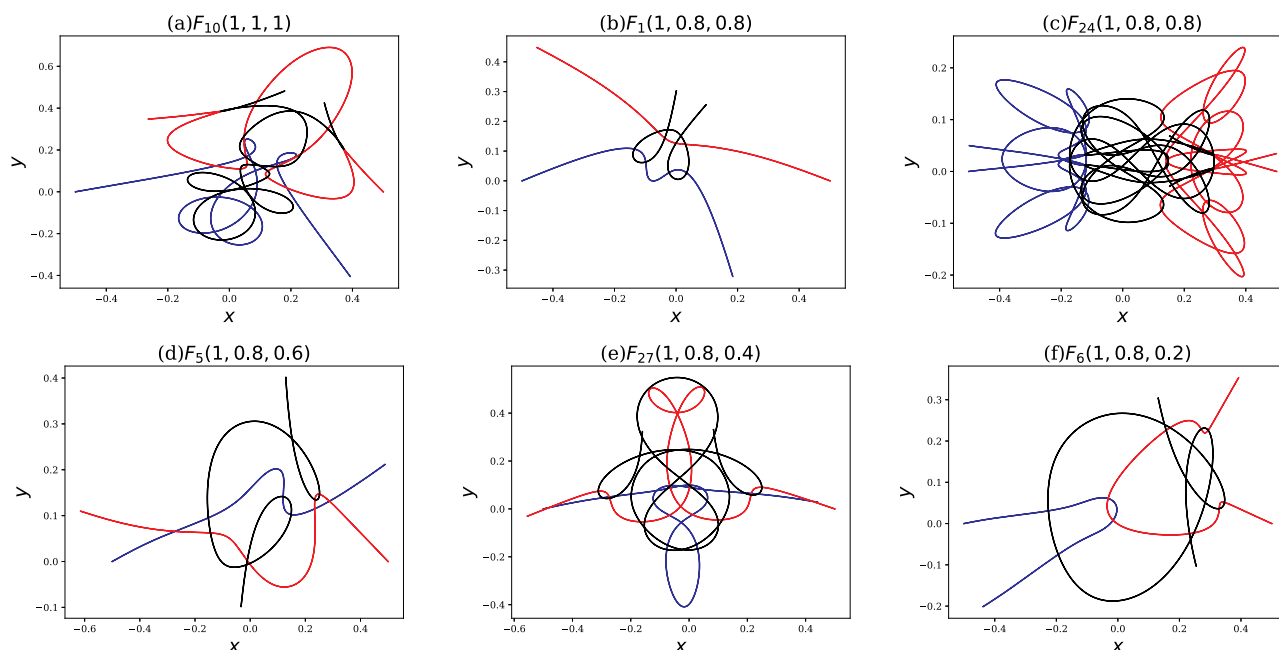
**Fig. 3.** (color online.) The trajectories of six new collisionless periodic orbits in the free-fall three-body problem. Blue line: body-1, red line: body-2, black line: body-3. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
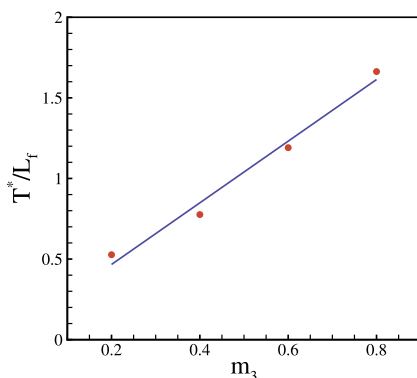


**Fig. 4.** (color online). The scale-invariant average period $T^*/L_f = T|E|^{3/2}/L_f$ versus $m_3$ in case of $m_1 = 1$ and $m_2 = 0.8$. Symbols: computed results; line: $T^*/L_f = 1.912m_3 + 0.084$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

for the periodic free-fall three-body system. All of these would enrich our knowledge and deepen our understanding about the famous three-body problem as a whole.

## Conflict of interest

The authors declare that they have no conflict of interest.

## Acknowledgments

## Supplementary material

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