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A new method for homoclinic solutions of ordinary differential equations

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Abstract

Consideration is given to the homoclinic solutions of ordinary differential equations. We first review the Melnikov analysis to obtain Melnikov function, when the perturbation parameter is zero and when the differential equation has a hyperbolic equilibrium. Since Melnikov analysis fails, using Homotopy Analysis Method (HAM, see [Liao SJ. Beyond perturbation: introduction to the homotopy analysis method. Boca Raton: Chapman & Hall/CRC Press; 2003; Liao SJ. An explicit, totally analytic approximation of Blasius' viscous flow problems. Int J Non-Linear Mech 1999;34(4):759–78; Liao SJ. On the homotopy analysis method for nonlinear problems. Appl Math Comput 2004;147(2):499–513] and others [Abbasbandy S. The application of the homotopy analysis method to nonlinear equations arising in heat transfer. Phys Lett A 2006;360:109–13; Hayat T, Sajid M. On analytic solution for thin film flow of a forth grade fluid down a vertical cylinder. Phys Lett A, in press; Sajid M, Hayat T, Asghar S. Comparison between the HAM and HPM solutions of thin film flows of non-Newtonian fluids on a moving belt. Nonlinear Dyn, in press]), we obtain homoclinic solution for a differential equation with zero perturbation parameter and with hyperbolic equilibrium. Then we show that the Melnikov type function can be obtained as a special case of this homotopy analysis method. Finally, homoclinic solutions are obtained (for nontrivial examples) analytically by HAM, and are presented through graphs.

1. Introduction

The exact or reduced order nonlinear dynamics of many mechanical models or infinite-dimensional systems can be described by single-degree-of-freedom oscillators: That is, by second-order ordinary nonlinear differential equations (for examples see [7,8]).

These systems are described by the equation

$$\frac{d^2x}{dt^2} = f(x) + \varepsilon g(x, \dot{x}, t), \quad x \in \mathbb{R}. \tag{1.1}$$

This is equivalent to the first-order system

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$$\begin{bmatrix} \dot{x} = y, \\ \dot{y} = f(x) + \varepsilon g(x, y, t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2. \tag{1.2}$$

We assume that $f(x)$ and $g(x, \dot{x}, t)$ are sufficiently smooth and bounded on a bounded set, which provide the existence of a solution on the bounded set.

In (1.1) the mechanical differences between various systems are taken into account by considering different nonlinear stiffness, that is, different restoring forces $f(x)$. These forces have strong consequences in terms of the dynamical response. For example, for softening versus hardening systems; for left versus right bending of the nonlinear resonance curve; and for escape versus scattered chaotic attractor for large excitation amplitude. In spite of these important distinctions, there are some common dynamical features which permit a unified approach to the analysis of various oscillators.

The formation of homoclinic orbit is taken as one of the criteria of transition from regular to chaotic behavior in solutions of a dynamical system (for details see [9,10]).

The aim of the present work is to find a homoclinic (or heteroclinic) solution of equation (1.1). By definition, homoclinic and heteroclinic orbits correspond to the intersection of the stable and unstable manifolds of certain types of invariant sets. As a consequence, one of the earliest method to determine the conditions for the existence of such orbits is very well known through the Melnikov method (see, for example, [11,12]) which uses a function equivalent to the distance between the relevant stable and unstable manifolds. Specific criteria for the existence are obtained by making this distance equal to zero. As we notice in the following section, in order to apply this method, in Eq. (1.1) ε must be a small parameter and exact analytical solution of (1.1) must be known for $\varepsilon = 0$. It is well known that generally we can not find the exact analytical solution of second-order nonlinear differential equation, and ε in Eq. (1.1) may not be small. In these cases, literature review reveals that there is no work dealing with the analytical approximation of homoclinic orbit except the work of the Manucharyan and Mikhlin [13]. However, even this work [13] is limited to some special type of differential equations.

Hence, in this paper, for a given differential equation with a hyperbolic equilibrium, a new approach is proposed to the construction of homoclinic orbit in nonlinear dynamical systems with two-dimensional phase space. Homotopy analysis method (HAM) is used to approximate both homoclinic orbit in the phase space and the corresponding solution as a function of time. Finally, this method is applied to nontrivial examples, for which homoclinic solutions are obtained analytically and evaluated numerically.

2. Melnikov analysis

For this method, we assume that ε is a small parameter, so perturbation analysis can be developed and for $\varepsilon = 0$, the system has an orbit $x^u(t)$ (unperturbed unstable manifold) backward-asymptotic to a hyperbolic saddle point $p_0 = [p_0, 0]^T$, that is, $\lim_{t\to-\infty} x^u(t) = p_0, f(p_0) = 0$, and, $f_x(p_0) = \lambda^2$, where, $f_x(\cdot)$ means the derivative of $f(x)$ with respect to its argument x. We also note that if $\varepsilon = 0$, the system is called "unperturbed". Also, we have Hamiltonian with $H(x, \dot{x}) = \frac{\dot{x}^2}{2} + \int f(\tau)\mathrm{d}\tau - C$, where the constant C is chosen in such a way that $H(x, \dot{x}) = 0$. Hence, the orbit $x^u(t)$ can be computed from the first-order equation

$$\dot{x} = \pm\sqrt{2\left(C - \int f(\tau)\mathrm{d}\tau\right)} \quad \text{and} \quad \lim_{t\to-\infty} x^u(t) = p_0. \tag{2.1}$$

Next, it is clear that the function $t \to q_0(t) = [x^u(t), \dot{x}^u(t)]^T$ in the phase space parametrically describes unstable manifold $W^u(p_0)$ of p_0. Now, in order to see the effect of the perturbation $\varepsilon g(x, \dot{x}, t)$on the unstable orbit, as in [9, Lemma 4.5.2], we express $x_h^u(t), f(x_h^u), g(x_h^u, \dot{x}_h^u, t)$ on the interval $-\infty < t \leqslant t_0$ in the following perturbative form

$$\begin{aligned} &x_h^u(t) = x^u(t - t_0) + \varepsilon x_1^u(t) + \cdots, \\ &f(x_h^u) = f(x^u) + \varepsilon f(x^u)x_1^u + \cdots, \\ &g(x_h^u, \dot{x}_h^u, t) = g(x^u, \dot{x}^u, t) \ldots \end{aligned} \tag{2.2}$$

Substitution (2.2) into (1.1) and collecting like power of ε, we obtain

$$\{\ddot{x}^u - f(x^u)\} + \varepsilon\{\ddot{x}_1^u - f_{,x}(x^u)x_1^u - g(x^u, \dot{x}^u, t)\} + \cdots = 0. \tag{2.3}$$

Because of (2.1), first term is zero. Equating zero and the terms multiplying successive powers of ε, we get

$$\ddot{x}_1^u(t) = f_{,x}(x^u)x_1^u(t) + g(x^u, \dot{x}^u, t). \tag{2.4}$$

Since $y_1^u = \dot{x}(t - t_0)$ is one of the solutions of the homogeneous part of the equation, second solution can be found by using the Wronskian expression and Abel theorem as

$$\dot{y}_1^u(t) y_2^u(t) - \dot{y}_2^u(t) y_1^u(t) = 1. \tag{2.5}$$

Since, this is a first-order linear equation in the unknown $y_2^u(t)$ we can solve easily to obtain

$$y_2^u(t) = -y_1^u(t) \int \frac{\mathrm{d}\tau}{(y_1^u(\tau))^2}. \tag{2.6}$$

Now, using the variation of parameter method, we obtain from (2.4)

$$x_1^u(t) = c_1 y_1^u(t) + c_2 y_2^u(t) - y_2^u(t) \int_{t_0}^{t} y_1^u(t) g(x^u, \dot{x}^u, \tau) \mathrm{d}\tau + y_1^u(t) \int_{t_0}^{t} y_2^u(t) g(x^u, \dot{x}^u, \tau) \mathrm{d}\tau. \tag{2.7}$$

Since $y_1^u(t) \cong \mathrm{e}^{\lambda t}, \lambda > 0$ for $t \to -\infty$ (because p_0 is a hyperbolic fixed point), hence $y_2^u(t) \cong \mathrm{e}^{-\lambda t}$. Thus $y_1^u(t)$ vanishes and $y_2^u(t)$ is unbounded for $t \to -\infty$. The constants c_1 and c_2 can be determined from the boundary conditions. The first one is the boundedness of $x_1^u(t)$ for, which require

$$c_2^u = \int_{t_0}^{-\infty} y_1^u(\tau) g(x^u, \dot{x}^u, \tau) \mathrm{d}\tau. \tag{2.8}$$

The analysis can be repeated analogously for the *stable* manifolds. In this case assumption requires the existence of an orbit $x^s(t)$ forward-asymptotic to p_0, that is, $\lim_{t\to\infty} x^s(t) = p_0$, and it is just $t \to q_0(t) = [x^s(t), \dot{x}^s(t)]^{\mathrm{T}}$ that parametrically describes the *stable* manifold $W^s(p_0)$ of p_0 on the time interval $t_0 \leqslant t < \infty$. The solution of equation can be expressed in the perturbative form (2.2), with a simple substitution of the apex "*s*" instead of "*u*". The remaining part of the analysis is basically identical, up to this change of label. In particular, the correction terms $x_1^s(t)$ are determined by solving "the same" variational problems (2.8). A slight difference is that the vanishing of $y_1^s(t)$ and the unboundedness of $y_2^s(t)$ for $t \to +\infty$, instead of $t \to -\infty$, are used in above, hence we have

$$c_2^s = \int_{t_0}^{\infty} y_1^s(\tau) g(x^s, \dot{x}^s, \tau) \mathrm{d}\tau. \tag{2.9}$$

The function c_2^s is conceptually and numerically distinct from c_2^u. Finally, Melnikov function is defined as distance between the relevant stable and unstable manifolds. Thus, $M(t_0) = c_2^s(t_0) - c_2^u(t_0)$, hence, we get

$$M(t_0) = \int_{-\infty}^{\infty} \dot{x}^\varepsilon(\tau) g(x^\varepsilon(\tau), \dot{x}^\varepsilon(\tau), \tau + t_0) \mathrm{d}\tau. \tag{2.10}$$

This is the classical Melnikov function (see [9,10,12]). However, in practice, the main difficulty involved in above analysis is to find the exact analytical solution for homogeneous part of (1.1). In this case, either, one can use numerical method to find numerical solution to that equation, or one can find approximate analytical solution. We also note that above analysis depend on ε which must be small parameter for the perturbation analysis. In this work, we use the Homotopy Analysis Method (HAM), for which, we do not require small parameter; hence, there is no restriction on h. Furthermore, Melnikov type function can be obtained as a special case of our study.

3. Homotopy analysis method

Consider the nonlinear differential equation in general form

$$\widetilde{N}[x(t)] = 0, \tag{3.1}$$

where $\widetilde{N}$ is a differential operator and $x(t)$ is a solution. Applying the HAM to solve it, we first need to construct the following family of equations:

$$(1 - q)\{L[\theta(t, q) - x_0(t)]\} = \hbar q \widetilde{N}[\theta(t, q)], \tag{3.2}$$

where L is a properly selected auxiliary linear operator satisfying

$$L(0) = 0, \tag{3.3}$$

$\hbar \neq 0$ is an auxiliary parameter, and $x_0(t)$ is an initial approximation. Obviously, Eq. (3.2) gives

$$\theta(t, 0) = x_0(t), \tag{3.4}$$

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when $q = 0$. Similarly, when $q = 1$, Eq. (3.2) is the same as Eq. (3.1) so that we have

$$\theta(t,1) = x(t). \tag{3.5}$$

Suppose that Eq. (3.2) has solution $\theta(t,q)$ that converges for all $0 \leqslant q \leqslant 1$ and for properly selected $\hbar$ and the auxiliary linear operator L. Suppose further that $\theta(t,q)$ is infinitely differentiable at $q = 0$, that is

$$\left.\frac{\partial^k \theta(t,q)}{\partial q^k}\right|_{q=0}, \quad k = 1,2,3,\ldots \tag{3.6}$$

exists. Thus, as q increases from 0 to 1, the solution $\theta(t,q)$ of Eq. (3.2) varies continuously from the initial approximation $x_0(t)$ to the solution $x(t)$ of the original Eq. (3.1). Clearly, Eqs. (3.4) and (3.5) give an indirect relation between the initial approximation $x_0(t)$ and the general solution $x(t)$. A direct relationship between the two solutions is described as follows. Consider the Maclaurin's series of $\theta(t,q)$ about q as

$$\theta(t,q) = \theta(t,0) + \sum_{k=1}^{\infty} x_k(t) q^k, \tag{3.7}$$

where

$$x_k(t) = \left.\frac{1}{k!}\frac{\partial^k \theta(t,q)}{\partial q^k}\right|_{q=0}. \tag{3.8}$$

Assume that the series (3.7) converges at $q = 1$. From Eqs. (3.4), (3.5) and (3.7), we have the relationship

$$x(t) = x_0(t) + \sum_{k=1}^{\infty} x_k(t). \tag{3.9}$$

Liao [1] provides a general approach to derive the governing equation of $x_m(t)$. Recently, an equivalent approach is given by Hayat et al. [5,6]. Substituting the series (3.7) into Eq. (3.2) and equating the coefficient of the like power of q, we get the mth-order deformation equations

$$L[x_m(t) - \chi_m x_{m-1}(t)] = \hbar R_m(t), \tag{3.10}$$

where

$$R_m(t) = \frac{1}{(m-1)!}\left.\frac{\mathrm{d}^{m-1}\widetilde{N}[\theta(t,q)]}{\mathrm{d}q^{m-1}}\right|_{q=0} \tag{3.11}$$

and

$$\chi_k = \begin{cases} 0, & k \leqslant 1, \\ 1, & k \geqslant 2. \end{cases} \tag{3.12}$$

It is very important to emphasize that Eq. (3.10) is linear. If the first $(m-1)$th-order approximations have been obtained, then the right-hand side $R_m(t)$ will be obtained. So, using the selected initial approximation $x_0(t)$, we can obtain $x_1(t), x_2(t), x_3(t), \ldots$, one after the other in order. Therefore, according to Eq. (3.9), we convert the original nonlinear problem into an infinite sequence of linear sub-problems governed by Eq. (3.10).

We now consider Eq. (1.1) for the following cases:

Case 1. If$f(x) = \lambda^2 x$, then we choose L operator as

$$L(x) = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - \lambda^2 x \tag{3.13}$$

and hence the nonlinear equation will be

$$\widetilde{N}[\theta(t,q)] = \ddot{\theta} - \lambda^2\theta - \varepsilon g(\theta, \dot{\theta}, t), \tag{3.14}$$

where the dot denotes the derivatives with respect to the time t . Besides, we select

$$x_0(t) = \begin{cases} p_0 \mathrm{e}^{\lambda t}, & -\infty < t \leqslant 0, \\ p_0 \mathrm{e}^{-\lambda t}, & 0 \leqslant t < \infty \end{cases} \tag{3.15}$$

as the initial guess of $x(t)$, where p_0 is a hyperbolic fixed point.

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Following the setting given by Eq. (3.2), one has the zero-order deformation equation:

$$(1-q)L[\theta(t,q)-x_0(t)] = p\hbar\widetilde{N}[\theta(t,q)].$$

Using (3.10), we have the kth-order deformation equation

$$L[x_k(t)-\chi_k x_{k-1}(t)] = \hbar R_k(t), \tag{3.16}$$

where

$$R_1 = \ddot{x}_0(t) - \lambda^2 x_0(t) - \varepsilon g(x_0(t),\dot{x}_0(t),t),$$

$$R_m = \ddot{x}_{m-1}(t) - \lambda^2 x_{m-1}(t) - \frac{\varepsilon}{(m-1)!}\left\{\frac{\mathrm{d}^{m-1}}{\mathrm{d}q^{m-1}} g\left(\sum_{k=0}^{+\infty} x_k(t)q^k, \sum_{k=0}^{+\infty} \dot{x}_k(t)q^k, t\right)\right\}\Bigg|_{q=0}, \quad m \geqslant 2.$$

For $k=1$, we obtain

$$x_1(t) = c_1 \mathrm{e}^{\lambda t} + c_2 \mathrm{e}^{-\lambda t} - \frac{\varepsilon}{2\lambda}\left[\mathrm{e}^{\lambda t}\int_{t_0}^{t} \mathrm{e}^{-\lambda\tau} g(x_0,\dot{x}_0,\tau)\mathrm{d}\tau - \mathrm{e}^{-\lambda t}\int_{t_0}^{t} \mathrm{e}^{\lambda\tau} g(x_0,\dot{x}_0,\tau)\mathrm{d}\tau\right]. \tag{3.17}$$

This equation is exactly the same as that in (2.10), exactly same analysis and same condition at infinity as in previous section, for $t_0 = 0$ will apply. Hence, we get

$$M_1(t_0) = \int_{-\infty}^{\infty} x^{\varepsilon}(\tau) g(x^{\varepsilon}(\tau),\dot{x}^{\varepsilon}(\tau),\tau)\mathrm{d}\tau, \tag{3.18}$$

where

$$x^{\varepsilon}(\tau) = \begin{cases} \mathrm{e}^{\lambda\tau}, & -\infty < \tau \leqslant 0, \\ \mathrm{e}^{-\lambda\tau}, & 0 \leqslant \tau < \infty. \end{cases}$$

We note that this analysis does not depend on small parameter like ε. If ε is small enough, Eq. (3.17) gives accurate information for homoclinic orbit, if there is no restriction on ε, we go on calculating consecutive terms in Eq. (3.9) until the last calculated term in small enough.

Case 2. If p_0 is a hyperbolic fixed point of (1.1) , without loosing generality, we may assume that $p_0 = 0$ then we need to choose L operator such as solution of $L(x)$ must satisfy condition at infinity. Hence, we can choose L operator as

$$L(x) = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - \lambda^2 x. \tag{3.19}$$

The nonlinear equation will be

$$\widetilde{N}[\theta(t,q)] = \ddot{\theta} - f(\theta) - \varepsilon g(\theta,\dot{\theta},t). \tag{3.20}$$

Following the setting given by Eq. (3.2), one has

$$(1-q)L[\theta(t,q)-x_0(t)] = q\hbar\widetilde{N}[\theta(t,q)]. \tag{3.21}$$

According to (3.10), we have the kth-order deformation equation

$$L[x_k(t)-\chi_k x_{k-1}(t)] = \hbar R_k(t), \tag{3.22}$$

where

$$R_k(t) = \frac{1}{(k-1)!}\left\{\frac{\mathrm{d}^{k-1}\widetilde{N}[\theta(t,q)]}{\mathrm{d}q^{k-1}}\right\}\Bigg|_{q=0}.$$

For $k=1$, we obtain

$$x_1(t) = c_1 \mathrm{e}^{\lambda t} + c_2 \mathrm{e}^{-\lambda t} - \frac{\hbar}{2\lambda}\left[\mathrm{e}^{\lambda t}\int_{t_0}^{t} \mathrm{e}^{-\lambda\tau}(\ddot{x}_0 - f(x_0) - \varepsilon g(x_0,\dot{x}_0,\tau))\mathrm{d}\tau - \mathrm{e}^{-\lambda t}\int_{t_0}^{t} \mathrm{e}^{\lambda\tau}(\ddot{x}_0 - f(x_0) - \varepsilon g(x_0,\dot{x}_0,\tau))\mathrm{d}\tau\right]. \tag{3.23}$$

It is obvious that the condition on ε does not affect the calculation, whether is small or not, we need to calculate several terms in Eq. (3.9) until the last calculated terms small enough which means we must obtain convergent series. In the

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following section, we give several nontrivial examples which show the Homotopy Analysis Method can be used to obtain the homoclinic orbits.

4. Examples

To illustrate the features of the proposed approach to the homoclinic solutions of differential equations, in this section we consider one preliminary example where computations can be done analytically. Two more interesting examples will be discussed. We initially investigate the simplest case, i.e. the unforced and undamped Duffing equation

$$\ddot{x} - x + x^3 = 0. \tag{4.1}$$

This equation has the exact homoclinic solution $x(t) = \sqrt{2}\sec h(t)$. We use HAM to this problem, for $\lim_{t\to\pm\infty}x(t) = 0$ and $\dot{x}(0) = 0$ as in [15]. Now, we consider Eq. (4.1) subject to following boundary conditions:

$$\dot{x}(0) = 0 \quad \text{and} \quad x(\pm\infty) = 0. \tag{4.2}$$

Write $x(0) = a$, where a is unknown and will be determined in the way described later. Substitute $x(t) = ay(t)$ into (4.1), for $t \geqslant 0$, we get

$$\ddot{y} - y + \gamma y^3 = 0 \tag{4.3}$$

subject to the initial/boundary conditions

$$y(0) = 1, \quad \dot{y}(0) = 0, \quad y(+\infty) = 0, \tag{4.4}$$

where $\gamma = a^2$ is to be determined later.

Obviously, the homoclinic solution can be expressed by the set of base functions

$$y(t) = \sum_{m=1}^{+\infty} c_m e^{-mt}, \tag{4.5}$$

where c_m is a coefficient. This provides us with the so-called *solution expression*, as described in [1]. According to the solution expression (4.5) and the initial/boundary conditions (4.4), it is straightforward to choose

$$y_0(t) = 2e^{-t} - e^{-2t} \tag{4.6}$$

as the initial guess of $y(t)$, and besides to choose

$$Ly = \ddot{y} - y \tag{4.7}$$

as the auxiliary linear operator, which has the property

$$L[C_1e^{-t} + C_2e^{t}] = 0 \tag{4.8}$$

for all C_1 and C_2. Let $\hbar$ denote an auxiliary parameter, $q \in [0, 1]$ an embedding parameter, respectively. For simplicity, write

$$N[Y(t;q), \Gamma(q)] = \frac{\partial^2 Y(t;q)}{\partial t^2} - Y(t;q) + \Gamma(q)Y^3(t;q). \tag{4.9}$$

Then, we construct the zeroth-order deformation equation

$$(1-q)L[Y(t;q) - y_0(t)] = q\hbar N[Y(t;q), \Gamma(q)] \tag{4.10}$$

subject to the boundary conditions

$$Y(0;q) = 1, \quad \dot{Y}(0,q) = 0, \quad Y(+\infty;q) = 0. \tag{4.11}$$

Obviously, we have

$$Y(t;0) = y_0(t) \tag{4.12}$$

and

$$Y(t;1) = y(t), \quad \Gamma(1) = \gamma. \tag{4.13}$$

Expanding $Y(t;q)$ and $\Gamma(q)$ into Maclaurin series of q, we have

$$Y(t;q) = y_0(t) + \sum_{k=1}^{+\infty} y_k(t)q^k, \tag{4.14}$$

$$\Gamma(q) = \gamma_0 + \sum_{k=1}^{+\infty} \gamma_k q^k, \tag{4.15}$$

where

$$y_k(t) = \frac{1}{k!}\frac{\partial^k Y(t;q)}{\partial q^k}\bigg|_{q=0}, \quad \gamma_k = \frac{1}{k!}\frac{\partial^k \Gamma(q)}{\partial q^k}\bigg|_{q=0}.$$

Here, $\gamma_0 = \Gamma(0)$ is an initial approximation of γ and is unknown. Then, we have the relationships

$$y(t) = y_0(t) + \sum_{k=1}^{+\infty} y_k(t), \tag{4.16}$$

$$\gamma = \gamma_0 + \sum_{k=1}^{+\infty} \gamma_k. \tag{4.17}$$

Substituting (4.14) and (4.15) into (4.10) and (4.11), and equating the like power of q, we have the mth-order deformation equation

$$L[y_m(t) - \chi_m y_{m-1}(t)] = \hbar R_m(t) \tag{4.18}$$

subject to the boundary conditions

$$y_m(0) = 0, \quad \dot{y}_m(0) = 0, \quad y_m(+\infty) = 0, \tag{4.19}$$

where

$$R_m(t) = \ddot{y}_{m-1}(t) - y_{m-1}(t) + \sum_{j=0}^{m-1} \alpha_j(t)\beta_{m-1-j}(t), \tag{4.20}$$

$$\alpha_n(t) = \sum_{j=0}^{n} \gamma_j y_{n-j}(t), \tag{4.21}$$

$$\beta_n(t) = \sum_{j=0}^{n} y_j(t) y_{n-j}(t). \tag{4.22}$$

When $m = 1$, we have the first-order deformation equation

$$L[y_1(t)] = \hbar[-3e^{-2t} + \gamma_0(8e^{-3t} - 12e^{-4t} + 6e^{-5t} - e^{-6t})] \tag{4.23}$$

subject to the boundary conditions

$$y_1(0) = 0, \quad \dot{y}_1(0) = 0, \quad y_1(+\infty) = 0. \tag{4.24}$$

The above equation has the general solution

$$y_1(t) = \hbar\left[-e^{-2t} + \gamma_0\left(e^{-3t} - \frac{4}{5}e^{-4t} + \frac{1}{4}e^{-5t} - \frac{1}{35}e^{-6t}\right)\right] + C_1 e^{-t} + C_2 e^{t}.$$

Using $y_1(0) = 0, y_1(+\infty) = 0$, we have $C_1 = \hbar\left(1 - \frac{59\gamma_0}{140}\right)$ and $C_2 = 0$. Then, enforcing $\dot{y}_1(0) = 0$, we obtain

$$\gamma_0 = \frac{35}{16}. \tag{4.25}$$

The corresponding solution is

$$y_1(t) = \hbar\left[\frac{5}{64}e^{-t} - e^{-2t} + \frac{35}{16}e^{-3t} - \frac{7}{4}e^{-4t} + \frac{35}{64}e^{-5t} - \frac{1}{16}e^{-6t}\right]. \tag{4.26}$$

Similarly, we can get $\gamma_1, y_2(t), \gamma_2, y_3(t), \ldots$ and so on.

It should be emphasized that our results contain the auxiliary parameter $\hbar$, which provides us with a simple way to ensure the convergence of our series solutions. Note that γ is a power series of $\hbar$. It is found that, at the 10th-order approximation, γ converges to the same value in the region $-2 < \hbar < 0$, as shown in Fig. 1 for the so-called γ–$\hbar$ curve. This is indeed true: γ converges to 2 when $\hbar = -1$, as shown in Table 1.

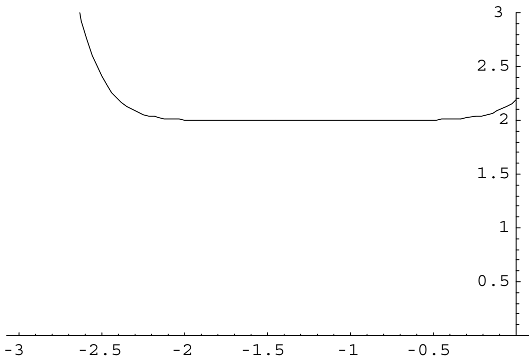

Fig. 1. The 10th-order approximation of γ versus $\hbar$.

Table 1
γ and $x(0)$ by the HAM when $\hbar = -1$

Order of approximation	γ	$x(0)$
0	2.18750	1.47902
2	2.02646	1.42354
4	2.00563	1.41620
6	2.00134	1.41469
8	2.00034	1.41433
10	2.00017	1.41425
12	2.00002	1.41422
14	2.00001	1.41422
16	2.00000	1.41421
18	2.00000	1.41421
20	2.00000	1.41421

Numerically, it is far more difficult to find homoclinic solution of the type of differential equation in (4.1). However, by using the homotopy analysis method (HAM), one can find homoclinic solution (if there is one). Exact and HAM series solution of the homoclinic orbit are given in Fig. 2. We notice that the difference between the exact and the HAM solution is less than 10^{-10} for 16th-order approximation, are shown in Fig. 2.

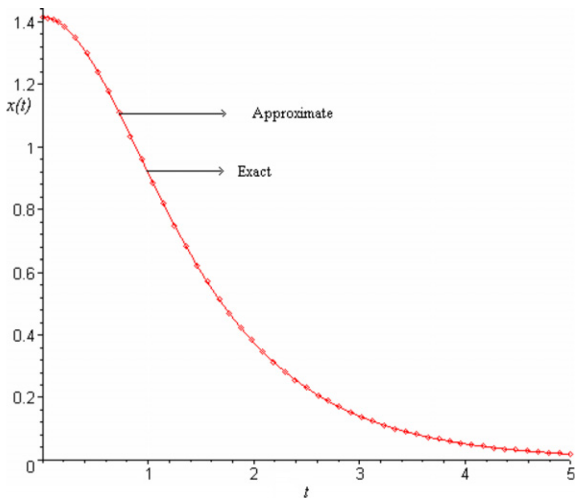

Fig. 2. Comparison of the exact and HAM sereis solution $x(t)$ versus t for Example 1.

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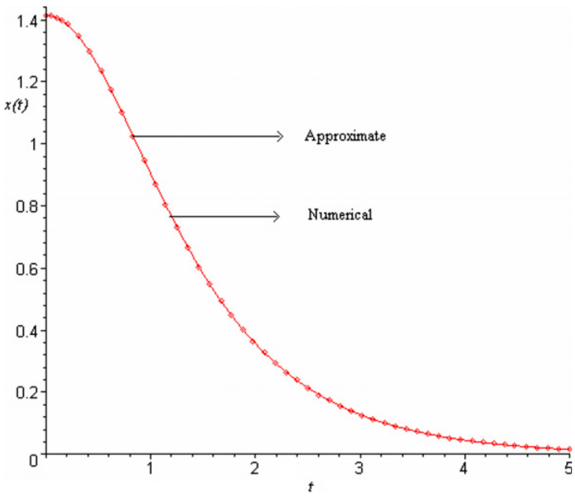

Fig. 3. Comparison of the numerical and HAM series solutions $x(t)$ versus t for Example 2 when $\delta = 0.1$, $\alpha = 0.8$ and $\beta = 1$.

Table 2
γ and $x(0)$ by the HAM when $\hbar = -1$

Order of approximation	γ	$x(0)$
0	2.16032	1.46980
2	2.02048	1.42144
4	2.00458	1.41583
6	2.00144	1.41472
8	2.00074	1.41447
10	2.00057	1.41444

Our second example is the well known Van Der Pol–Duffing equation [13]

$$\ddot{x} + \delta(\alpha - \beta x^2)\dot{x} - x + x^3 = 0; \quad \alpha, \beta > 0, \ 0 < \delta \ll 1. \tag{4.27}$$

Once again, our boundary conditions for the homoclinic solution are $\lim_{t\to\pm\infty} x(t) = 0$ and $\dot{x}(0) = 0$. The result can be seen in Fig. 3 for the parametric values of $\delta = 0.1$, $\alpha = 0.8$ and $\beta = 1$. Again as in Example 1, we look at following problem:

$$\begin{aligned} &\ddot{x} + \delta(\alpha - \beta x^2)\dot{x} - x + x^3 = 0, \\ &x(0) = a, \quad \dot{x}(0) = 0 \quad \text{and} \quad x(\infty) = 0 \end{aligned} \tag{4.28}$$

and we change the dependent variable as $x(t) = ay(t)$ to obtain

$$\ddot{y} + \delta(\alpha - \beta\gamma x^2)\dot{y} - y + \gamma y^3 = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0 \quad \text{and} \quad y(\infty) = 0, \tag{4.29}$$

where $\gamma = a^2$ is unknown. Similarly, when $h = -1$, we have the results listed in Table 2.

Obviously, $x(0)$ tends to $\sqrt{2}$. We also carried out numerical computation for which initial conditions are taken from our analytical solution of the homoclinic orbit, i.e. $x(0) = \sqrt{2}, \dot{x}(0) = 0$. The numerical solution agrees well with the HAM result, as shown in Fig. 3.

Our last example is

$$\ddot{x} - x + 2x^3 - \varepsilon\dot{x}\sin(\varepsilon t) = 0$$

for details see Ref. [14]. Similarly, we get the HAM series solution of the homoclinic orbit, which agrees well with the corresponding numerical result by means of Runge–Kutta's method and the analytic value of $x(0)$ and the initial condition $\dot{x}(0) = 0$, as shown in Fig. 4. This problem demonstrates that the homotopy analysis method is effective in solving the infinite interval problems. In Figs. 2–4, we considered the interval $[0, \infty)$; however, the analysis can be extended easily to the interval $(-\infty, 0]$ as well. Another main advantage of HAM method is that it gives the analytical solution.

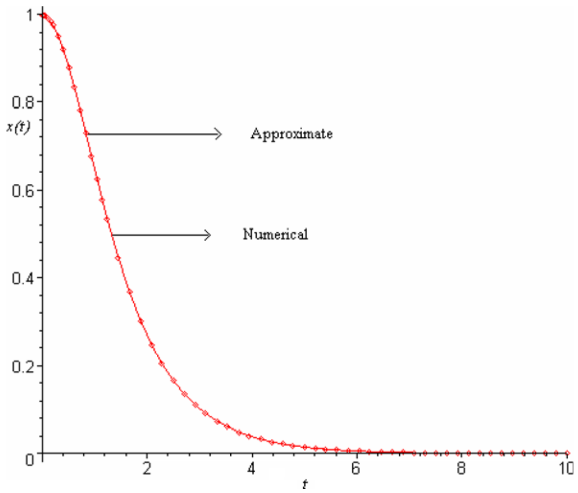

Fig. 4. Comparison of the numerical and HAM series solution $x(t)$ versus t for Example 3 when $\varepsilon = 0.1$.

5. Conclusions

In this paper, we consider the homoclinic solutions of ordinary differential equations. We first review the Melnikov analysis to obtain Melnikov function, when the perturbation parameter is zero and when the differential equation has a hyperbolic equilibrium. Since Melnikov analysis fails, using Homotopy Analysis Method [1–6], we obtain homoclinic solution for a differential equation with zero perturbation parameter and with hyperbolic equilibrium. Then we show that the Melnikov type function can be obtained as a special case of the homotopy analysis method. Finally, homoclinic solutions are obtained (for nontrivial examples) analytically by HAM, and are presented through graphs.

This work verifies the validity and the potential of the homotopy analysis method for investigating homoclinic orbits of nonlinear oscillation systems. This analytic approach can be easily extended to give heteroclinic solutions of a nonlinear differential equation.

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