A new modification of false position method based on homotopy analysis method [*]

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Abstract   A new modification of false position method for solving nonlinear equations is presented by applying homotopy analysis method (HAM). Some numerical illustrations are given to show the efficiency of algorithm.

Key words   false position method, regula falsi method, homotopy analysis method

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Introduction

In numerical analysis, the false position method or regula falsi method is a root-finding algorithm that combines features from the bisection method and the secant method[1]. The method of false position is a thousands-of-years-old method for solving nonlinear equations. Recently, a modified formula based on homotopy continuation technique was presented by Wu[2]. In this paper, the example problems posed apply the false position method to nonlinear equations including the high order algebraic equations.

Like the bisection method, the false position method starts with two points a and b such that $f(a)$ and $f(b)$ are of opposite signs, which implies by the intermediate value theorem that the continuous function f has a root in the interval $[a, b]$. The method proceeds by producing a sequence of shrinking intervals such that all contain a root of f, i.e., the number

$$x_0 = a - \frac{b - a}{f(b) - f(a)} f(a) \tag{1}$$

is computed, where x_0 is the root of the secant line, $y(x)$, through $(a, f(a))$ and $(b, f(b))$. If $f(a)$ and $f(x_0)$ have the same sign, then we set $a_{\text{new}} = x_0$ and $b_{\text{new}} = b$, otherwise we set $a_{\text{new}} = a$ and $b_{\text{new}} = x_0$. This process is repeated until the root is approximated sufficiently well. It is clear that $y(x) = y(x_0) + (x - x_0) y\prime(x_0)$.

In 1992, Liao[3] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely the homotopy analysis method (HAM)[4–9]. This method has been successfully applied to solve many types of nonlinear problems[10–13].

1 Homotopy analysis method

In this section, the ideas of homotopy analysis method (HAM) are applied[7]. Consider the nonlinear algebraic equation,

$$f(x) = 0, \tag{2}$$

where α is a simple root of it and f is a continuous function on its domain, and we suppose $\alpha \in [a, b]$ such that $f(a)f(b) < 0$. Let $y(x)$ be the secant line through $(a, f(a))$ and $(b, f(b))$.

Let $q \in [0, 1]$ denote an embedding parameter, $\hbar \neq 0$ an auxiliary parameter, $H(q) \neq 0$ an auxiliary function. We construct the zero-order deformation equation[7],

$$(1 - q)[y(x(q)) - y(x_0)] - q\hbar H(q)f(x(q)) = 0, \tag{3}$$

where x_0 is obtained by (1). It should be emphasized that one has great freedom to choose the auxiliary parameter $\hbar$, and the auxiliary function $H(q)$. Obviously, when $q = 0$ and $q = 1$, it holds that

$$x(0) = x_0, \quad x(1) = \alpha,$$

respectively. When q increases from 0 to 1, $x(q)$ varies from the x_0 to the solution of equation (2), α. Such a kind of continuous variation is called deformation in homotopy, and for this reason we call (3) the zero-order deformation equation. Defining

$$x_m = \frac{1}{m!}\frac{d^m x(q)}{dq^m}\Big|_{q=0},$$

and expanding $x(q)$ in Taylor series with respect to the embedding parameter q, we have

$$x(q) = x(0) + \sum_{m=1}^{\infty} x_m q^m. \tag{4}$$

If the auxiliary parameter $\hbar$, and the auxiliary function $H(q)$ are so properly chosen that the series (4) converges at $q = 1$, one has

$$\alpha = x_0 + \sum_{m=0}^{\infty} x_m, \tag{5}$$

which must be the solution of (2), as proved by Liao[7]. It is very important to ensure the convergence of series (4) at $q = 1$, otherwise, the series (5) has no meaning.

For simplicity, we put $H(\delta) \equiv 1$. Let $\alpha_M = x_0 + \cdots + x_M$ denote the $(M + 1)$-term approximation of α. Hence, by putting (4) in (3) and using the Taylor series, we have the following iterative relations for various M.

For $M = 0$,

$$\alpha \approx \alpha_0 = x_0,$$

which is the false position method.

For $M = 1$,

$$x_1 = \hbar\frac{b - a}{f(b) - f(a)}f(x_0),$$

$$\alpha \approx \alpha_1 = x_0 + \hbar\frac{b - a}{f(b) - f(a)}f(x_0), \tag{6}$$

which is the same as false position method for $\hbar = 0$.

For $M = 2$,

$$x_2 = x_1 \left[1 + \hbar \frac{(b-a)f'(x_0)}{f(b)-f(a)} \right],$$

$$\alpha \approx \alpha_2 = x_0 + 2\hbar \frac{(b-a)f(x_0)}{f(b)-f(a)} + \hbar^2 \frac{(b-a)^2 f(x_0)f'(x_0)}{(f(b)-f(a))^2}, \tag{7}$$

which is the same as false position method for $\hbar = 0$.

Hence, we have the modified false position method as follows. Suppose $\alpha_M \in [a, b]$, otherwise we choose another $\hbar$, for example halved $\hbar$, and again α_M computed until our assumption occurred, or we can ignore this $\hbar$ and choose another, which we do in our examples. If $f(a)$ and $f(\alpha_M)$ have the same sign, then we set $a_{\text{new}} = \alpha_M$ and $b_{\text{new}} = b$, otherwise we set $a_{\text{new}} = a$ and $b_{\text{new}} = \alpha_M$. This process is repeated until the root is approximated sufficiently well. As mentioned, the new method for $\hbar = 0$ is traditional false position method.

2   Examples

Here we consider some examples to compare the number of iterations needed in the traditional false position method, N_{f}, and the modified false position method, $N(M)$ for various M, and also we plot the $\hbar$-curve[7]. In all examples the precision of calculation is 10^{-7}.

Example 2.1[14, 15]   Consider the equation,

$$x^2 - (1-x)^5 = 0,$$

with the solution $\alpha = 0.345\,955$, where $[a, b] = [0, 2]$. Here $N_{\text{f}} = 18$ and for $\hbar = -1.84$, $N(1) = 2$, see Fig. 1. Also, for $\hbar \in [-1.92, -1.84]$, $N(2) = 2$, see Fig. 2.

Example 2.2   Consider the equation,

$$x^3 + 4x^2 - 10 = 0,$$

with the solution $\alpha = 1.365\,23$, where $[a, b] = [0, 3]$. Here $N_{\text{f}} = 28$ and for $\hbar \in [-2.04, -1.92]$, $N(1) = 5$, see Fig. 3. Also, for $\hbar = -1.24$, $N(2) = 4$, see Fig. 4.

Example 2.3   Consider the equation,

$$\sin^2 x - x^2 + 1 = 0,$$

with the solution $\alpha = 1.404\,49$, where $[a, b] = [1, 2]$. Here $N_{\text{f}} = 15$ and for $\hbar = -1.32$, $N(1) = 4$, see Fig. 5. Also, for $\hbar \in [-1.08, -1.0]$, $N(2) = 3$, see Fig. 6.
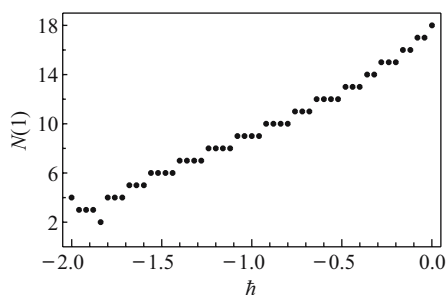

Fig. 1   The $\hbar$-curve of Example 2.1 (In false position method $N_{\text{f}} = 18$ and in HAM, $N(1) = 2$ for $\hbar = -1.84$)
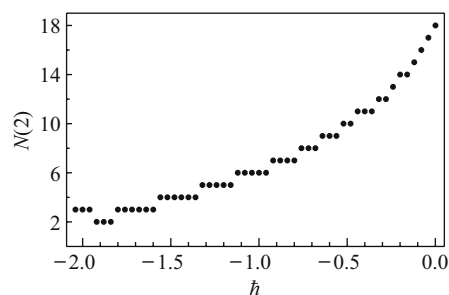
Fig. 2   The $\hbar$-curve of Example 2.1 (In false position method $N_{\text{f}} = 18$ and in HAM, $N(2) = 2$ for $\hbar \in [-1.92, -1.84]$)
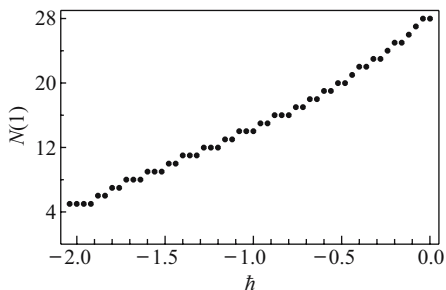
Fig. 3   The $\hbar$-curve of Example 2.2 (In false position method $N_{\mathrm{f}} = 28$ and in HAM, $N(1) = 5$ for $\hbar \in [-2.04, -1.92]$)
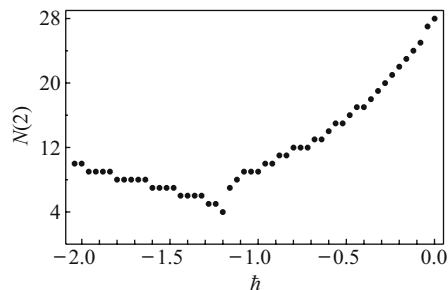
Fig. 4   The $\hbar$-curve of Example 2.2 (In false position method $N_{\mathrm{f}} = 28$ and in HAM, $N(2) = 4$ for $\hbar = -1.24$)
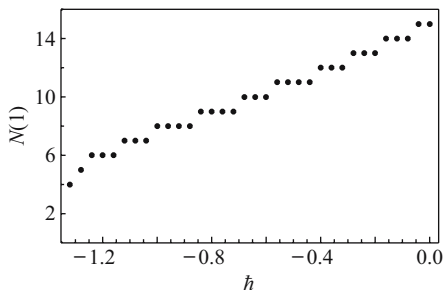

Fig. 5   The $\hbar$-curve of Example 2.3 (In false position method $N_{\mathrm{f}} = 15$ and in HAM, $N(1) = 4$ for $\hbar = -1.32$)
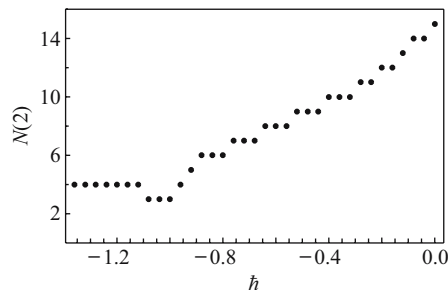
Fig. 6   The $\hbar$-curve of Example 2.3 (In false position method $N_{\mathrm{f}} = 15$ and in HAM, $N(2) = 3$ for $\hbar \in [-1.08, -1.0]$)

Example 2.4   Consider the equation,

$$x^2 - \mathrm{e}^x - 3x + 2 = 0,$$

with the solution $\alpha = 0.257\,53$, where $[a, b] = [0, 4]$. Here $N_{\mathrm{f}} = 47$ and for $\hbar = -3.42$, $N(1) = 3$, see Fig. 7. Also, for $\hbar \in [-3.6, -3.24]$, $N(2) = 3$, see Fig. 8.

Example 2.5   Consider the equation,

$$\cos(x) - x = 0,$$

with the solution $\alpha = 0.739\,085$, where $[a, b] = [0, 3]$. Here $N_{\mathrm{f}} = 7$ and for $\hbar = -1$, $N(1) = 3$, see Fig. 9. Also, for $\hbar \in [-1.32, -0.8]$, $N(2) = 3$, see Fig. 10.

As shown in Figs. 1 to 10, the number of iteration is the largest by means of the traditional false position method, corresponding to $\hbar = 0$. And much less iteration number is needed if a proper value of $\hbar$ is chosen.

Remark 2.1   Assume that we gain the solution in the auxiliary parameter $\hbar$, like Eqs. (6) and (7), by means of HAM. How does one then choose the value of $\hbar$ to ensure that the solution is obtained fast enough? In this case, we can gain convergent solution by means of the homotopy-padé technique[7].
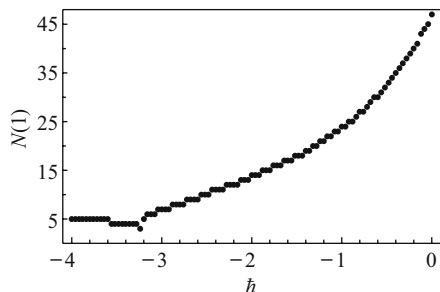
Fig. 7 The $\hbar$-curve of Example 2.4 (In false position method $N_{\mathrm{f}} = 47$ and in HAM, $N(1) = 3$ for $\hbar = -3.24$)
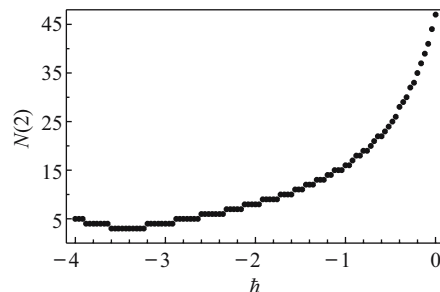

Fig. 8 The $\hbar$-curve of Example 2.4 (In false position method $N_{\mathrm{f}} = 47$ and in HAM, $N(2) = 3$ for $\hbar \in [-3.6, -3.24]$)
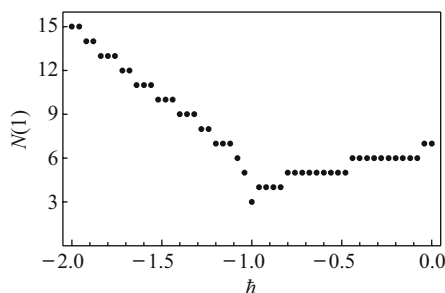

Fig. 9 The $\hbar$-curve of Example 2.5 (In false position method $N_{\mathrm{f}} = 7$ and in HAM, $N(1) = 3$ for $\hbar = -1.0$)
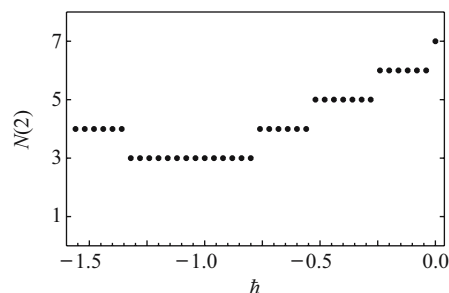

Fig. 10 The $\hbar$-curve of Example 2.5 (In false position method $N_{\mathrm{f}} = 7$ and in HAM, $N(2) = 3$ for $\hbar \in [-1.32, -0.8]$)

3 Homotopy-Padé approach

As verified in previous publications[6, 7, 16], it is the auxiliary parameter $\hbar$ which gives us with a simple way to adjust or control the number of iterations, $N(M)$, by the homotopy analysis method. Employing the traditional $[1, 1]$ Padé approximant to the $\alpha_2(q) = x_0 + x_1 q + x_2 q^2$, we have

$$\alpha_2(q)\Big|_{q=1} \approx x_0 - \frac{f(x_0)}{f'(x_0)}, \tag{8}$$

which is independent upon the auxiliary parameter $\hbar$. Thus, the result given by the homotopy-Padé approach is independent upon the auxiliary parameter $\hbar$. Table 1 shows the number of iterations needed by (8), N_{p}, in our examples. Note that N_{p} is much less than N_{f}.

Table 1 Results for the $[1, 1]$ homotopy-Padé approach

Example	2.1	2.2	2.2	2.3	2.4	2.4	2.5
$[a, b]$	$[0,2]$	$[0,3]$	$[1,2]$	$[1,2]$	$[0,4]$	$[0,3]$	$[0,3]$
N_{f}	18	28	14	15	47	19	7
N_{p}	2	7	3	3	3	2	3

Remark 3.1 Let $g(x) = x - \frac{f(x)}{f'(x)}$. In the homotopy-Padé method, Eq. (8), if $g(x_0) \notin [a, b]$,

we must choose another interval $[a, b]$. The following lemma ensures us the existence of a good interval.

Lemma 3.1[1]   *Let $f(x)$ have only one simple root in $[a, b]$, and $f \in C^2[a, b]$. Then there exists an interval $[c, d] \subset [a, b]$, which contains the simple root $f(x)$ and we can use it in the homotopy-Padé method, i.e., g maps $[c, d]$ into $[c, d]$.*

4   Conclusions

In this paper, an easy-to-use modification of traditional false position method is obtained by the homotopy analysis method. Besides, the homotopy analysis method provides us with freedom to choose the auxiliary parameter $\hbar$ and the auxiliary function $H(q)$. Also, with $\hbar$-curves we can control the speed of convergence. Examples showed the validity and potential of the homotopy analysis method for nonlinear problems in the science and engineering.

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