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A SHORT NOTE ON HIGH-ORDER STREAMFUNCTION–VORTICITY FORMULATIONS OF 2D STEADY STATE NAVIER–STOKES EQUATIONS

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SUMMARY

In this paper the high-order formulations described by Liao (*Int. j. numer. methods fluids*, **15**, 595–612 (1992)) are proved to be stable for viscous flow under high Reynolds number. As an example, results for shear-driven flow in a square cavity at Reynolds numbers up to 10,000 are given.

KEY WORDS: 2D Navier–Stokes equations; convergence under high *Re*; homotopy; BEM; FDM

1. INTRODUCTION

Many researchers are now applying the boundary element method (BEM) to solve non-linear problems such as the Navier–Stokes equations.[1–9] For 2D viscous flow, Rodriguez-Prada *et al.*[7] described a method for the 2D Navier–Stokes equations which is based on a set of fundamental solutions providing a complete coupling between the streamfunction and vorticity equations so that iteration is not needed in the case $Re = 0$. The non-linear terms are considered as inhomogeneities and treated by simple direct iteration. However, this numerical scheme is unstable in the case $Re > 300$ for shear-driven flow in a square cavity.[7]

Applying the homotopy technique together with Taylor series theory, Liao[10] described high-order streamfunction–vorticity BEM formulations of the 2D steady state Navier–Stokes equations. His first-order formulations are the same as those described in Reference 7 and also do not give stable results in the case $Re > 300$ for cavity flow. However, convergent results at Re up to 2000 for shear-driven cavity flow have been obtained by means of his second-order formulations.[10]

In this paper we will go along the same way as described in Reference 10. We will prove that the high-order formulations given in Reference 10 are still stable at high Reynolds numbers up to 10,000 for shear-driven cavity flow.

2. MAIN MATHEMATICAL FORMULATIONS

The two-dimensional Navier–Stokes equations in terms of the streamfunction ψ and vorticity ω are

$$\nabla^2\omega = Re\left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right), \quad (x, y) \in \Omega, \tag{1}$$

$$\nabla^2\psi + \omega = 0, \quad (x, y) \in \Omega, \tag{2}$$

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with boundary conditions

$$\psi = \psi_b \quad \text{on } \Gamma, \tag{3}$$

$$\frac{\partial \psi}{\partial n} = \left(\frac{\partial \psi}{\partial n}\right)_b \quad \text{on } \Gamma, \tag{4}$$

where Ω denotes the domain of the flow and Γ denotes the boundary. All the above equations are expressed in dimensionless form. *Re* is the Reynolds number.

In Reference 10 Liao derived the high-order iterative formulations

$$\omega_{k+1}(x,y) = \omega_k(x,y) + \sum_{m=1}^{M} \left(\frac{\omega_k^{[m]}(x,y)}{m!}\right) p^m, \quad (x,y) \in \Omega, \quad p \in (0,1], \tag{5}$$

$$\psi_{k+1}(x,y) = \psi_k(x,y) + \sum_{m=1}^{M} \left(\frac{\psi_k^{[m]}(x,y)}{m!}\right) p^m, \quad (x,y) \in \Omega, \quad p \in (0,1], \tag{6}$$

where $\psi_k^{[m]}(x,y)$ and $\omega_k^{[m]}(x,y)$ ($m \geqslant 1$), called *mth-order deformation derivatives*, satisfy the equations

$$\nabla^2 \omega_k^{[m]}(x,y) = \mathscr{H}_m(x,y) - \mathscr{D}_m \nabla^2 \omega_k(x,y), \quad (x,y) \in \Omega, \tag{7}$$

$$\nabla^2 \psi_k^{[m]}(x,y) + \omega_k^{[m]}(x,y) = -\mathscr{D}_m[\nabla^2 \psi_k(x,y) + \omega_k(x,y)], \quad (x,y) \in \Omega, \tag{8}$$

and the corresponding boundary conditions

$$\psi_k^{[m]}(x,y) = (\psi_b - \psi_k)\mathscr{D}_m, \quad (x,y) \in \Gamma, \tag{9}$$

$$\frac{\partial \psi_k^{[m]}(x,y)}{\partial n} = \left[\left(\frac{\partial \psi}{\partial n}\right)_b - \frac{\partial \psi_k}{\partial n}\right]\mathscr{D}_m, \quad (x,y) \in \Gamma. \tag{10}$$

Here

$$\mathscr{D}_m = \begin{cases} 1 & \text{when } m = 1, \\ 0 & \text{when } m > 1, \end{cases} \tag{11}$$

and for any $(x,y) \in \Omega$ we have

$$\mathscr{H}_1(x,y) = Re\left(\frac{\partial \psi_k}{\partial y}\frac{\partial \omega_k}{\partial x} - \frac{\partial \psi_k}{\partial x}\frac{\partial \omega_k}{\partial y}\right), \tag{12}$$

$$\mathscr{H}_m(x,y) = mRe\sum_{i=0}^{m-1}\binom{i}{m-1}\left(\frac{\partial \psi_k^{[i]}}{\partial y}\frac{\partial \omega_k^{[m-1-i]}}{\partial x} - \frac{\partial \psi_k^{[i]}}{\partial x}\frac{\partial \omega_k^{[m-1-i]}}{\partial y}\right) \quad (m > 1), \tag{13}$$

where $\psi_k(x, y)$ and $\omega_k(x, y)$ are approximations of the streamfunction and vorticity respectively after k time iterations ($k = 0, 1, 2, \ldots$). Note that for any value of m ($m \geqslant 1$), $\mathscr{H}_m(x, y)$ is a known function. Thus, as mentioned in References 7 and 10, equations (7)–(10) can be solved by the BEM. The corresponding BEM formulations are described in detail in Reference 10.

3. NUMERICAL RESULTS

As mentioned in Reference 10, in the case $M=1$ the formulations described above will give the same BEM expressions as those given in Reference 7, which are unstable at $Re > 300$ for shear-driven cavity flow. However, the second-order formulations can give convergent results at Re up to 2000. Naturally, we would like to know whether or not the high-order formulations can give convergent results at higher Re.

The high-order iterative formulations (5) and (6) and the corresponding equations (7)–(10) for *deformation derivatives* $\psi_k^{[m]}(x,y)$ and $\omega_k^{[m]}(x,y)$ $(m \geqslant 1)$ are the key to the proposed method. Obviously equations (7) and (8) are linear and $\aleph_m(x, y)$ $(m \geqslant 1)$ are known functions, so that either the BEM or the finite difference method (FDM) can be easily applied to solve these equations.

In Reference 10 the BEM is used with a rather coarse numerical grid for the corresponding 2D integral. However, a very fine numerical grid is necessary for high-Re flow. Thus, for the sake of computational efficiency, we use in this paper the FDM to solve equations (7)–(10). (The computational efficiency is especially important for us because we use a PC (COMPAQ P4/50, 8 MB RAM) as a computational tool.) Multigrid techniques[13–18] are used to accelerate the iterations. The nine-point restriction operator and the nine-point prolongation operator are applied for the multigrid method. Second-order-accurate central finite difference approximations are used for the first- and second-order derivatives of equations (7) and (8). The second-order approximation formulation for the vorticity on the boundary Γ, as described in Reference 11, is used.

We define

$$(RMS_\omega)^2 = \frac{1}{N \times N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[\frac{1}{Re} \nabla^2 \omega_k(x_i, y_j) - \left(\frac{\partial \psi_k(x_i, y_j)}{\partial y} \frac{\partial \omega_k(x_i, y_j)}{\partial x} - \frac{\partial \psi_k(x_i, y_j)}{\partial x} \frac{\partial \omega_k(x_i, y_j)}{\partial y}\right)\right]^2 \tag{14}$$

and

$$(RMS_\psi)^2 = \frac{1}{N \times N} \sum_{i=1}^{N} \sum_{j=1}^{N} [\nabla^2 \psi_k(x_i, y_j) + \omega_k(x_i, y_j)]^2 \tag{15}$$

as the convergence criteria, where $N \times N$ is the grid for shear-driven cavity flow. Similarly as in Reference 11, we use a 129×129 grid for $Re \leqslant 3200$ but a 257×257 grid for higher Reynolds numbers. As mentioned in Reference 10, the radius of convergence of the Taylor series (5) and (6) reduces as Re increases. Thus we select a value of Δp as the step spacing and then find the best value of p, which is some multiple of Δp.

Using the FDM and the same values of Δp (at the corresponding Re) as those in Reference 10, we find that the first-order formulations ($M=1$) are stable at $Re \leqslant 300$ but unstable at $Re \geqslant 400$. However, similarly as in Reference 10, the second-order formulations ($M=2$) are stable even at Re up to 10,000. The history of the errors during the iterative algorithm using the first- and second-order formulations at $Re=400$, 1000 and 3200 (127×127 grid) is shown in Figure 1. This result seems to imply that from the viewpoint of iteration convergence, whether or not the high-order formulations are applied is much more important than whether the BEM or FDM is selected to solve the corresponding linear equations (7)–(10). Note that these formulations are also suited to the BEM, as described in Reference 10. Thus we can be reasonably confident that we would also obtain convergent results in the case $Re \leqslant 10{,}000$ by applying the BEM.

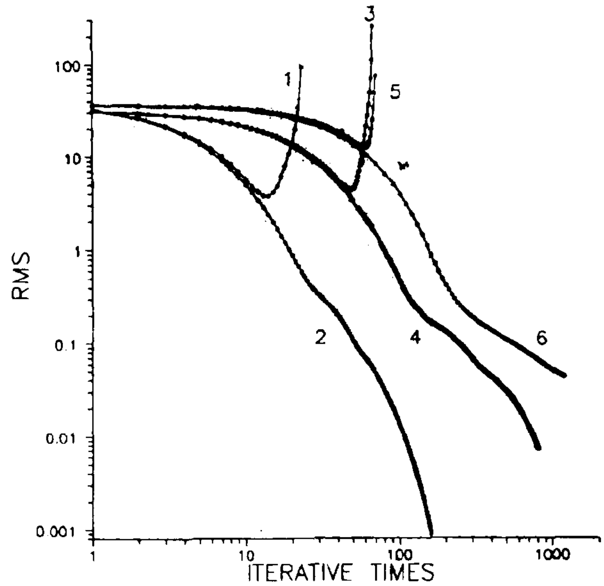

Figure 1. History of errors during iterative algorithm: curve 1, $Re=400$, $M=1$; curve 2, $Re=400$, $M=2$; curve 3, $Re=1000$, $M=1$; curve 4, $Re=1000$, $M=2$; curve 5, $Re=3200$, $M=1$; curve 6, $Re=3200$, $M=2$. $RMS=\sqrt{\{[(RMS_\omega)^2+(RMS_\psi)^2]/2\}}$

All results described below are obtained by the second-order formulations ($M=2$). The values of Δp, the corresponding iteration numbers and the root-mean-square errors RMS_ψ and RMS_ω at $Re=400$, 1000, 3200, 5000, 7500 and 10,000 are given in Table I. The results for the streamfunction $\psi_{v,c}$ and vorticity $\omega_{v,c}$ at the vorticity centre (x_c, y_c) are given in Table II. The contours of the streamfunction ψ and vorticity ω at $Re=3200$, 7500 and 10,000 are shown in Figures 2–7 respectively. The velocity distributions of u at $x=0{\cdot}5$ and v at $y=0{\cdot}5$ are shown in Figures 8 and 9 respectively, where the symbols denote the results given in Reference 11. All our results agree well with those of Ghia *et al.*[11]

It should be emphasized that for shear-driven cavity flow the high-order iterative formulations (5) and (6) can give convergent results at Re up to 10,000. This means that the formulations (5) and (6) are stable at high Reynolds numbers. In this paper, for the sake of computational efficiency, we use the FDM to solve the linear equations (7)–(10). However, as mentioned before, whether the BEM or FDM

Table I. Parameters, number of iterations and errors of corresponding results

Re	Δp	Number of iterations	RMS_ψ	RMS_ω
400	0·200	161	$9{\cdot}7 \times 10^{-4}$	$5{\cdot}1 \times 10^{-4}$
1000	0·050	779	$8{\cdot}7 \times 10^{-3}$	$1{\cdot}0 \times 10^{-3}$
3200	0·030	1196	$2{\cdot}7 \times 10^{-2}$	$5{\cdot}0 \times 10^{-2}$
5000	0·020	1366	$2{\cdot}9 \times 10^{-2}$	$5{\cdot}0 \times 10^{-2}$
7500	0·010	2695	$1{\cdot}9 \times 10^{-2}$	$5{\cdot}5 \times 10^{-2}$
10000	0·010	3744	$2{\cdot}5 \times 10^{-2}$	$5{\cdot}7 \times 10^{-2}$

Table II. Comparison of present results for primary vortex with those of Ghia *et al.*[11]

Re	Present results			Results of Ghia *et al.*[11]		
	ψ_{min}	$\lvert\omega_{v,c}\rvert$	$(x, y)_{v,c}$	ψ_{min}	$\lvert\omega_{v,c}\rvert$	$(x, y)_{v,c}$
400	−0·1130	2·2819	(0·5547,0·6093)	−0·11391	2·2947	(0·5547,0·6055)
1000	−0·1160	2·0234	(0·5313,0·5625)	−0·11793	2·0497	(0·5313,0·5625)
3200	−0·1168	1·8791	(0·5156,0·5469)	−0·12038	1·9886	(0·5165,0·5469)
5000	−0·1186	1·9375	(0·5117,0·5430)	−0·11897	1·8602	(0·5117,0·5352)
7500	−0·1201	1·9630	(0·5078,0·5469)	−0·11998	1·8799	(0·5117,0·5322)
10000	−0·1201	1·9426	(0·5078,0·5430)	−0·11973	1·8808	(0·5117,0·5333)

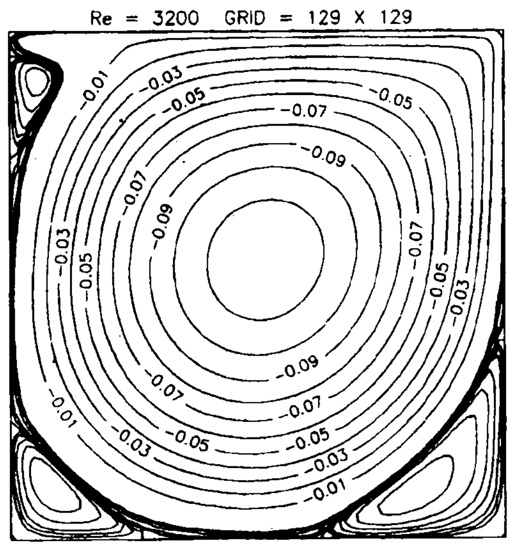

Figure 2. Contours of streamfunction ψ at $Re = 3200$

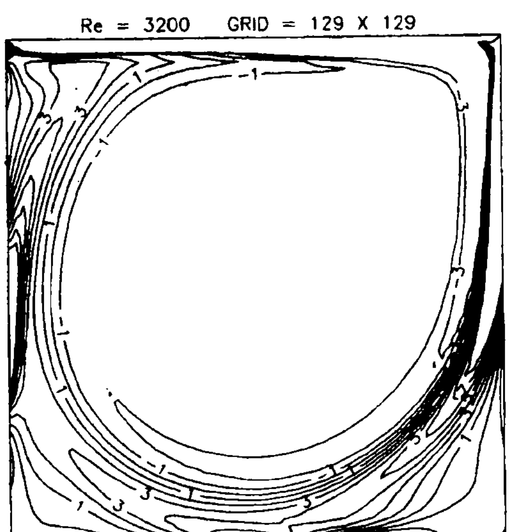

Figure 3. Contours of vorticity ω at $Re = 3200$

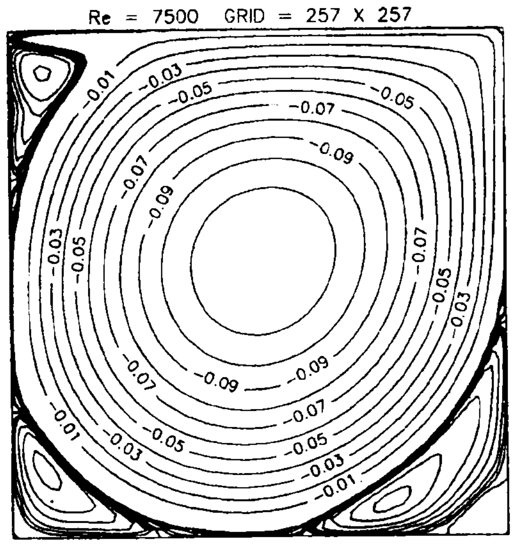

Figure 4. Contours of streamfunction ψ at $Re = 7500$

is applied to solve these equations seems not very important for the convergence of the proposed method. Thus the present work should give us confidence that these formulations might also provide a kind of stable BEM scheme for 2D steady state viscous flow at high Reynolds number. We will attempt to prove this directly as soon as we have the chance to use a supercomputer.

4. CONCLUSIONS

In this paper we have shown that the high-order iterative formulations described in Reference 10 are stable and can give convergent results for 2D viscous flows at high Reynolds numbers. As an example,

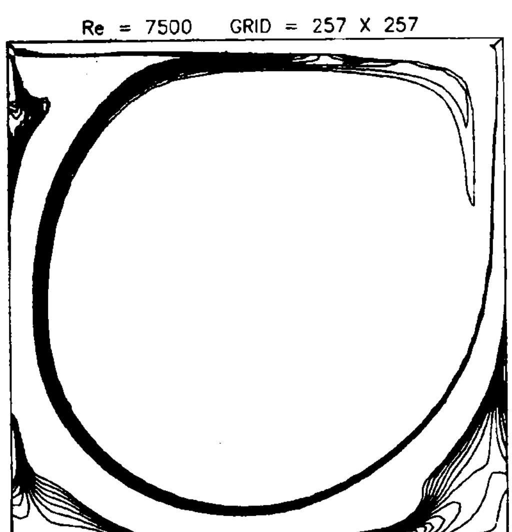

Figure 5. Contours of vorticity ω at $Re = 7500$

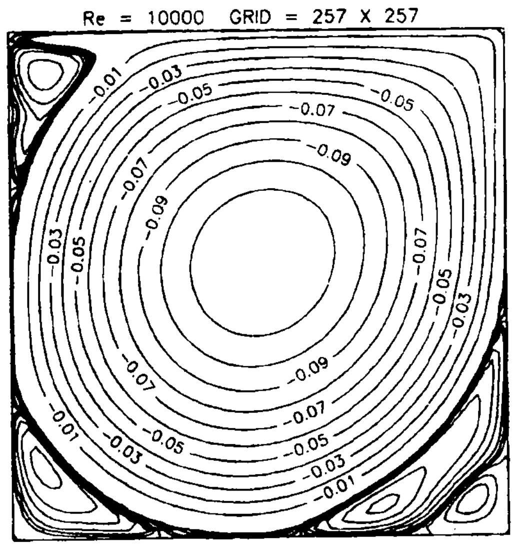

Figure 6. Contours of streamfunction ψ at $Re = 10{,}000$

convergent results for 2D shear-driven cavity flow at *Re* up to 10,000 are obtained. Note that these formulations are also suited to the BEM, but, as proved in this paper, whether the BEM or FDM is used to solve the corresponding linear equations is not important for iteration convergence. Thus we have reason to believe that these high-order formulations might give a kind of stable BEM scheme for viscous flow at high Reynolds number. However, this must be proved by directly applying the BEM and we will do so as soon as possible.

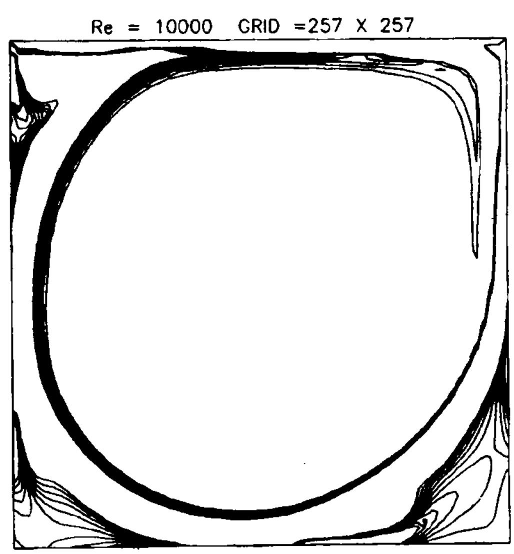

Figure 7. Contours of vorticity ω at $Re = 10{,}000$

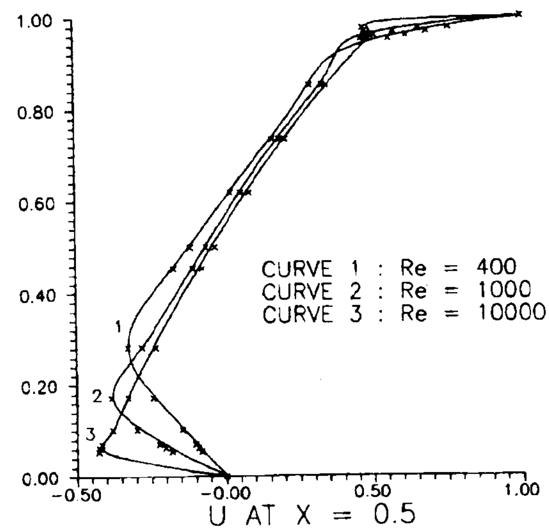

Figure 8. Velocity distributions of u at $x = 0{\cdot}5$

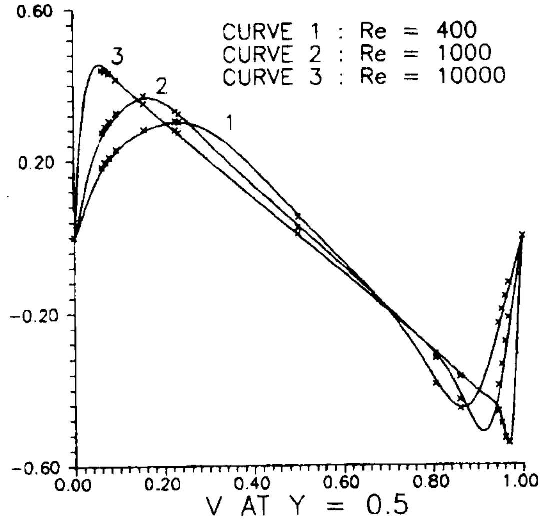

Figure 9. Velocity distributions of v at $y = 0{\cdot}5$

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