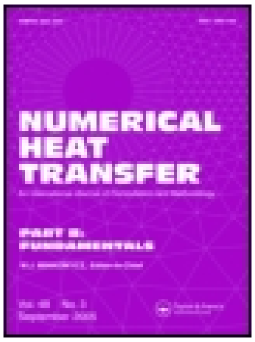

Taylor & Francis
Taylor & Francis Group

Numerical Heat Transfer, Part B: Fundamentals

ISSN: 1040-7790 (Print) 1521-0626 (Online) Journal homepage: http://china.tandfonline.com/loi/unhb20

GENERAL BOUNDARY-ELEMENT METHOD FOR UNSTEADY NONLINEAR HEAT TRANSFER PROBLEMS

Shi-Jun Liao

To cite this article: Shi-Jun Liao (1999) GENERAL BOUNDARY-ELEMENT METHOD FOR UNSTEADY NONLINEAR HEAT TRANSFER PROBLEMS, Numerical Heat Transfer, Part B: Fundamentals, 35:2, 225-242, DOI: 10.1080/104077999275965

To link to this article: http://dx.doi.org/10.1080/104077999275965

Published online: 29 Oct 2010.

Numerical Heat Transfer, Part B, 35:225–242, 1999

1040–7790/99 $12.00 + .00

GENERAL BOUNDARY-ELEMENT METHOD FOR UNSTEADY NONLINEAR HEAT TRANSFER PROBLEMS

Shi-Jun Liao and Allen T. Chwang
Department of Mechanical Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong

The general boundary-element method (BEM) for strongly nonlinear problems proposed in [1–3] is further applied to solve a two-dimensional (2D), unsteady, nonlinear heat transfer problem in the time domain, governed by the parabolic heat conduction equation with temperature-dependent thermal conductivity coefficients that are different in the x and y directions. This article shows that the general BEM is valid to solve even those nonlinear unsteady heat transfer problems whose governing equations do not contain any linear terms in the spatial derivatives. This demonstrates the validity and the great potential of the general BEM.

INTRODUCTION

Let us consider a 2D, unsteady, nonlinear heat transfer problem with inhomogeneous materials in the time domain, governed by

$$\frac{\partial}{\partial x}\left[k_1(\theta)\frac{\partial \theta}{\partial x}\right]+\frac{\partial}{\partial y}\left[k_2(\theta)\frac{\partial \theta}{\partial y}\right]+S(x,y,\theta)-\frac{\partial \theta}{\partial t}=0 \qquad (x,y)\in\Omega,\ t\geqslant 0 \tag{1}$$

with boundary conditions

$$\begin{aligned}\theta &= g_1(x,y,t) \qquad (x,y)\in\Gamma_1,\ t\geqslant 0\\ \frac{\partial\theta}{\partial n} &= g_2(x,y,t) \qquad (x,y)\in\Gamma_2,\ t\geqslant 0\end{aligned} \tag{2}$$

and the initial condition

$$\theta(x,y,0)=g_0(x,y) \qquad (x,y)\in\Omega \tag{3}$$

where $\theta(x, y, t)$ denotes the temperature distribution, $k_1(\theta)$ and $k_2(\theta)$ are thermal conductivity coefficients in the x and y directions, $S(x, y, \theta)$ is the heat source

Received 7 March 1997; accepted 27 August 1998
The authors are grateful to the reviewers for their helpful suggestions.
The present address of Shi-Jun Liao is School of Naval Architecture & Ocean Engineering, Shanghai Jiao Tong University, Shanghai 200030, China.
Address correspondence to Prof. Allen T. Chwang, The University of Hong Kong, Department of Mechanical Engineering, Pokfulam Road, Hong Kong. E-mail: atchwang@hkucc.hku.hk

NOMENCLATURE

k	thermal conductivity coefficient	θ^n	temperature distribution at the nth time step
k_1, k_2	thermal conductivity coefficients in the x and y directions	$\theta^n_{i,j}$	temperature distribution at the nth time step at position $(i\,\Delta x, j\,\Delta y)$
p	auxiliary variable	Θ	a kind of homotopy
S	heat source term	ρ	convergence radius
t	time	φ	auxiliary function defined by Eq. (6)
x, y	spatial coordinates	ω	fundamental solution
γ	thermal absorptivity	Ω	spatial domain of the temperature distribution
Γ	boundary of the spatial domain Ω		
θ	temperature distribution		

term that is also dependent on temperature $\theta(x, y, t)$, Ω is the domain of the temperature distribution, $\Gamma = \Gamma_1 \cup \Gamma_2$ denotes the boundary of the domain Ω, and t denotes time. All variables in Eqs. (1)–(3) are nondimensional.

If $k_1(\theta) = k_2(\theta) = 1$ and $S(x, y, \theta) = 0$, Eq. (1) is a linear diffusion problem that can be solved by the classical boundary–element method (BEM) based on time-dependent fundamental solutions [4, 5] or the Laplace transformation [6]. When $S(x, y, \theta) = 0$, $k_1(\theta) = k_2(\theta) = k(\theta)$ and the temperature distribution θ is independent of t, Eq. (1) becomes a steady-state one,

$$\frac{\partial}{\partial x}\left[k(\theta)\,\frac{\partial\theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[k(\theta)\,\frac{\partial\theta}{\partial y}\right] = 0 \qquad (x, y) \in \Omega,\ t \geqslant 0 \tag{4}$$

which can be rewritten by the Kirchhoff transformation [7] as

$$\nabla^2\varphi(x, y) = 0 \qquad (x, y) \in \Omega,\ t \geqslant 0 \tag{5}$$

where

$$\varphi(x, y) = \int_{\theta_0}^{\theta} k(\theta)\, d\theta \tag{6}$$

Equation (5) can also be solved by the classical BEM. However, when $k_1(\theta) \neq k_2(\theta)$ and the process is unsteady, the above-mentioned techniques based on the Kirchhoff transformation or the time-dependent fundamental solution are invalid. In this case, one can try to move all the nonlinear terms of Eq. (1) to the right-hand side and then find the fundamental solution of the linear operator still remaining on the left-hand side. This kind of BEM approach implies that both the linear operator and the corresponding fundamental solution are very important and absolutely necessary for the classical BEM. However, there exists a possibility that nothing is left on the left-hand side after moving all the nonlinear terms to the right-hand side of the equation. For example, when the thermal conductivity coefficients $k_1(\theta)$ and $k_2(\theta)$ are not only dependent on temperature θ but also different in the x and y directions, say,

$$k_1(\theta) = \exp(\alpha_1\theta) \qquad k_2(\theta) = \exp(\alpha_2\theta) \qquad (\alpha_1 \neq \alpha_2) \tag{7}$$

then nothing is left on the left-hand side of Eq. (1) after moving all nonlinear terms and the heat source term $S(x, y, \theta)$ to the right-hand side. In this case, the classical BEM for nonlinear problems does not work at all. Therefore, it is necessary to develop a more general BEM that works well even in the above-mentioned case.

Liao [1] and Liao and Chwang [2] proposed a general BEM for nonlinear problems. This general BEM was applied by Liao and Chwang [3] to solve a 2D steady nonlinear heat transfer problem with inhomogeneous materials, governed by the dimensionless equation

$$\frac{\partial}{\partial x}\left[k_1(\theta)\frac{\partial \theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_2(\theta)\frac{\partial \theta}{\partial y}\right] + S(x, y) = 0 \qquad (x, y) \in \Omega \tag{8}$$

It should be noted that when

$$k_1(\theta) = \exp(\alpha_1 \theta) \qquad k_2(\theta) = \exp(\alpha_2 \theta) \qquad (\alpha_1 \neq \alpha_2)$$

nothing is left on the left-hand side of Eq. (8) after moving all the nonlinear terms and the known heat source term $S(x, y)$ to its right-hand side. Even under this circumstance, the general BEM was successfully applied to solve the nonlinear heat transfer problem. In this article, we further extend the approach of Liao and Chwang [2] to solve the above-mentioned unsteady nonlinear (parabolic) heat transfer problem with inhomogeneous materials and show its validity by an example.

FORMULATION OF THE GENERAL BEM

Let Δt denote the time step and θ^n denote $\theta(x, y, n\Delta t)$. At the nth time step $t = n\Delta t$ $(n \geqslant 1)$, Eq. (1) can be approximately rewritten as

$$\frac{\partial}{\partial x}\left[k_1(\theta^n)\frac{\partial \theta^n}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_2(\theta^n)\frac{\partial \theta^n}{\partial y}\right] + S(x, y, \theta^n) - \frac{\theta^n - \theta^{n-1}}{\Delta t} = 0$$

$$(x, y) \in \Omega \tag{9}$$

with boundary conditions

$$\theta^n = g_1(x, y, n\Delta t) \qquad (x, y) \in \Gamma_1 \tag{10}$$

$$\frac{\partial \theta^n}{\partial n} = g_2(x, y, \Delta t) \qquad (x, y) \in \Gamma_2 \tag{11}$$

Here, we use the fully implicit expression of Eq. (1) in the time domain. Note that the temperature distribution $\theta^{n-1} = \theta(x, y, n\Delta t - \Delta t)$ is known when we solve Eqs. (9)–(11) at the nth $(n \geqslant 1)$ time step. For simplicity, we define a nonlinear operator

$$A(\theta^n) = \frac{\partial}{\partial x}\left[k_1(\theta^n)\frac{\partial \theta^n}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_2(\theta^n)\frac{\partial \theta^n}{\partial y}\right] + S(x, y, \theta^n) - \frac{\theta^n - \theta^{n-1}}{\Delta t} \tag{12}$$

Note that, at each time step, if we consider

$$S(x, y, \theta^n) - \frac{\theta^n - \theta^{n-1}}{\Delta t}$$

as a new heat source term, the nonlinear boundary-value problem governed by Eqs. (9)–(11) can be seen as a steady-state heat transfer problem that can be solved successfully by means of the general BEM [3].

Following Liao and Chwang [3], we first construct a family of partial differential equations for $\Theta(x, y; p)$ as

$$(1 - p)\nabla^2[\Theta(x, y; p) - \theta_0(x, y)] = -pA[\Theta(x, y; p)] \qquad (x, y) \in \Omega \qquad p \in [0,1] \tag{13}$$

with boundary conditions

$$\Theta(x, y; p) = pg_1(x, y, n\,\Delta t) + (1 - p)\theta_0(x, y) \qquad (x, y) \in \Gamma_1,\ p \in [0,1] \tag{14}$$

$$\frac{\partial\Theta(x, y; p)}{\partial n} = pg_2(x, y, n\,\Delta t) + (1 - p)\frac{\partial\theta_0}{\partial n} \qquad (x, y) \in \Gamma_2,\ p \in [0,1] \tag{15}$$

where p is an embedding parameter and $\theta_0(x, y)$ is an initial approximation of the temperature distribution θ^n at the nth time step. Note that $\Theta(x, y; p)$ is also a function of the embedding parameter p.

When $p = 0$, we obtain from Eqs. (13)–(15) that

$$\nabla^2\Theta(x, y; 0) = \nabla^2\theta_0(x, y) \qquad (x, y) \in \Omega \tag{16}$$

with boundary conditions

$$\Theta(x, y; 0) = \theta_0(x, y) \qquad (x, y) \in \Gamma_1 \tag{17}$$

$$\frac{\partial\Theta(x, y; 0)}{\partial n} = \frac{\partial\theta_0(x, y)}{\partial n} \qquad (x, y) \in \Gamma_2 \tag{18}$$

whose solution is obviously

$$\Theta(x, y; 0) = \theta_0(x, y) \tag{19}$$

When $p = 1$, we obtain from Eqs. (13) and (15) that

$$A[\Theta(x, y; 1)] = \frac{\partial}{\partial x}\left[k_1(\Theta)\frac{\partial\Theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_2(\Theta)\frac{\partial\Theta}{\partial y}\right] + S(x, y, \Theta) - \frac{\Theta - \theta^{n-1}}{\Delta t} = 0 \qquad (x, y) \in \Omega,\ p = 1 \tag{20}$$

with boundary conditions

$$\Theta(x, y; 1) = g_1(x, y, n\,\Delta t) \qquad (x, y) \in \Gamma_1 \tag{21}$$

$$\frac{\partial \Theta(x, y; 1)}{\partial n} = g_2(x, y, n\,\Delta t) \qquad (x, y) \in \Gamma_2 \tag{22}$$

By comparing Eqs. (20)–(22) with Eqs. (9)–(11), $\Theta(x, y, 1)$ is obviously the solution θ^n,

$$\Theta(x, y; 1) = \theta^n = \theta(x, y, n\,\Delta t) \tag{23}$$

Therefore, Eqs. (13)–(15) form a family of equations in parameter $p \in [0, 1]$, whose solution at $p = 0$ is equal to the initial approximation $\theta_0(x, y)$ and at $p = 1$ is the temperature distribution $\theta(x, y, n\,\Delta t)$ at the nth time step. The process of the continuous change of the imbedding parameter p from 0 to 1 is just the process of the continuous variation of solution $\Theta(x, y; p)$ from $\theta_0(x, y)$ to $\theta(x, y, n\,\Delta t)$. This kind of continuous variation is called *deformation* in topology, $\Theta(x, y; p)$ is called *homotopy*, $\theta_0(x, y)$ and $\theta^n = \theta(x, y, n\,\Delta t)$ are *homotopic*. Notice that this kind of continuous deformation is completely governed by Eqs. (13)–(15), which are called the *zeroth-order deformation equations*.

Expanding $\Theta(x, y; p)$ at $p = 0$ by the Taylor formula and using Eq. (19), we obtain

$$\begin{aligned}\Theta(x, y; p) &= \Theta(x, y, 0) + \sum_{m=1}^{\infty} \left[\frac{\theta_0^{[m]}(x, y)}{m!}\right] p^m \\ &= \theta_0(x, y) + \sum_{m=1}^{\infty} \left[\frac{\theta_0^{[m]}(x, y)}{m!}\right] p^m\end{aligned} \tag{24}$$

where $\theta_0^{[m]}(x, y)$ $(m \geqslant 1)$, called the *mth-order deformation derivative* at $p = 0$, is defined by

$$\theta_0^{[m]}(x, y) = \left.\frac{\partial^m \Theta(x, y; p)}{\partial p^m}\right|_{p=0} \qquad (m \geqslant 1) \tag{25}$$

Assume that the convergence radius of series (24) is not less than 1. Then, at $p = 1$, we obtain from Eqs. (23) and (24) that

$$\theta(x, y, n\,\Delta t) = \theta_0(x, y) + \sum_{m=1}^{\infty} \frac{\theta_0^{[m]}(x, y)}{m!} \tag{26}$$

which gives a relationship between the initial approximation $\theta_0(x, y)$ and the unknown temperature distribution $\theta^n = \theta(x, y, n\,\Delta t)$ at the nth time step. In general, the temperature distribution $\theta^{n-1} = \theta(x, y, n\,\Delta t - \Delta t)$ is used as the initial approximation $\theta_0(x, y)$ at the nth time step. Therefore, Eq. (26) gives, in fact, a relationship between $\theta(x, y, n\,\Delta t - \Delta t)$ and $\theta(x, y, n\,\Delta t)$.

Differentiating the zeroth-order deformation equations (13)–(15) m times with respect to p and then setting $p = 0$, we obtain the following *mth-order deformation equation* at $p = 0$:

$$\nabla^2 \theta_0^{[m]}(x, y) = R_m(x, y) \qquad m \geq 1,\ (x, y) \in \Omega \tag{27}$$

with boundary conditions

$$\theta_0^{[m]} = [g_1(x, y, n\,\Delta t) - \theta_0(x, y)]\,\delta_{1m} \qquad (x, y) \in \Gamma_1 \tag{28}$$

$$\frac{\partial \theta_0^{[m]}}{\partial n} = \left[g_2(x, y, n, \Delta t) - \frac{\partial \theta_0(x, y)}{\partial n} \right] \delta_{1m} \qquad (x, y) \in \Gamma_2 \tag{29}$$

where δ_{1m} is the Kronecker delta and

$$R_1(x, y) = -A(\theta_0) \tag{30}$$

$$R_m(x, y) = m \left[\nabla^2 \theta_0^{[m-1]} - \left. \frac{d^{m-1} A[\Theta(x, y; p)]}{dp^{m-1}} \right|_{p=0} \right] \qquad (m \geq 2) \tag{31}$$

Moreover, we have by Eq. (12) that

$$\begin{aligned}
\left. \frac{dA[\Theta(x, y; p)]}{dp} \right|_{p=0} = {} & \left[k_1'(\theta_0) \frac{\partial^2 \theta_0}{\partial x^2} + k_2'(\theta_0) \frac{\partial^2 \theta_0}{\partial y^2} \right. \\
& \left. + k_1''(\theta_0) \left(\frac{\partial \theta_0}{\partial x} \right)^2 + k_2''(\theta_0) \left(\frac{\partial \theta_0}{\partial y} \right)^2 \right] \theta_0^{[1]} \\
& + 2k_1'(\theta_0) \left(\frac{\partial \theta_0}{\partial x} \right) \frac{\partial \theta_0^{[1]}}{\partial x} + 2k_2'(\theta_0) \left(\frac{\partial \theta_0}{\partial y} \right) \frac{\partial \theta_0^{[1]}}{\partial y} \\
& + k_1(\theta_0) \frac{\partial^2 \theta_0^{[1]}}{\partial x^2} + k_2(\theta_0) \frac{\partial^2 \theta_0^{[1]}}{\partial y^2} \\
& + \left(\frac{\partial S(x, y, \theta_0)}{\partial \theta} - \frac{1}{\Delta t} \right) \theta_0^{[1]}
\end{aligned} \tag{32}$$

and so on.

Note that the mth-order deformation equation (27) with the corresponding boundary conditions (28) and (29) are *linear* with respect to the mth-order deformation derivative $\theta_0^{[m]}(x, y)$ $(m \geq 1)$. The linear equation (27) contains the well-known Laplace operator, whose fundamental solution is familiar to us so that it can be easily solved by the classical BEM. Precisely speaking, one can solve the integral equation

$$\oint_{\Gamma}\left(\omega\frac{\partial\theta_0^{[m]}}{\partial n}-\frac{\partial\omega}{\partial n}\theta_0^{[m]}\right)d\Gamma=\iint_{\Omega}\omega(\mathbf{r}',\mathbf{r})\,R_m(x,y)\,dx\,dy\quad(m\geqslant 1)\tag{33}$$
$$\mathbf{r}'=(\xi,\eta)\in\Omega^c,\ \mathbf{r}=(x,y)\in\Omega$$

to determine the unknown values of $\theta_0^{[m]}(x,y)$ on Γ_2 and $\partial\theta_0^{[m]}(x,y)/\partial n$ on Γ_1, where

$$\omega(\mathbf{r}',\mathbf{r})=-\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-\eta)^2}\tag{34}$$

is the fundamental solution of the 2D Laplace operator, and Ω^c denotes the exterior of domain Ω excluding its boundary $\Gamma=\Gamma_1\cup\Gamma_2$. The linear equation (33) can easily be solved by some well-known techniques of the BEM. For example, the boundary can be divided into N_Γ equal parts and within each boundary element the unknown $[\partial\theta_0^{[m]}(x,y)/\partial n]$ $(m\geqslant 1)$ can be linearly distributed and (33) is satisfied at end points of each element. For the domain integral, the domain can be divided into $N_\Omega\times N_\Omega$ equal subdomains and the four-point Gauss-integral formula can be used. For more details about the element technology and numerical integration technology, refer to Brebbia et al. [8]. Note that the left-hand matrix related to Eq. (33) is the same for all time steps, so its inverse matrix can be used in each iteration at every time step $(n\geqslant 1)$. After obtaining the unknown values of $\theta_0^{[m]}(x,y)$ on Γ_2 and $\partial\theta_0^{[m]}(x,y)/\partial n$ $(m\geqslant 1)$ on Γ_1, we have at point $(\xi,\eta)\in\Omega\cup\Gamma$ that

$$c(\xi,\eta)\,\theta_0^{[m]}(\xi,\eta)=\oint_{\Gamma}\left(\omega\frac{\partial\theta_0^{[m]}}{\partial n}-\frac{\partial\omega}{\partial n}\theta_0^{[m]}\right)d\Gamma$$
$$-\iint_{\Omega}\omega(\mathbf{r}',\mathbf{r})\,R_m(x,y)\,dx\,dy\tag{35}$$
$$\mathbf{r}'=(\xi,\eta)\in\Omega\cup\Gamma,\mathbf{r}=(x,y)\in\Omega,\ m\geqslant 1$$

where

$$c(\xi,\eta)=\begin{cases}1 & \text{if }(\xi,\eta)\in\Omega\\ \frac{1}{2} & \text{if }(\xi,\eta)\in\Gamma\end{cases}\tag{36}$$

Note that only a finite number of deformation derivatives $\theta_0^{[m]}(x,y)$ $(m\geqslant 1)$ can be obtained. If the convergence radius ρ of the Taylor series (24) is greater than or equal to 1, we can use

$$\theta(x, y, n\,\Delta t) \approx \theta_0(x, y) + \sum_{m=1}^{M} \frac{\theta_0^{[m]}(x, y)}{m!} \tag{37}$$

to obtain a new approximation better than $\theta_0(x, y)$, where M denotes the order of approximation. However, the convergence radius ρ of series (24) may be less than 1, so that (26) does not hold. Even in this case, $\Theta(x, y; \lambda)$ $(0 < \lambda < \rho)$ is in most cases still better than the initial approximation $\theta_0(x, y)$, so we can use the iterative formula

$$\theta_{k+1}(x, y, n\,\Delta t) = \theta_k(x, y, n\,\Delta t) + \sum_{m=1}^{M} \frac{\lambda^m \theta_0^{[m]}(x, y)}{m!} \qquad (k = 0, 1, 2, 3, \ldots) \tag{38}$$

where λ $(0 < \lambda < \rho)$ is treated as an iterative parameter and M is the order of approximation. We call Eq. (38) the Mth-order iterative formula. At the beginning of each iteration process, we simply use the known temperature distribution $\theta^{n-1} = \theta(x, y, n\,\Delta t - \Delta t)$ as the initial approximation for $\theta^n = \theta(x, y, n\,\Delta t)$. However, we should keep in mind that $\theta_0(x, y)$ appearing in all of the above expressions should be set new values before each new iteration.

By means of the general BEM mentioned above, we can use the initial condition $\theta(x, y, 0) = g_0(x, y)$ as an initial approximation to obtain the temperature distribution $\theta(x, y, \Delta t)$ at the first time step. Then, in a similar way, we can further use $\theta(x, y, \Delta t)$ to obtain the temperature distribution $\theta(x, y, 2\,\Delta t)$ at the second time step, and so on. In this way, we can obtain the solution of the original equations (1)–(3) in the whole time domain.

Finally, we mention that, in Eq. (33), R_m contains a term $\nabla^2\theta_0^{[m-1]}$ which can be converted to the boundaries by using the BEM. In fact, the term R_m is a known function so that it can be converted to the boundaries if the dual reciprocity BEM [7, 9, 10] is applied. We emphasize that the left-hand matrix related to Eq. (33) is the same for all time steps, so its inverse matrix can be used in each iteration at every time step $(n \geqslant 1)$.

NUMERICAL EXAMPLES

In order to show the validity of the above-mentioned general BEM formulation for unsteady, nonlinear heat transfer problems with inhomogeneous materials, we consider a 2D microwave heating problem. Generally, a microwave heating process is quite complicated: it contains the absorption and diffusion of heat, governed by a forced heat equation, and also the propagation and decay of the microwave radiation through a given material, governed by the Maxwell equation.

The absorption and diffusion of heat is modeled by a nondimensional equation,

$$\frac{\partial}{\partial x}\left[k_1(\theta)\frac{\partial \theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_2(\theta)\frac{\partial \theta}{\partial y}\right] + \gamma(\theta)|E|^2 - \frac{\partial \theta}{\partial t} = 0 \tag{39}$$

where θ denotes temperature, $k_1(\theta)$ and $k_2(\theta)$ are thermal conductivity coefficients of the material in the x and y directions, respectively, $\gamma(\theta)$ is the thermal absorptivity of the material, and $|E|$ is the amplitude of the electric field. Equation (39) is generally coupled with the Maxwell equation because of the temperature dependence of material properties such as electric conductivity, magnetic permeability, and electric permeability. However, if these material properties are assumed constant, the amplitude of the electric field is exponentially dependent on the spatial variables, i.e.,

$$|E| = \exp\left(-\frac{\varepsilon x}{2}\right) \tag{40}$$

for the decay from an incident boundary at $x = 0$, where ε is the decay constant. Here, the electric field has been assumed to decay exponentially in the x direction, whereas it is uniform in the y direction. Moreover, for many materials, the thermal absorptivity $\gamma(\theta)$ has a power-law dependence on temperature θ, say,

$$\gamma(\theta) = \beta\theta^{\nu} \tag{41}$$

Hill et al. [11–13] gave some examples of materials for which this power-law form of the thermal absorptivity is valid. For these materials, the so-called thermal runaway is possible if $\nu \geqslant 1$. Then, substituting Eqs. (40) and (41) into (39) leads to a simplified model equation,

$$\frac{\partial}{\partial x}\left[k_1(\theta)\frac{\partial \theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_2(\theta)\frac{\partial \theta}{\partial y}\right] + \beta\theta^{\nu}\exp(-\varepsilon x) - \frac{\partial \theta}{\partial t} = 0 \tag{42}$$

The corresponding boundary conditions may be of essential, natural, or mixed type. For more details, refer to [14–16]. For simplicity, we consider in the present paper a microwave heating problem of a 2D unit plate made of an inhomogeneous material. The corresponding boundary conditions are $\theta = 1$ on four sides of the unit plate and the initial condition is $\theta(x, y, 0) = 1$. For simplicity, we consider only the case of $k_1(\theta) = \exp(\alpha\theta)$ and $k_2(\theta) = 1$.

By Eqs. (13)–(15), the corresponding zeroth-order deformation equation for the temperature distribution $\theta^n = \theta(x, y, n\,\Delta t)$ is

$$\begin{gathered}(1-p)\nabla^2[\Theta(x, y; p) - \theta_0(x, y)] = -p\tilde{A}[\Theta(x, y; p)] \\ (x, y) \in [0,1] \times [0,1], \quad p \in [0,1]\end{gathered} \tag{43}$$

with boundary condition

$$\Theta(x, y; p) = p + (1-p)\theta_0(x, y) \qquad (x, y) \in \Gamma, \quad p \in [0,1] \tag{44}$$

where the nonlinear operator $\tilde{A}(\Theta)$ is defined by

$$\tilde{A}(\Theta) = k_1(\Theta)\frac{\partial^2\Theta}{\partial x^2} + k_2(\Theta)\frac{\partial^2\Theta}{\partial y^2} + k_1'(\Theta)\left[\frac{\partial\Theta}{\partial x}\right]^2 + k_2'(\Theta)\left[\frac{\partial\Theta}{\partial y}\right]^2$$
$$+ \beta\Theta^{\nu}\exp(-\varepsilon x) - \frac{\Theta - \theta^{n-1}}{\Delta t}$$
$$(x, y) \in [0,1]\times[0,1],\ p \in [0,1] \tag{45}$$

According to Eqs. (27)–(29), $\theta_0^{[m]}(x, y)$ $(m \geqslant 1)$ are governed by

$$\nabla^2\theta_0^{[m]}(x, y) = \tilde{R}_m(x, y) \qquad (x, y) \in [0,1]\times[0,1],\ m \geqslant 1 \tag{46}$$

with the boundary condition

$$\theta_0^{[m]}(x, y) = \delta_{1m}[1 - \theta_0(x, y)] \qquad (x, y) \in \Gamma \tag{47}$$

where δ_{1m} is the Kronecker delta and

$$\tilde{R}_1(x, y) = -\tilde{A}(\theta_0) \tag{48}$$

$$\tilde{R}_m(x, y) = m\left[\nabla^2\theta_0^{[m-1]} - \left.\frac{d^{m-1}\tilde{A}[\Theta(x, y; p)]}{dp^{m-1}}\right|_{p=0}\right] \qquad (m \geqslant 2) \tag{49}$$

The linear equation (46) with linear boundary condition (47) can easily be solved by the classical BEM. Note that the boundary now has four sides, each of which is divided into N_Γ equal parts. At each corner, two very close points, each belonging to a different boundary, are used to deal with the discontinuation. Within each boundary element, the unknown $[\partial\theta_0^{[m]}(x, y)/\partial n]$ $(m \geqslant 1)$ is linearly distributed and boundary condition (47) is satisfied at two end points of each element so that we have all together $4(N_\Gamma + 1)$ unknowns on the four sides of the unit plate. For the domain integral, the domain $[0,1]\times[0,1]$ is divided into $N_\Omega \times N_\Omega$ equal subdomains and the four-point Gauss-integral formula is used. We emphasize once again that the left-hand matrix related to Eq. (33) is the same for all time steps, so its inverse matrix can be used in each iteration at every time step $(n \geqslant 1)$.

By Eq. (38), we can obtain iterative formulas at different orders. Our numerical calculations indicate that the higher the order of the iterative formulas, the faster the corresponding iteration process converges, as shown in Figure 1. The same trend was reported by Liao and Chwang [2] and will not be discussed further. When the first-order iterative formula

$$\theta_{k+1}(x, y, n\Delta t) = \theta_k(x, y, n\Delta t) + \lambda\theta_0^{[1]}(x, y) \tag{50}$$

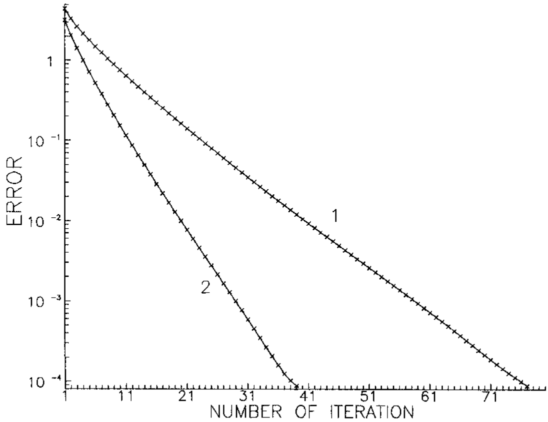

Figure 1. RMS errors Δ versus the number of iterations for $\alpha = 0.25$, $\beta = 14.1$, $\varepsilon = 2$, and $\nu = 2$ ($\Delta t = 0.01$, $\lambda = 0.1$, $N_\Gamma = N_\Omega = 20$): curve 1, the first-order formula ($M = 1$); curve 2, the second-order formula ($M = 2$).

is used and the value of λ is properly selected, the iteration converges quickly, as shown in Figure 1 for the case $k_1(\theta) = \exp(\theta/4)$, $k_2(\theta) = 1$, $\beta = 14.1$, $\varepsilon = 2$, and $\nu = 2$, where $N_\Gamma = 20$, $N_\Omega = 20$, $\lambda = 0.1$, and $\Delta t = 0.01$.

According to Eq. (45), $\tilde{A}[\theta(x_i, y_j, n\,\Delta t)]$ denotes the residual error of the governing equation (42) at point $x_i = i\,\Delta x$, $y_j = j\,\Delta y$ and the nth time step. Clearly, the smaller the root mean square of the residual error,

$$\Delta = \sqrt{\frac{\sum_{i=0}^{N_\Omega} \sum_{j=0}^{N_\Omega} \left| \tilde{A}\left[\theta(x_i, y_j, n\,\Delta t)\right] \right|^2}{(N_\Omega + 1)^2}}$$

the better the approximation $\theta(x_i, y_j, n\,\Delta t)$ is. Hence, the convergence criterion is given by

$$\Delta = \sqrt{\frac{\sum_{i=0}^{N_\Omega} \sum_{j=0}^{N_\Omega} \left| \tilde{A}\left[\theta(x_i, y_j, n\,\Delta t)\right] \right|^2}{(N_\Omega + 1)^2}} \leqslant 10^{-4} \tag{51}$$

where the nonlinear operator $\tilde{A}$ is defined by Eq. (45).

Without loss of generality, we consider in this article only four cases, that is, $\alpha = 0.25$, 0.50, 0.75, and 1.0 with $\nu = 2$ and $\varepsilon = 2$. Here, we use $\lambda = 0.01$, $\Delta t = 0.01$, $N_\Omega = 20$, $N_\Gamma = 20$, and the first-order iterative formula (50) ($M = 1$).

In each case, the temperature distributions of the unit plate in the time domain with different values of β can be solved by the general BEM formulation. If the value of β is small enough, $\theta(x, y, n\Delta t)$ tends to a steady-state solution as n is large enough. However, "thermal runaway" occurs as β is greater than a critical value, denoted by β_c. In engineering, it is very important to predict the circumstance under which thermal runaway occurs. Using the general BEM formulation for unsteady nonlinear heat transfer problems, we successfully determine the critical value β_c for the occurrence of thermal runaway, as listed in Table 1, where $\theta_{\max}$ is the maximum temperature corresponding to the critical value β_c. Thermal runaway occurs when $\beta > \beta_c$. The corresponding temperature distributions of a unit plate under microwave heating with different critical values of β_c are shown in Figures 2–7.

If thermal runaway does not occur, the temperature distribution $\theta(x, y, n\Delta t)$ tends to a steady-state solution that can also be obtained by the general BEM formulation given by Liao and Chwang [3]. On the other hand, if thermal runaway occurs, the steady-state equation cannot be obtained, because the corresponding iteration diverges.

For comparison, we also solve Eq. (42) by means of the finite-difference method (FDM). By writing $\theta^n_{i,j} = \theta(i\Delta x, j\Delta y, n\Delta t)$, where $\Delta x = \Delta y = 1/N$ and Δt denote the spatial and time steps, respectively, the governing equation (42) can be discretized as

$$k_1(\theta^n_{i,j})\left[\frac{\theta^n_{i+1,j} - 2\theta^n_{i,j} + \theta^n_{i-1,j}}{(\Delta x)^2}\right] + k_2(\theta^n_{i,j})\left[\frac{\theta^n_{i,j+1} - 2\theta^n_{i,j} + \theta^n_{i,j-1}}{(\Delta y)^2}\right]$$

$$+ k_1'(\theta^n_{i,j})\left[\frac{\theta^n_{i+1,j} - \theta^n_{i-1,j}}{2\Delta x}\right]^2 + k_2'(\theta^n_{i,j})\left[\frac{\theta^n_{i,j+1} - \theta^n_{i,j-1}}{2\Delta y}\right]^2$$

$$+ \beta(\theta^n_{i,j})^{\nu}\exp(-\varepsilon i\Delta x) = \frac{\theta^n_{i,j} - \theta^{n-1}_{i,j}}{\Delta t} \qquad (1 \leqslant i, j \leqslant N - 1)$$

and

$$\theta^n_{0,i} = \theta^n_{N,i} = \theta^n_{i,0} = \theta^n_{i,N} = 1 \qquad (0 \leqslant i \leqslant N)$$

Table 1. Values of β_c and $\theta_{\max}$ with $\varepsilon = 2$ and $\nu = 2$ for different values of α

	Present BEM approach		FDM approach	
α	β_c	$\theta_{\max}$	β_c	$\theta_{\max}$
0.25	14.1	2.53	14.1	2.54
0.50	33.0	8.33	33.1	8.34
0.75	55.5	5.93	55.4	5.91
1.00	84.0	4.69	84.2	4.71

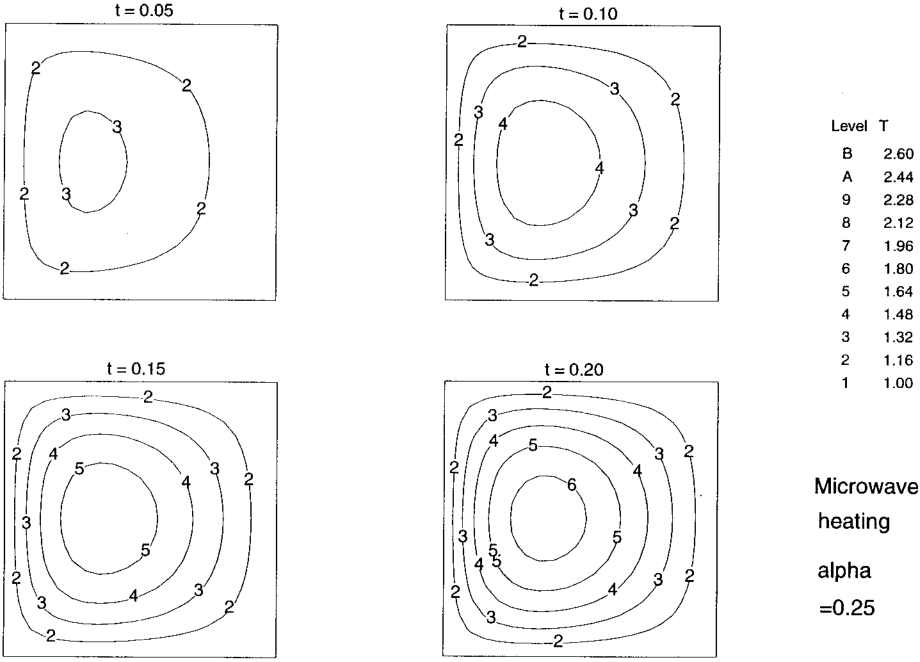

Figure 2. Temperature distribution of a unit plate under microwave heating for $\alpha = 0.25$ and $\beta = 14.1$ ($\varepsilon = 2$, $\nu = 2$) with $k_1(\theta) = \exp(\theta/4)$ and $k_2(\theta) = 1$.

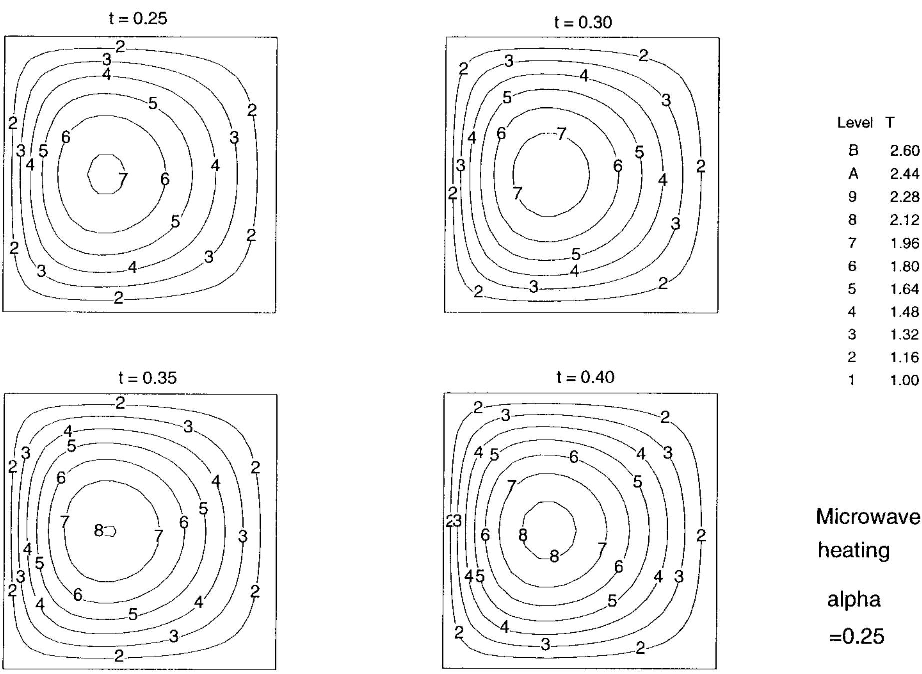

Figure 3. Temperature distribution of a unit plate under microwave heating for $\alpha = 0.25$ and $\beta = 14.1$ ($\varepsilon = 2$, $\nu = 2$) with $k_1(\theta) = \exp(\theta/4)$ and $k_2(\theta) = 1$.

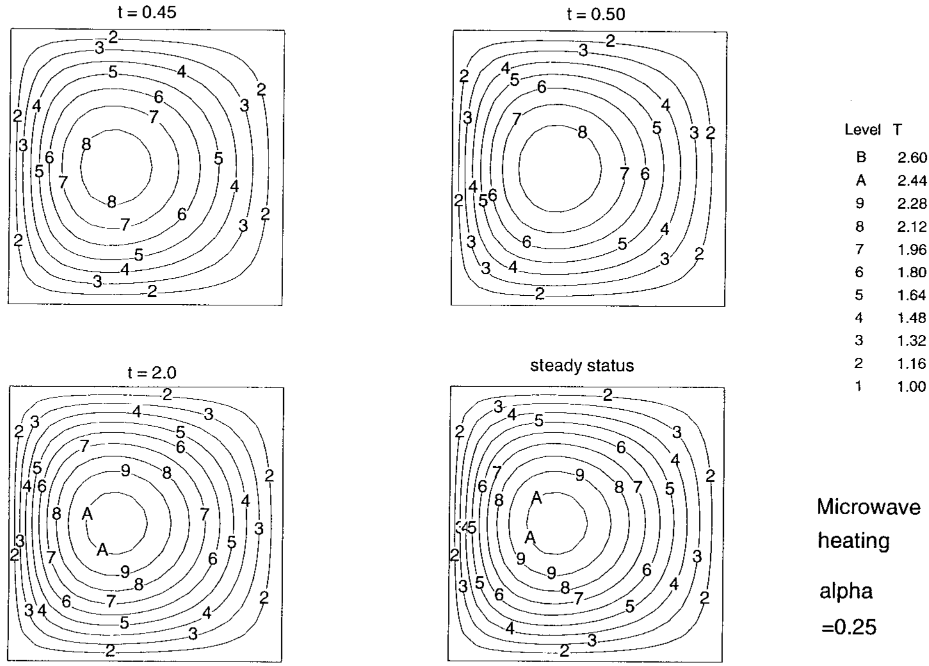

Figure 4. Temperature distribution of a unit plate under microwave heating for $\alpha = 0.25$ and $\beta = 14.1$ ($\varepsilon = 2$, $\nu = 2$) with $k_1(\theta) = \exp(\theta/4)$ and $k_2(\theta) = 1$.

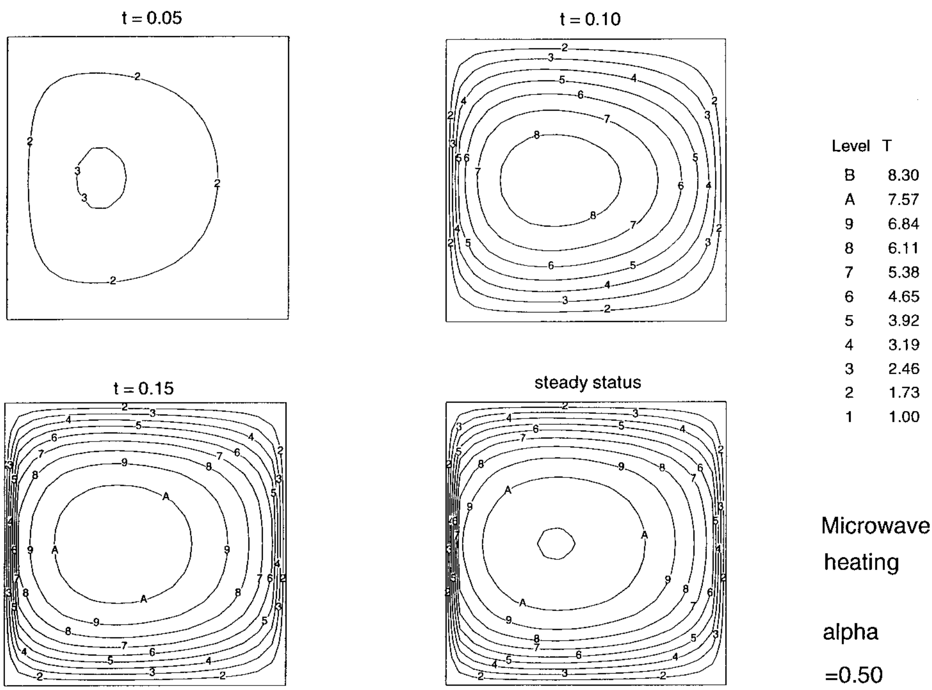

Figure 5. Temperature distributions of a unit plate under microwave heating for $\alpha = 0.50$ and $\beta = 33.0$ ($\varepsilon = 2$, $\nu = 2$) with $k_1(\theta) = \exp(\theta/2)$ and $k_2(\theta) = 1$.

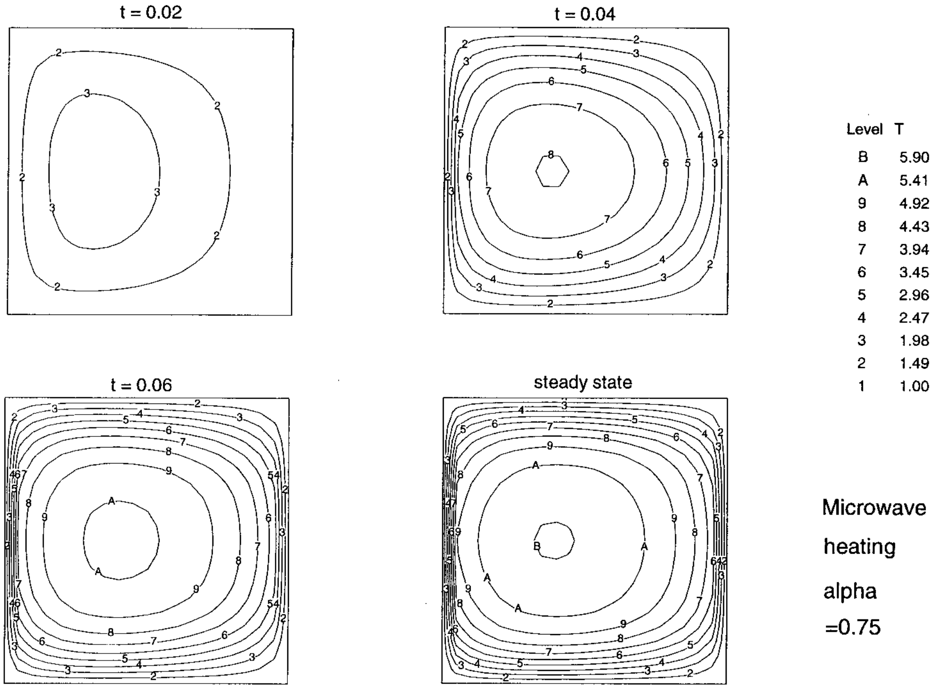

Figure 6. Temperature distribution of a unit plate under microwave heating for $\alpha = 0.75$ and $\beta = 55.5$ ($\varepsilon = 2$, $\nu = 2$) with $k_1(\theta) = \exp(3\theta/4)$ and $k_2(\theta) = 1$.

We solve the above set of nonlinear algebraic equations by the iteration technique, and the result $\theta_{i,j}^{n-1}$ at the $(n-1)$th time step is used as the initial approximation. The results given by the FDM for the case of $\beta = 14.1$, $\varepsilon = 2$, $\nu = 2$, $\Delta x = \Delta y = 0.025$, $\Delta t = 0.01$, and $\lambda = 0.1$ are shown in Figure 8. Comparing Figure 8 with Figures 2–4, we note that the results obtained by the present BEM agree very well with those given by the FDM at every time step. Moreover, by the above-mentioned FDM, we also determine the corresponding values of $\theta_{\max}$ and β_c related to thermal runaway. As listed in Table 1, the values of $\theta_{\max}$ and β_c given by the FDM agree well with those given by the present BEM approach. This verifies the validity of the present general BEM.

CONCLUSIONS

In this article, we apply the general BEM [1–3] to solve a 2D, unsteady, nonlinear heat transfer problem with inhomogeneous materials. We consider here the 2D parabolic heat conduction equation with temperature-dependent heat source and thermal conductivity coefficients different along the x and y directions. This problem cannot be solved by the classical BEM. However, the general BEM works well for this strongly nonlinear problem.

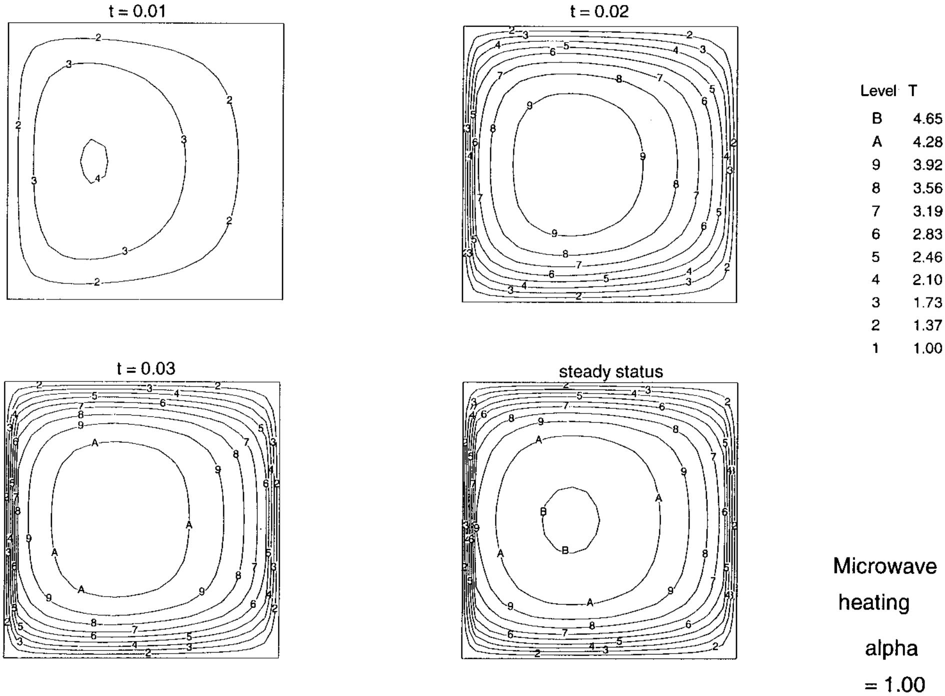

Figure 7. Temperature distribution of a unit plate under microwave heating for $\alpha = 1.00$ and $\beta = 84.0$ ($\varepsilon = 2$, $\nu = 2$) with $k_1(\theta) = \exp(\theta)$ and $k_2(\theta) = 1$.

Thermal runaway occurs as the value of β becomes large. In the four cases under consideration, the general BEM for unsteady nonlinear heat transfer problems successfully predicts the critical values of β_c for the occurrence of thermal runaway, which agree very well with those obtained by the FDM approach, as shown in Table 1 and Figure 8.

We note that the domain integral appears in the general BEM formulation, which might decrease the numerical efficiency. However, a boundary-element technique called the dual reciprocity BEM [7, 9, 10] has been developed to avoid the domain integration by transforming it to a surface integration. Moreover, the integral is suitable for parallel computation. Besides, all of the left-hand matrices for the system are the same at every time step for each iteration, so the inverse matrices can be used many times. Thus, the general BEM may become more efficient if combined with the dual reciprocity BEM and parallel computation.

Finally, we emphasize that the classical BEM is invalid to solve nonlinear unsteady heat transfer problems with inhomogeneous materials. However, the general BEM is still valid for the problem under consideration. This demonstrates once again the validity and the great potential of the general BEM. We believe that the general BEM can be applied to solve strongly nonlinear unsteady problems that cannot be solved by the classical BEM.

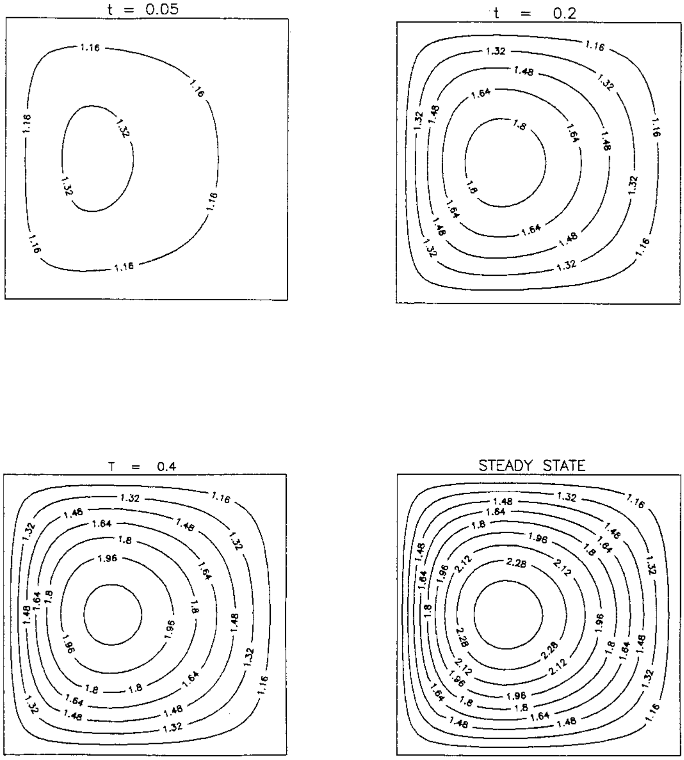

Figure 8. Temperature distribution (given by the FDM) of a unit plate under microwave heating for $\alpha = 0.25$ and $\beta = 14.1$ ($\varepsilon = 2$, $\nu = 2$) with $k_1(\theta) = \exp(\theta/4)$ and $k_2(\theta) = 1$.

REFERENCES

1. S. J. Liao, The Quite General BEM for Strongly Nonlinear Problems, in C. A. Brebbia, S. Kim, T. A. Osswald, and H. Power (eds.), *Boundary Elements XVII*, pp. 67–74, Computational Mechanics Publications, Southampton, UK, 1995.
2. S. J. Liao and A. T. Chwang, The General Boundary Element Method for Nonlinear Problems, *Int. J. Numer. Meth. Fluids*, vol. 23, pp. 467–483, 1996.
3. S. J. Liao and A. T. Chwang, The General BEM and Its Applications in Nonlinear Heat Transfer, in R. C. Ertekin, M. Tanaka, and C. A. Brebbia (eds.), *Boundary Element Technology XI*, pp. 83–92, Computational Mechanics Publications, Southampton, UK, 1996.
4. L. C. Wrobel, On the Use of Time-Dependent Fundamental Solutions for the Boundary Element Formulation of Diffusion Problems, in C. A. Brebbia, (ed.), *Boundary Elements X*, Vol. 2: Heat Transfer, Fluid Flow and Electrical Applications, pp. 3–13, Computational Mechanics Publications, Springer-Verlag, Berlin, 1988.
5. L. C. Wrobel and C. A. Brebbia, Time-Dependent Potential Problems, in C. A. Brebbia (ed.), Progress in Boundary Element Methods, Pentech Press, London, pp. 192–212, 1981.
6. F. J. Rizzo and D. J. Shippy, A Method of Solution for Certain Problems of Transient Heat Conduction, *AIAA J.*, vol. 8, pp. 2004–2009, 1970.

7. P. Satravaha and S. P. Zhu, An Application of the LTDRM to Non-linear Transient Heat Conduction Problems, in C. A. Brebbia, S. Kim, T. A. Osswald, H. Power (eds.), *Boundary Element XI*, pp. 297–304, Computational Mechanics Publications, Southampton, UK, 1989.
8. C. A. Brebbia, J. C. F. Telles, and L. C. Wrobel, *Boundary Element Techniques*, Springer-Verlag, Berlin, 1984.
9. Partridge, P. W., New Development in the Dual Reciprocity Method, in C. A. Brebbia, S. Kim, T. A. Osswald, and H. Power (eds.), *Boundary Elements XVII*, pp. 11–18, Computational Mechanics Publications, Southampton, U.K., 1995.
10. S. P. Zhu, Y. Zhang, and T. R. Marchant, A DRBEM Model for Microwave Heating Problems, *Appl. Math. Modelling*, vol. 19, pp. 287–297, 1995.
11. J. M. Hill and A. H. Pincombe, Some Similarity Profiles for the Microwave Heating of a Half Space, *J. Math. Soc. Ser. B.*, vol. 33, pp. 29–30, 1991.
12. J. M. Hill and N. F. Smyth, On the Mathematical Analysis of Hot-Spots Arising from Microwave Heating, *Math. Eng. Ind.*, vol. 17, pp. 1823–1834, 1990.
13. J. M. Hill and M. J. Jennings, Formulation of Model Equations for Heating by Microwave Radiation, *Appl. Math. Modelling*, vol. 17, pp. 1823–1834, 1993.
14. T. R. Marchant, Microwave Heating of Materials with Impurities, *J. Eng. Math.*, vol. 28, no. 5, pp. 379–400, 1994.
15. C. J. Coleman, On the Microwave Hot-Spot Problem, *J. Austral. Math. Soc. Ser. B.*, vol. 300, pp. 1–8, 1991.
16. A. S. Mexatax and R. J. Meredity, *Industrial Microwave Heating*, London, Pereginus, 1983.