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Series Solution of Non-similarity Boundary-Layer Flows Over a Porous Wedge

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Abstract Solution of non-similarity boundary-layer flows over a porous wedge is studied. The free stream velocity $U_w(x) \sim a\,x^m$ and the injection velocity $V_w(x) \sim b\,x^n$ at the surface are considered, which result in the corresponding non-similarity boundary-layer flows governed by a nonlinear partial differential equation. An analytic technique for strongly nonlinear problems, namely, the homotopy analysis method (HAM), is employed to obtain the series solutions of the non-similarity boundary-layer flows over a porous wedge. An auxiliary parameter is introduced to ensure the convergence of solution series. As a result, series solutions valid for all physical parameters in the whole domain are given. Then, the effects of the physical parameters on the skin friction coefficient and displacement thickness are investigated. To the best of our knowledge, it is the first time that the series solutions of this kind of non-similarity boundary-layer flows are reported.

Keywords Non-similarity · Boundary-layer flow · Porous wedge · Homotopy analysis method

1 Introduction

A milestone contribution in fluid mechanics was made by Prandtl (1904), who proposed the revolutionary concept of the boundary-layer of viscous fluid. Since then, the boundary-layer theory (Blasius 1908; Rosenhead 1963; Howarth 1938; Van Dyke 1962a,b, 1964, 1969, 1975; Tani 1977; Schlichting and Gersten 2000; Sobey 2000) has been greatly developed and widely applied to many fields of science and engineering. In the frame of boundary-layer theory, the original partial differential equations (PDEs) are often transferred into nonlinear

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ordinary differential equations (ODEs) by defining some similarity variables, corresponding to the so-called similarity flows. Unfortunately, such kinds of transformations exist only for some special flows and/or special boundary conditions. If the boundary-layer conditions or the flows are so general that such kind of similarity transformation does not exist, one had to solve nonlinear PDEs. Mathematically, it is easier to solve nonlinear ODEs than nonlinear PDEs. So, it is easier to solve similarity boundary-layer flows than non-similarity ones. This is the reason why most of publications for boundary-layer flows are related to similarity flows.

Recently, the study of boundary-layer flow along surfaces embedded in fluid saturated porous media has received considerable interest, especially in the enhanced recovery of petroleum resources, packed bed reactors, and geothermal industries. The boundary-layer behavior over a moving continuous solid surface is an important type of flow, occurring in several engineering processes. For example, the thermal processing of sheet-like materials is a necessary operation in the production of paper, linoleum, polymeric sheets, wire drawing, drawing of plastic films, metal spinning, roofing shingles, insulating materials, and fine-fiber mats. A main reason for the interest in analysis of boundary-layer flows along solid surfaces is the possibility of applying the theory to the efficient design of supersonic and hypersonic flights. Besides, the mathematical model considered in the present research has importance in studying many problems of engineering, meteorology, and oceanography. Due to study of heat and mass transfer in moving fluids, the applications of boundary layers extended to different engineering branches. Examples include boundary-layer control on airfoils, transpiration cooling of turbine blades, lubrication of ceramic machine parts, food processing, electronics cooling, the extraction of geothermal energy, nuclear reactor cooling system, and filtration process.

Some boundary-layer problems such as the stagnation flow (Xu et al. 2008), the flow over a flat plate (Liao 2006), and the flow over a stretching sheet (Liao 2003a) are reported. The steady laminar flow passing a fixed wedge was first analyzed in the early 1930s by Falkner and Skan (1931)[1] to illustrate Prandtl's boundary-layer theory. With a similarity transformation, the boundary-layer equations are reduced to a nonlinear ODE, which is called now the Falkner–Skan equation. This equation includes a non-uniform outer flow with the velocity in the form ax^m, where x is the coordinate measured along the wedge wall, with the constant a and m. Hsu et al. (1997) studied the combined effects of the shape factor, suction/injection rates, and viscoelasticity on the flow and temperature fields of the flow past a wedge. Magyari and Keller (2000) obtained the exact solutions for the two-dimensional similarity boundary-layer flows induced by permeable stretching surfaces. Rajagopal and Gupta (1984) gave non-similarity solutions for the flow of a second grade fluid over wedge. This work was followed by Rajagopal et al. (1987), where non-similarity solutions were obtained for the flow over stretching sheet with uniform flow stream. Koh and Hartnett (1961) have solved the skin friction and heat transfer for incompressible laminar flow over porous wedges with suction and variable wall temperature.

Generally speaking, it is difficult to solve nonlinear PDEs, especially by means of the analytic method. Using the perturbation methods or the traditional non-perturbation methods such as Lyapunov's small parameter method (Lyapunov 1992), the δ-expansion method (Karmishin et al. 1990), and Adomian's decomposition method (Adomian 1994), it is difficult to get analytic approximations convergent for all physical parameters in the infinite domain of the flows, because all of these techniques cannot ensure the convergence of approximation

[1] See http://demonstrations.wolfram.com/NumericalSolutionOfTheFalknerSkan EquationForVariousWedgeAngl/ and http://library.wolfram.com/infocenter/MathSource/6003/.

series. The analytic method most frequently used in non-similarity boundary layer flows is the so-called 'method of local similarity'. Sparrow et al. (1970) introduced this method under some additional assumptions and applied it to some non-similarity flows (Sparrow et al. 1971). Massoudi (2001) used it to solve a non-similarity flow of non-Newtonian fluid over a wedge. In some cases, the results given by the method of local non-similarity agree with numerical solutions, as mentioned in Wanous and Sparrow (1965) and Catherall et al. (1965). However, the results given by the method of local similarity are not very accurate, as pointed out by Sparrow et al. (1971), and besides are valid only for small ξ in general, as mentioned by Massoudi (2001).

The purpose of this article is to give the convergent series solution of the non-similarity boundary-layer flows over a porous wedge by means of the homotopy analysis method (HAM), an analytic approach for strongly nonlinear problems. Unlike the perturbation techniques, the HAM is independent of small/large physical parameters. Besides, it provides us with a simple way to control and adjust the convergence of solution series. Furthermore, it provides great freedom to choose the so-called auxiliary linear operator so that one can approximate a nonlinear problem more effectively by means of better base functions. The HAM has been successfully applied to solve many strongly nonlinear problems, such as Thomas–Fermi equation (Liao 2003b), non-linear water wave problems (Liao and Cheung 2003), magnetohydrodynamic (MHD) flows of non-Newtonian fluids over the stretching sheet (Liao 2003a), non-Darcy natural convection (Wang et al. 2003), MHD flows of an Oldroyd-6 constant fluids (Hayat et al. 2004), free oscillations of self-excited systems (Liao 2004), boundary-layer flows (Liao and Pop 2004; Liao 2009), and so on. Especially, a few new solutions of some non-linear problems (Liao 2005; Liao and Magyari 2006) are found, which were neglected by all other analytic methods and even by numerical techniques. The HAM has also been applied to solve some non-linear PDEs, such as unsteady similarity boundary-layer flows (Liao 2006), Black–Scholes type equation in finance for American put option (Zhu 2006a,b), and so on. All of these successful applications verified the validity, effectiveness, and flexibility of the HAM. These previous work provide us a good background for our work mentioned in this article.

2 Mathematical Formulations

Consider a steady-state, two-dimensional, and incompressible boundary-layer flow. We formulate the problem in a coordinate system as shown in Fig. 1, where x is measured along the surface of the wedge from the apex, and y is normal to the wedge surface, respectively. In the frame of boundary-layer theory (Schlichting and Gersten 2000), the flow is governed by

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{\mathrm{d}U}{\mathrm{d}x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2}$$

subject to the boundary conditions

$$u=0,\quad v=V_w(x)=b\,x^n,\ \text{ at } y=0,\quad \text{and}\quad u=U(x)=a\,x^m,\ \text{ as } y\to\infty, \tag{3}$$

where u and v are the velocity components along and cross the flow, respectively, and ν the kinematic viscosity of the fluid. Here, $V_w(x)=bx^n$ represents the suction ($b>0$) or injection ($b<0$) velocity, and $U(x)=ax^m$ denotes the outer flow velocity, where $a>0$

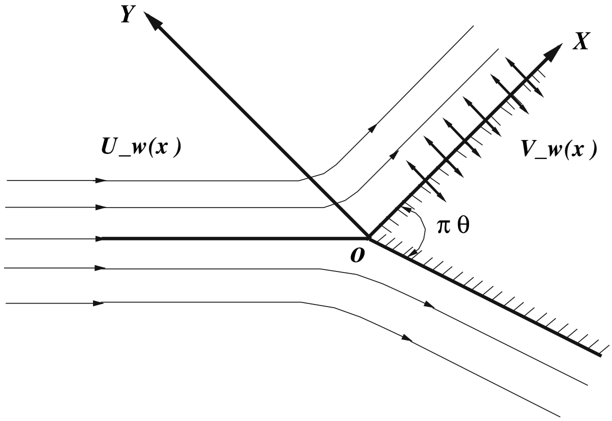


Fig. 1 The coordinate system

is an arbitrary constant and m is related to the angle of the wedge $\pi\theta$ by $m = \theta/(2-\theta)$, as shown in Fig. 1. The first term on the right-hand side of Eq. 1 represents the effect of the axial pressure gradient which exists due to the variable velocity $U(x)$ of the external flow stream.

First, we transform the governing equations from the x, y coordinate system to ξ, η system by means of the Görtler Transformations (Schlichting and Gersten 2000)

$$\xi = \frac{1}{\nu}\int_0^x U(x)\mathrm{d}x \quad \text{and} \quad \eta = \frac{U(x)y}{\nu\sqrt{2\xi}}. \tag{4}$$

The coordinate η, which involves both x and y, becomes a similarity variable if the flow is a similarity ones. However, ξ is related to x only. Let ψ denote the stream function satisfying

$$u = \frac{\partial\psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial\psi}{\partial x}. \tag{5}$$

Following the Görtler Transformations, we write the dimensionless variables and stream function in Eq. 3 as

$$\psi(x,y) = \sqrt{\frac{2a\nu}{(m+1)}}x^{\frac{(m+1)}{2}} f(\xi,\eta) \tag{6}$$

$$\xi = \frac{ax^{m+1}}{\nu(m+1)} \quad \text{and} \quad \eta = \sqrt{\frac{a(m+1)}{2\nu}}x^{\frac{(m-1)}{2}}y. \tag{7}$$

Then, the governing Eqs. (1) and (2) become

$$f_{\eta\,\eta\,\eta} + f\ f_{\eta\,\eta} + \beta\ (1 - f_\eta^2) = 2\ \xi\ (f_\eta\ f_{\xi\,\eta} - f_\xi\ f_{\eta\,\eta}) \tag{8}$$

with the so-called principal function

$$\beta(\xi) = 2\,\frac{U^{'}(x)}{U^2(x)}\int_0^x U(x)\mathrm{d}x, \tag{9}$$

subject to the boundary conditions

$$f_\eta = 0, \quad f(\xi, 0) + 2\,\xi\, f_\xi(\xi, 0) = \gamma\, \xi^\kappa, \text{ at } \eta = 0, \quad \text{and} \quad f_\eta = 1 \text{ as } \eta \to \infty, \tag{10}$$

where $\gamma = -\sqrt{2}\, a^{-\kappa-1/2}\, b\ \ [\nu\,(m+1)]^{\kappa-1/2}$ and $\kappa = (m+1)^{-1}\ [n - (m-1)/2]$ are constants. Note that, when $U(x) = ax^m$, the principal function (9) is reduced to the constant $\beta = 2m/(m+1)$. Here, γ is the suction/injection parameter, where $b \in R$ and $a > 0$, and κ defines the relation between the injection index n and the wedge angle parameter m. Note that $\gamma = 0$ is the necessary condition of the existence of similarity solutions.

2.1 HAM Solution

In order to find the analytic approximations of Eq. 8 subject to the boundary condition (10), we apply the HAM, a widely used analytic method for highly nonlinear equations.

2.1.1 The Zeroth-Order Deformation Equation

The HAM is based on such a continuous mapping $\Phi(\xi, \eta; q)$ that, as the embedding parameter q increases from 0 to 1, $\Phi(\xi, \eta; q)$ varies (or deforms) from the initial approximation $f_0(\xi, \eta)$ to the exact solution $f(\xi, \eta)$ of Eqs. 8 and 10, i.e. $\Phi(\xi, \eta; q) : f_0(\xi, \eta) \sim f(\xi, \eta)$. According to Eq. 8, we define a nonlinear operator

$$\begin{aligned}\mathcal{N}[\Phi(\xi, \eta; q)] = {} & \frac{\partial^3 \Phi(\xi, \eta; q)}{\partial \eta^3} + \Phi\, \frac{\partial^2 \Phi(\xi, \eta; q)}{\partial \eta^2} + \beta \left[1 - \left(\frac{\partial \Phi(\xi, \eta; q)}{\partial \eta}\right)^2\right] \\ & -2\,\xi \left[\frac{\partial \Phi(\xi, \eta; q)}{\partial \eta}\, \frac{\partial^2 \phi(\xi, \eta; q)}{\partial \xi\, \partial \eta} - \frac{\partial \Phi(\xi, \eta; q)}{\partial \xi}\, \frac{\partial^2 \Phi(\xi, \eta; q)}{\partial \eta \partial \eta}\right],\end{aligned} \tag{11}$$

where $q \in [0, 1]$ is the embedding parameter. Let $\hbar$ denote a nonzero auxiliary parameter, $\mathcal{L}$ an auxiliary linear operator, and $f_0(\xi, \eta)$ an initial guess that satisfies the boundary conditions (10). Note that we have great freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $f_0(\xi, \eta)$, and both $\mathcal{L}$ and $f_0(\xi, \eta)$ are unknown right now but will be chosen later. Then, we construct a homotopy

$$\mathcal{H}[\Phi(\xi, \eta; q); \hbar, q] = (1-q)\mathcal{L}[\Phi(\xi, \eta; q) - f_0(\xi, \eta)] - q\,\hbar\, \mathcal{N}[\Phi(\xi, \eta; q)].$$

Enforcing $\mathcal{H}[\Phi(\xi, \eta; q); \hbar, q] = 0$, we have a family of equations

$$(1-q)\ \mathcal{L}[\Phi(\xi, \eta; q) - f_0(\xi, \eta)] = q\,\hbar\, \mathcal{N}[\Phi(\xi, \eta; q)]\,, \tag{12}$$

subject to two boundary conditions on the wedge surface

$$\Phi_\eta(\xi, 0; q) = 0, \tag{13}$$

$$(1-q)\,[\Phi(\xi, 0; q) - f_0(\xi, 0)] = q\,\hbar_b\, \left[\Phi(\xi, 0; q) + 2\,\xi\, \Phi_\xi - \gamma\, \xi^\kappa\right], \tag{14}$$

and one boundary condition at infinity

$$\Phi_\eta(\xi, \infty; q) \to 1. \tag{15}$$

Since the initial guess $f_0(\xi, \eta)$ satisfies the boundary conditions, it is obvious that, when $q = 0$, the solution of Eqs. 12–15 reads

$$\Phi(\xi, \eta; 0) = f_0(\xi, \eta). \tag{16}$$

When $q = 1$, since $\hbar \neq 0$, Eqs. 12–15 are equivalent to Eqs. 8–10, respectively, provided

$$\Phi(\xi, \eta; 1) = f(\xi, \eta). \tag{17}$$

Therefore, as the embedding parameter q increases from 0 to 1, $\Phi(\xi, \eta; q)$ varies from the initial approximation $f_0(\xi, \eta)$ to the exact solution $f(\xi, \eta)$. In topology, this kind of variation is called *deformation*, and Eqs. 12–15 construct the homotopy $\Phi(\xi, \eta; q)$. For brevity, Eqs. 12–15 are called the zeroth-order deformation equations.

Using (16) and Taylor's theorem, we expand $\Phi(\xi, \eta; q)$ in power series of q as

$$\Phi(\xi, \eta; q) = f_0(\xi, \eta) + \sum_{m=1}^{+\infty} f_m(\xi, \eta)\, q^m , \tag{18}$$

where

$$f_m(\xi, \eta) = \frac{1}{m!} \left. \frac{\partial^m \Phi(\xi, \eta; q)}{\partial q^m} \right|_{q=0} .$$

It should be emphasized that there exists an auxiliary parameter $\hbar$ in the zeroth-order deformation Eq. (12) so that $\Phi(\xi, \eta; q)$ and thus $f_m(\xi, \eta)$ are dependent upon $\hbar$. Besides, we have great freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $f_0(\xi, \eta)$. As a result, the convergence region of the series (18) is dependent upon $\hbar$, $f_0(\xi, \eta)$ and $\mathcal{L}$. Assuming that all of them are so properly chosen that the series (18) is convergent at $q = 1$, we have from Eq. 17 that

$$f(\xi, \eta) = f_0(\xi, \eta) + \sum_{m=1}^{+\infty} f_m(\xi, \eta). \tag{19}$$

This provides us a relationship between the initial guess $f_0(\xi, \eta)$ and the exact solution $f(\xi, \eta)$.

2.1.2 The High-Order Deformation Equation

For brevity, define the vector

$$\vec{f}_m = \{f_0(\xi, \eta), f_1(\xi, \eta), f_2(\xi, \eta), \ldots, f_m(\xi, \eta)\}. \tag{20}$$

Differentiating the zeroth-order deformation Eqs. 12–15 m times with respect to the embedding parameter q, then setting $q = 0$, and finally dividing by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}[f_m(\xi, \eta) - \chi_m f_{m-1}(\xi, \eta)] = \hbar\, R_m(\vec{f}_{m-1}) , \tag{21}$$

subject to the boundary conditions on the wedge

$$\frac{\partial f_m}{\partial \eta} = 0, \qquad \text{at } \eta = 0, \tag{22}$$

$$f_m - \chi_m\, f_{m-1} = \hbar_b \left[f_{m-1} + 2\,\xi\, \frac{\partial f_{m-1}}{\partial \xi} - \gamma\, \xi^{\kappa}(1 - \chi_m) \right], \qquad \text{at } \eta = 0, \tag{23}$$

and the boundary condition at infinity

$$\frac{\partial f_m}{\partial \eta} = 0 \quad \text{as } \eta \to \infty, \tag{24}$$

where

$$R_m(\vec{f}_{m-1}) = \frac{\partial^3 f_{m-1}}{\partial \eta^3} + \sum_{n=0}^{m-1} f_{m-1-n} \frac{\partial^2 f_n}{\partial \eta^2} - \beta \sum_{n=0}^{m-1} \frac{\partial f_{m-1-n}}{\partial \eta} \frac{\partial f_n}{\partial \eta}$$
$$+ 2\, \xi \sum_{n=0}^{m-1} \left[\frac{\partial f_n}{\partial \xi} \frac{\partial^2 f_{m-1-n}}{\partial \eta^2} - \frac{\partial f_n}{\partial \eta} \frac{\partial^2 f_{m-1-n}}{\partial \xi\, \partial \eta} \right] + \beta\, (1 - \chi_m) \tag{25}$$

with the definition

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{26}$$

2.1.3 Solution Expression

As mentioned earlier, we have great freedom to choose the auxiliary linear operator $\mathcal{L}$, the initial guess $f_0(\xi, \eta)$ and the auxiliary parameter $\hbar$. As a starting point, we should find a proper set of base functions to fit the solutions. Physically, it is well known that most viscous flows decay exponentially at infinity (i.e. as $\eta \to +\infty$). So, without loss of generality, $f(\xi, \eta)$ can be expressed by the set of base functions

$$\{\xi^n\, \eta^k\, \mathrm{e}^{-m\, \lambda \eta} \mid n \ge 0,\ \ k \ge 0,\ \ m \ge 0,\ \ \lambda > 0\}$$

in the form

$$f(\xi, \eta) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{m,n}^k\, \xi^n\, \eta^k\, \mathrm{e}^{-m\, \lambda \eta}, \tag{27}$$

where $a_{m.n}^k$ is a coefficient to be determined. Our purpose is to give a convergent series solution for the nonlinear PDEs (8) and (10), expressed in the so-called solution expression (27). In order to obey the solution expression (27) and the boundary conditions (10), we choose the initial guess $f_0(\xi, \eta)$

$$f_0(\xi, \eta) = \eta - \frac{\mathrm{e}^{-\lambda\, \eta}}{\lambda} + \frac{\mathrm{e}^{-2\, \lambda\, \eta}}{\lambda} + \frac{\gamma\, \xi^\kappa}{(1 + 2\, \kappa)}. \tag{28}$$

Besides, physically speaking, for boundary-layer flows, the velocity variation across the flow direction is much larger than that in the flow direction. Therefore, the derivatives across the flow direction (i.e. with respect to η) are considerably larger and thus physically more important than the derivatives $\frac{\partial f}{\partial \xi}$ and $\frac{\partial^2 f}{\partial \xi \partial \eta}$. Considering that the original equation is third order, we can choose such an auxiliary linear operator

$$\mathcal{L} w = \frac{\partial^3 w}{\partial \eta^3} + a_2(\xi) \frac{\partial^2 w}{\partial \eta^2} + a_1(\xi)\, \frac{\partial w}{\partial \eta} + a_0(\xi) w, \tag{29}$$

where $a_0(\xi)$, $a_1(\xi)$ and $a_2(\xi)$ are real functions to be determined later. Let $w_1(\xi, \eta)$, $w_2(\xi, \eta)$ and $w_3(\xi, \eta)$ denote the three solutions of $\mathcal{L}w = 0$, i.e.

$$\mathcal{L}[w_1(\xi, \eta)] = 0, \quad \mathcal{L}[w_2(\xi, \eta)] = 0, \quad \text{and } \mathcal{L}[w_3(\xi, \eta)] = 0. \tag{30}$$

According to the solution expression (27) and considering the boundary condition (10) at infinity, we could choose

$$w_1 = 1, \quad w_2 = \exp(-\lambda\, \eta), \quad w_3 = \exp(\lambda\, \eta), \tag{31}$$

where $\lambda > 0$ is an auxiliary parameter. Then, the general solution of (21) reads

$$f_m(\xi,\eta) = f_m^*(\xi,\eta) + C_1 + C_2\ \exp(-\lambda\,\eta) + C_3\ \exp(\lambda\,\eta). \tag{32}$$

where $f_m^*(\xi,\eta)$ is the special solution of Eq. 21. To satisfy the boundary condition (24) at infinity, it holds $C_3 = 0$. Then, C_1 and C_2 are determined by the boundary conditions (22) and (23). Substituting (31) into (30) gives the auxiliary linear operator

$$\mathcal{L} f = \frac{\partial^3 f}{\partial \eta^3} - \lambda^2\ \frac{\partial f}{\partial \eta}. \tag{33}$$

Note that, if we choose $w_3 = \exp(-2\lambda\,\eta)$, then the boundary condition (24) is automatically satisfied by any values of C_1, C_2 and C_3, and thus C_3 has multiple solutions. This can be avoided by choosing $w_3 = \exp(+\lambda\,\eta)$. Besides, it should be emphasized that the original nonlinear PDE (8) is transferred into an infinite number of linear ODEs (21) by means of choosing the auxiliary linear operator (33). Obviously, it is much easier to solve a linear ODE than a nonlinear PDE! It also should be pointed out that the auxiliary linear operator (33) has not close relationship to the original governing equation (8). In fact, the same auxiliary linear operator (33) has been used for different types of similarity flows (Liao 2003c; Liao and Pop 2004).

It is easy to solve the *linear* ODE (21), subject to the *linear* boundary conditions (22)–(24). The special solution of (21) reads

$$f_m^*(\xi,\eta) = \chi_m\ f_{m-1}(\xi,\eta) + \hbar\ \mathcal{L}^{-1}\ [R_m(\vec{f}_{m-1})], \tag{34}$$

where $\mathcal{L}^{-1}$ denotes the inverse operator of $\mathcal{L}$. Thus, the solution of the high-order deformation equations (21)–(24) is

$$f_m(\xi,\eta) = f_m^*(\xi,\eta) + C_1 + C_2\ \mathrm{e}^{-\lambda\,\eta}, \tag{35}$$

where

$$\begin{aligned} C_1 = {} & \chi_m f_{m-1}(\xi,0) + \hbar_b \left[f_{m-1}(\xi,0) + 2\xi \frac{\partial f_{m-1}(\xi,0)}{\partial \xi} - \gamma\xi^{\kappa}(1-\chi_m) \right] \\ & -C_2 - f_m^*(\xi,0). \end{aligned} \tag{36}$$

and

$$C_2 = \frac{1}{\lambda}\ \left.\frac{\partial f_m^*}{\partial \eta}\right|_{\eta=0} \tag{37}$$

are determined by the boundary conditions (22) and (23). Therefore, high-order approximations of $f(\xi,\eta)$ can be obtained, especially by means of symbolic computation such as Mathematica, Maple, and so on.

The physical quantities of interest are the coefficient of the skin friction and the displacement thickness, defined by

$$C_f = \frac{2\ \tau_w}{\rho\ U_w^2}, \tag{38}$$

where

$$\tau_w = \mu\ \left.\frac{\partial u}{\partial y}\right|_{y=0}. \tag{39}$$

Under the transformations (7), the coefficient of the skin friction for the non-similarity flow is

$$C_f = \frac{\sqrt{2\nu(m+1)}}{\sqrt{a}\; x^{\frac{m+1}{2}}} \quad \frac{\partial^2 f}{\partial \eta^2}\bigg|_{\eta=0},$$

or

$$C_f = \frac{2}{\sqrt{\xi}} \quad \frac{\partial^2 f}{\partial \eta^2}\bigg|_{\eta=0}. \tag{40}$$

The boundary-layer thickness of the non-similarity flow reads

$$\delta(\xi) = \frac{1}{U_\infty} \int_0^\infty [U_\infty - u(\xi, \eta)] \, \mathrm{d}y.$$

Hence, we have in view of Eq. 7 that

$$\delta(\xi) = \sqrt{\frac{\nu}{a}} \int_0^\infty \left[1 - f_\eta(\xi, \eta)\right] \mathrm{d}\eta. \tag{41}$$

3 Result Analysis

Liao (2003c) proved in general that, as long as a solution series given by the HAM converges, it must be one of solutions. Therefore, it is very important to ensure the convergence of the solution series given by HAM. Note that solution series (19) contains two auxiliary parameters $\hbar_b$ and $\hbar$. For simplicity, we take here $\hbar_b = \hbar$. Note that κ is dependent upon the injection index n and the wedge parameter m, and γ is related to the injection velocity $V_w(x)$ through constant b. Note that $\gamma > 0$ when b is negative (suction), and $\gamma < 0$ when b is positive (injection).

The case $\gamma = 0$ corresponds to the boundary-layer flows past a non-porous wedge, which is the famous Falkner–Skan problem[2] with similarity solutions. So, in the case of $\gamma = 0$, $f(\xi, \eta)$ is independent of ξ. Without the loss of generality, let us consider the case of $\beta = 1$, and check the influence of λ and $\hbar$ on the solution series of $f''(\xi, 0)$. First, we set $\hbar = -1$ so that the solution series depends on the λ only, and then we investigate the influence of λ on $f''(\xi, 0)$ by plotting the curves of $f''(\xi, 0) \sim \lambda$. It is found that, when $\lambda \geq 5$, the series $f''(\xi, 0)$ converges to the same value. Second, we set $\lambda = 5$ and regard $\hbar$ as a unknown variable. In this case, the square residual error of the governing equation is dependent upon $\hbar$. It is found that, in the case of $\gamma = 0$, the residual square error decreases when $\hbar = -1$ and $\lambda = 5$. Besides, the so-called homotopy-Padé technique (Liao 2003c) is used to accelerate the convergence, and our results agree well with the numerical one given by Hartree's (1937), as shown in Table 1. This verifies the validity of the proposed analytic approach.

Secondly, we consider the case $\gamma = 1$ (suction) with $\beta = 1$, and $\kappa = 1/2$. Similarly, for given β, κ and γ, our solution series (19) depends upon two auxiliary parameters λ and $\hbar$. Figure 2 shows the influence of the auxiliary parameter λ on the solution series when $\hbar = -1$. It is found that the series solution converges for small as well as for large ξ when $\lambda \geq 5$. So, we set $\lambda = 5$. Then, it is found that the region for the convergence of the solution series (19)

[2] See http://demonstrations.wolfram.com/NumericalSolutionOfTheFalknerSkanEquationForVariousWedgeAngl/.

Table 1 Comparison between Hartree's numerical result and homotopy-Padé approximation of $f''(\xi, 0)$ when $\gamma = 0$ by means of $\lambda = 5$ and $\hbar = -1$

β	[20, 20]	[30, 30]	[40, 40]	Hartree's results
1	1.2327	1.2326	1.2326	1.2326
2	1.6872	1.6872	1.6872	1.6872

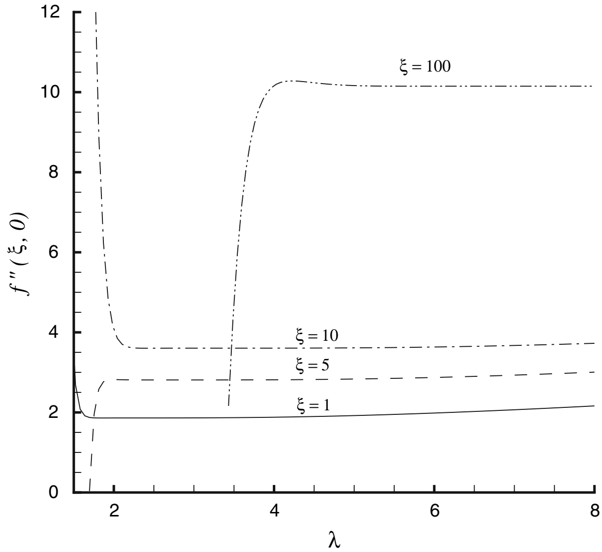


Fig. 2 The 15th-order approximation of $f''(\xi, 0)$ when $\beta = 1$, $\gamma = 1$, $\kappa = 1/2$ by means of $\hbar = -1$

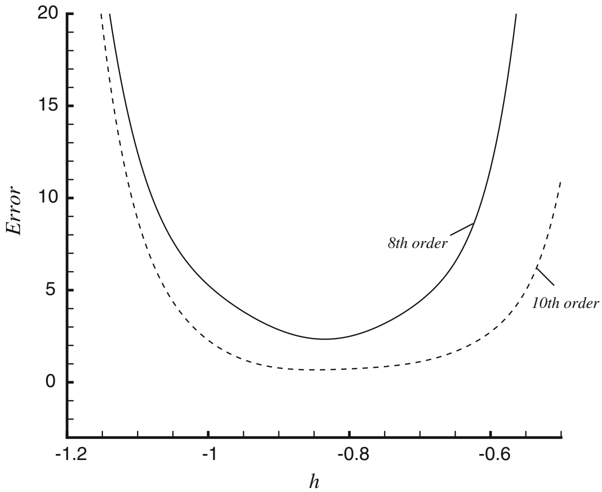


Fig. 3 The square residual error versus $\hbar$ when $\beta = 1$, $\kappa = 1/2$, $\gamma = 1$ by means of $\lambda = 5$. *Solid line* the 8th-order approximation; *dashed line* the 10th-order approximation

is about $-6/5 \leq \hbar \leq -1/2$, as shown in Fig. 3. It is clear from Fig. 4 that, when $\beta = 1$, $\gamma = 1$, and $\kappa = 1/2$, our series solution given by $\lambda = 5$ and $\hbar = -1$ converges in the domain $0 \leq \xi < \infty$ and $0 \leq \eta < \infty$, corresponding to $0 \leq x < \infty$ and $0 \leq y < \infty$, respectively. Besides, as shown in Fig. 4, our 20th and 25th order HAM results agree well with the 8th and

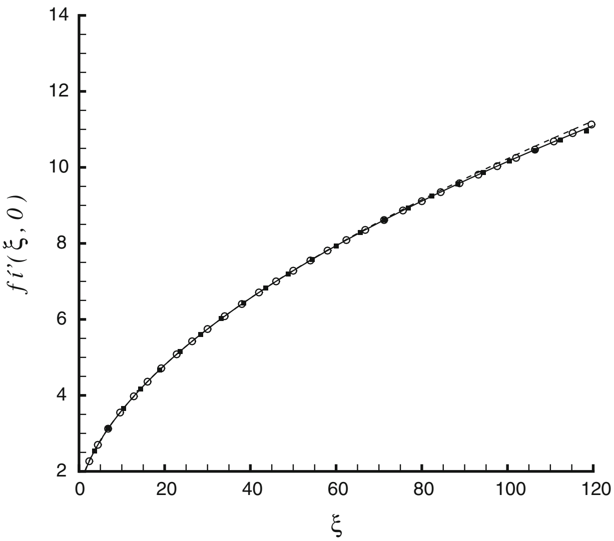


Fig. 4 Comparison of the analytic approximation of $f''(\xi, 0)$ with homotopy-Padé approximation when $\gamma = 1$, $\beta = 1$, $\kappa = 1/2$ by means of $\hbar = -1$ and $\lambda = 5$. *Solid line* [6,6] homotopy-Padé approximation; *dashed line* [4,4] homotopy-Padé approximation; *squares* 25th-order approximation; *circles* 30th-order approximation

Table 2 Square residual error when $\beta = 1$, $\gamma = 1$ and $\kappa = 1/2$ by means of $\lambda = 5$ and $\hbar = -1$

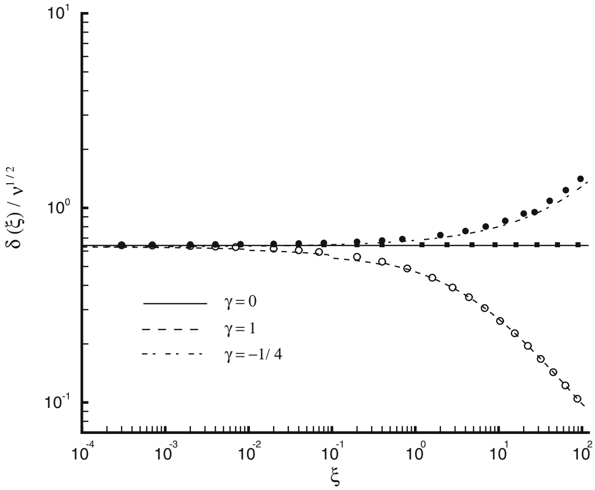
Order of approximation	Residual error
1st	2519.99
5th	2.70965
10th	0.137226
15th	0.0207309

12th order homotopy-Padé approximation. In order to ensure the convergence of the solution given by HAM, we substitute the solutions given by HAM into the original equations and evaluate the error. It is clear from Table 2 that error is decreasing by increasing the order of the original equation. This indicates that our HAM series solution is indeed convergent. Similarly, in the case of $\gamma = -1/4$ with $\beta = 1$ and $\kappa = 1/2$, our series solution converges by means of $\lambda = 5$ and $\hbar = -1$. It is found that, in general cases, our series solutions converge by means of $\lambda = 5$ and $\hbar = -1$ in the whole spatial domain.

The interesting quantities are the skin friction and the displacement thickness. Their variation with the suction/injection velocity is of importance. The injection increases the boundary-layer thickness and thus decreases the skin friction. On the other hand, the suction decreases the boundary-layer thickness and therefore gives rise to the velocity gradient, which in turn increases the skin friction.

According to Eq. 40, $f_{\eta\eta}(\xi, 0)$ is related with the skin friction. Figures 5 and 6 represent the effect of the parameter γ on the skin friction coefficient and displacement thickness for $\beta = 1, \kappa = 1/2$ by means of $\lambda = 5$ and $\hbar = -1$. It is clear from the figures that the injection increases the displacement thickness and decreases the shear stress (skin friction), but the suction decreases boundary-layer thickness and increases the skin friction. It is also noted that the non-similarity solutions are very close to the similarity ones as $\xi \to 0$ for $\kappa > 0$.

Fig. 5 Influence of γ on the displacement thickness $\delta(\xi)/\sqrt{\nu}$ when $\beta = 1$, $\kappa = 1/2$ by means of $\hbar = -1$ and $\lambda = 5$. *Solid line* 40th-order approximation for $\gamma = 0$; *dashed line* 30th-order approximation for $\gamma = 1$; *dash-dotted line* 30th-order approximation for $\gamma = -1/4$; *squares* [15,15] homotopy-Padé approximation for $\gamma = 0$; *open circles* [6,6] homotopy-Padé approximation for $\gamma = 1$, *filled circles* [6,6] homotopy-Padé approximation for $\gamma = -1/4$



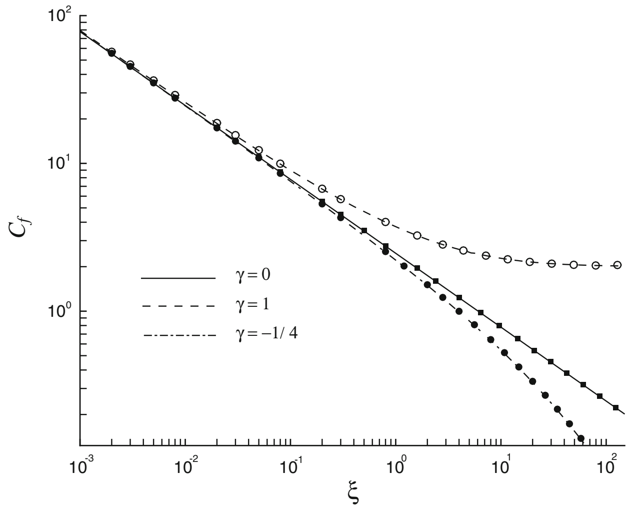


Fig. 6 Influence of γ on the skin friction coefficient C_f when $\beta = 1$, $\kappa = 1/2$ by means of $\hbar = -1$ and $\lambda = 5$. *Solid line* 40th-order approximation for $\gamma = 0$; *dashed line* 30th-order approximation for $\gamma = 1$; *dash-dotted line* 30th-order approximation for $\gamma = -1/4$; *squares* [12,12] homotopy-Padé approximation for $\gamma = 0$; *open circles* [6,6] homotopy-Padé approximation for $\gamma = 1$; *filled circles* [6,6] homotopy-Padé approximation for $\gamma = -1/4$

Figures 7 and 8 show the influence of γ on the displacement thickness and the skin friction coefficient when $\kappa = -1/4$, $\beta = 1$ by means of $\lambda = 5$ and $\hbar = -1$. Again, it is observed from these figures that the suction decreases the thickness but increases the shear stress, and the injection increases the thickness but decreases the wall shear stress. It is also observed that the non-similarity flows are close to the similarity ones as $\xi \to \infty$. Hence, the non-similarity flows in the region $\xi \to 0$ for $\kappa > 0$ and $\xi \to \infty$ for $\kappa < 0$ are very close to the similarity ones, respectively.

The influence of the parameter β on the skin friction coefficient and the boundary-layer thickness is as shown in Figs. 9 and 10 when $\gamma = 1$, $\kappa = 1/2$ by means of $\lambda = 5$ and $\hbar = -1$.

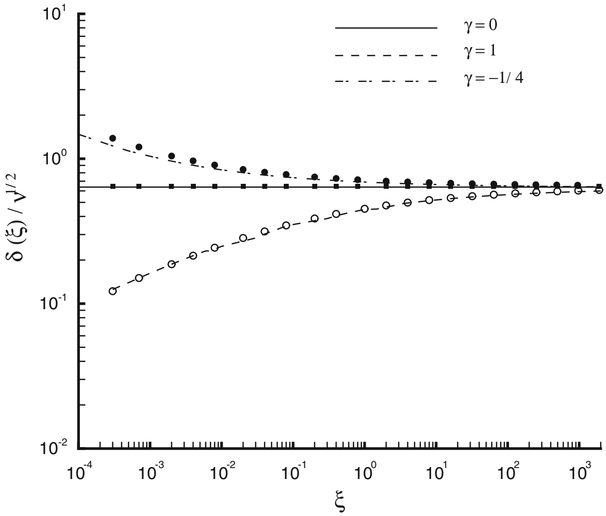


Fig. 7 Influence of γ on the displacement thickness $\delta(\xi)/\sqrt{\nu}$ when $\beta = 1$, $\kappa = -1/4$ by means of $\hbar = -1$ and $\lambda = 5$. *Solid line* 35th-order approximation for $\gamma = 0$; *dashed line* 30th-order approximation for $\gamma = 1$; *dash-dotted line* 30th-order approximation for $\gamma = -1/4$; *squares* [12,12] homotopy-Padé approximation for $\gamma = 0$; *open circles* [6,6] homotopy-Padé approximation for $\gamma = 1$; *filled circles* [6,6] homotopy-Padé approximation for $\gamma = -1/4$

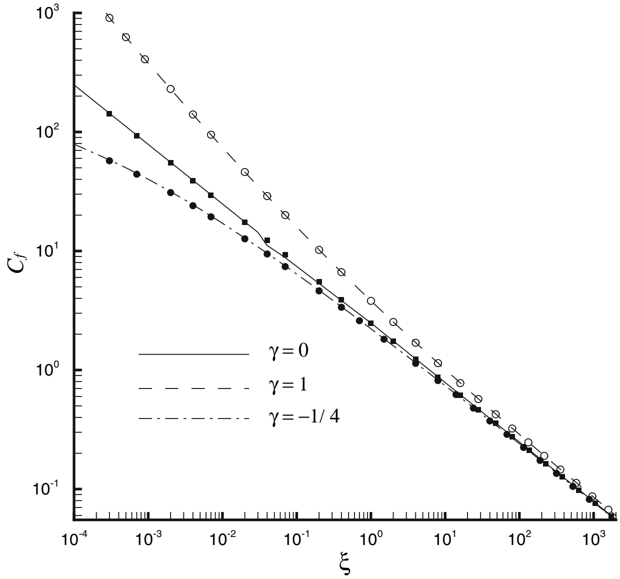
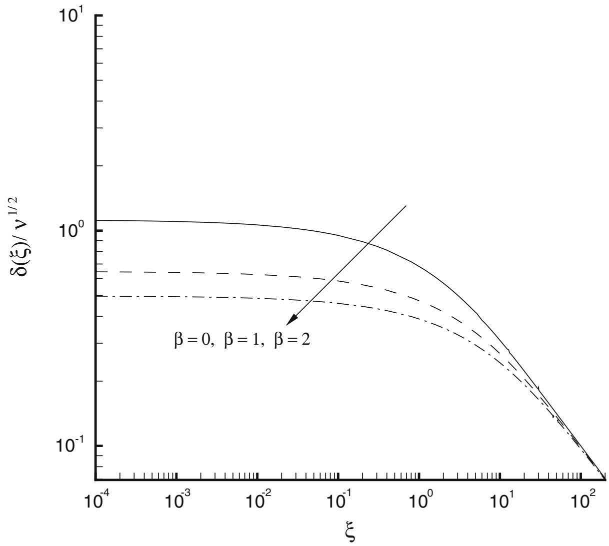


Fig. 8 Influence of γ on the skin friction coefficient C_f when $\beta = 1$, $\kappa = -1/4$ by means of $\hbar = -1$ and $\lambda = 5$. *Solid line* 35th-order approximation for $\gamma = 0$; *dashed line* 30th-order approximation for $\gamma = 1$; *dash-dotted line* 30th-order approximation for $\gamma = -1/4$; *squares* [12,12] homotopy-Padé approximation for $\gamma = 0$; *open circles* [6,6] homotopy-Padé approximation for $\gamma = 1$; *filled circles* [6,6] homotopy-Padé approximation for $\gamma = -1/4$

Note that, as β varies from 0 to 2, the friction coefficient increases but the displacement thickness decreases. In the similar way, we investigate the influence of β on the displacement thickness and skin friction in the case of $\gamma < 0$, and it is observed that the thickness increases and shear stress decreases in the case of injection. Thus by means of HAM, we obtain the analytic solutions which are uniformly valid for all the physical parameters in the whole domain.

Fig. 9 Influence of β on the displacement thickness $\delta(\xi)/\sqrt{\nu}$ when $\gamma = 1$, $\kappa = 1/2$ by means of $\hbar = -1$ and $\lambda = 5$. *Solid line* [6,6] homotopy-Padé approximation for $\beta = 0$; *dashed line* [6,6] homotopy-Padé approximation for $\beta = 1$; *dash-dotted line* [6,6] homotopy-Padé approximation for $\beta = 2$



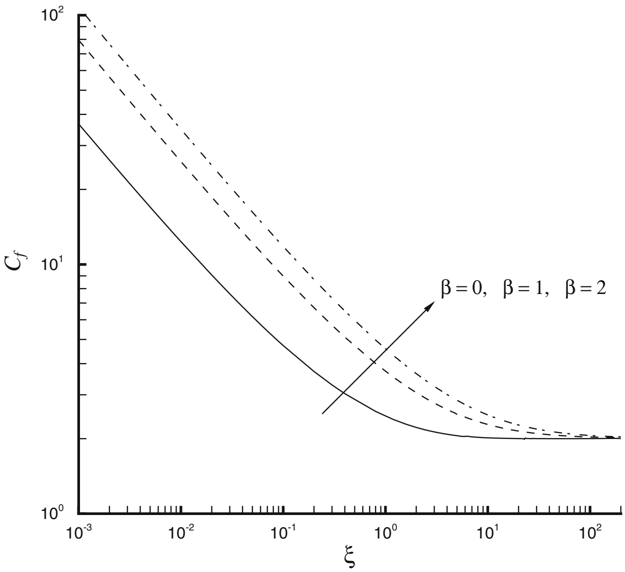


Fig. 10 Influence of β on the skin friction coefficient C_f when $\gamma = 1$, $\kappa = 1/2$ by means of $\hbar = -1$ and $\lambda = 5$. *Solid line* [6,6] homotopy-Padé approximation for $\beta = 0$; *dashed line* [6,6] homotopy-Padé approximation for $\beta = 1$; *dash-dotted line* [6,6] homotopy-Padé approximation for $\beta = 2$

4 Conclusion and Discussion

In this article, the non-similarity boundary-layer flows over a porous wedge are studied. The free stream velocity $U_w(x) \sim a\,x^m$ and the injection velocity $V_w(x) \sim b\,x^n$ at the surface are considered, which result in the corresponding non-similarity boundary-layer flows governed by a nonlinear PDE. An analytic technique for strongly nonlinear problems, namely, the HAM, is employed to obtain the series solutions of the non-similarity boundary-layer flows over a porous wedge. An auxiliary parameter is introduced to ensure the convergence of solution series. As a result, series solutions valid for all physical parameters in the whole domain are given. The effects of the physical parameters on the skin friction coefficient and displacement thickness are then investigated. To the best of our knowledge, it is the first time that the series solutions of this kind of non-similarity boundary-layer flows are reported.

Mathematically, non-similarity boundary-layer flows are governed by nonlinear PDEs. However, by means of the HAM, the nonlinear PDE related to the non-similarity boundary-layer flows is transferred into an infinite number of linear ODEs, which are easy to solve, especially by means of symbolic computation software. In physics, this is mainly because, for the boundary-layer flows, the variation cross the flow is much more larger and thus more important than the variation along the flow. So, this analytic approach has general meanings and can be used to solve other non-similarity boundary-layer flows in a similar way.

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