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# Comparison between the homotopy analysis method and homotopy perturbation method

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**Abstract**

In this paper, we show that the so-called "homotopy perturbation method" is only a special case of the homotopy analysis method. Both methods are in principle based on Taylor series with respect to an embedding parameter. Besides, both can give very good approximations by means of a few terms, if initial guess and auxiliary linear operator are good enough. The difference is that, "the homotopy perturbation method" had to use a good enough initial guess, but this is not absolutely necessary for the homotopy analysis method. This is mainly because the homotopy analysis method contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region and rate of solution series. So, the homotopy analysis method is more general. Besides, the update of the concept of the "analytical solution" is discussed.


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## 1. The sameness

In 1992 Liao [1] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely the homotopy analysis method [2], and then modified it, step by step (see [3] and its references). This method has been successfully applied to solve many types of nonlinear problems (for example, please refer to [4–10]). In 2003, a book entitled "Beyond Perturbation: Introduction to the Homotopy Analysis Method" [2] was published by Chapman & Hall/CRC Press, Boca Raton (USA), which systematically describes the basic ideas of the homotopy analysis method, its relationships with other analytic techniques, and some of its applications in science and engineering. To show its basic ideas, let us consider a differential equation

$$\mathcal{N}[f(\vec{r},t)] = 0,$$

where $\mathcal{N}$ is a nonlinear operator, $\vec{r}$ is a vector of spatial variables, $t$ denotes time, $f(\vec{r},t)$ is an unknown function, respectively. Note that, for the simplicity, we ignore all related initial/boundary conditions, which can be treated in the similar way.

By means of generalizing the traditional concept of homotopy, Liao [2] constructs the so-called zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(\vec{r},t;p) - f_0(\vec{r},t)] = \hbar H(\vec{r},t)\mathcal{N}[\phi(\vec{r},t;p)], \tag{1}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a non-zero auxiliary parameter, $H(\vec{r},t) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $f_0(\vec{r},t)$ is an initial guess of $f(\vec{r},t)$, $\phi(\vec{r},t;p)$ is a unknown function, respectively. It should be emphasized that one has great freedom to choose the initial guess, the auxiliary linear operator, the auxiliary parameter $\hbar$, and the auxiliary function $H(\vec{r},t)$. Obviously, when $p = 0$ and $p = 1$, it holds

$$\phi(\vec{r},t;0) = f_0(\vec{r},t), \quad \phi(\vec{r},t;1) = f(\vec{r},t),$$

respectively. So, as $p$ increases from 0 to 1, $\phi(\vec{r},t;p)$ varies (or deforms) from the initial guess $f_0(\vec{r},t)$ to the solution $f(\vec{r},t)$. Expanding $\phi(\vec{r},t;p)$ in Taylor series with respect to the embedding parameter $p$, one has

$$\phi(\vec{r},t;p) = f_0(\vec{r},t) + \sum_{k=1}^{+\infty} f_k(\vec{r},t)p^k, \tag{2}$$

where

$$f_k(\vec{r},t) = \frac{1}{k!}\frac{\partial^k \phi(\vec{r},t;p)}{\partial p^k}\bigg|_{p=0}.$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen that the above series converges at $p = 1$, one has

$$f(\vec{r},t) = f_0(\vec{r},t) + \sum_{k=1}^{+\infty} f_k(\vec{r},t), \tag{3}$$

which must be one of solutions of original nonlinear equation, as proved by Liao [2]. It should be emphasized that it is very important to ensure that the series (2) converges at $p = 1$. Otherwise, the series (3) has no meanings.

As $\hbar = -1$ and $H(\vec{r},t) = 1$, Eq. (1) becomes

$$(1-p)\mathscr{L}[\phi(\vec{r},t;p) - f_0(\vec{r},t)] + \mathscr{N}[\phi(\vec{r},t;p)] = 0, \tag{4}$$

which is used mostly in the so-called "homotopy perturbation method" proposed in 1998 by Dr. He [11–13]. Different from Liao [2], Dr. He [13] writes

$$\phi(\vec{r},t;p) = f_0(\vec{r},t) + \sum_{k=1}^{+\infty} f^{(k)}(\vec{r},t)p^k \tag{5}$$

and uses

$$f(\vec{r},t) = f_0(\vec{r},t) + \sum_{k=1}^{+\infty} f^{(k)}(\vec{r},t) \tag{6}$$

to approximate the solution of the original nonlinear equation. However, it is easy to prove that

$$f^{(k)}(\vec{r},t) = f_k(\vec{r},t) = \frac{1}{k!}\frac{\partial^k \phi(\vec{r},t;p)}{\partial p^k}\bigg|_{p=0}.$$

Dr. He [13] also admitted this point. Thus, if the *same* initial guess and the *same* auxiliary linear operator are chosen, the approximations given by the "homotopy perturbation method" are exactly a special case of those given by the homotopy analysis method when $\hbar = -1$ and $H(\vec{r},t) = 1$. Therefore, the homotopy analysis method [2] logically contains the so-called "homotopy perturbation method" [13] in principle. Besides, like the series (3), the series (6) itself is in principle a kind of Taylor series (at $p = 1$), too. So, mathematically speaking, "homotopy perturbation method" itself is also a kind of "generalized Taylor technique", as called by Dr. He himself [13], even if Dr. He announces that he needs only a few terms.

Consider the Taylor series of an analytic function $g(z)$, i.e.

$$g(z) \sim g(z_0) + \sum_{k=1}^{+\infty} \frac{g^{(k)}(z_0)}{k!}(z-z_0)^k.$$

Obviously, *if and only if* $z_0$ is close enough to $z$, one needs only a few terms to get an accurate enough approximation of $g(z)$. If not, many terms are necessary, and more importantly, convergence *must* be considered. So, *if and only if* the initial guess is good approximation and the auxiliary linear operator is properly chosen, one needs a few of the first terms to get an accurate enough result by means of *both* the homotopy analysis method [2] and the so-called "homotopy perturbation method" [13]. For example, the first-order approximation given by Liao [2, Chapter 11, Example 11.2.2, Fig. 11.2] is very accurate. More examples are given in Liao's book [2], for example, Fig. 6.4 (third-order approximation) in Chapter 6; Fig. 11.1 (first-order approximation) and Fig. 11.3 (first-order approximation) in Chapter 11, Figs. 12.1–12.4 (first- and third-order approximation) in Chapter 12, Figs. 13.2–13.4 (fifth-order approximation) in Chapter 13, and so on. All of these examples illustrate that one can also obtain very good approximations in a few terms by means of the homotopy analysis method for some simple nonlinear problems. Even so, we do not think the homotopy analysis method has relationships with perturbation techniques.

## 2. The difference

The convergence of Liao's solution series (3) is dependent upon four factors, i.e. the initial guess, the auxiliary linear operator, the auxiliary function $H(\vec{r}, t)$, and the auxiliary parameter $\hbar$. However, as a special case of homotopy analysis method when $\hbar = -1$ and $H(\vec{r}, t) = 1$, the convergence of Dr. He's solution series (6) is only dependent upon two factors: the auxiliary linear operator, and the initial guess. So, given the initial guess and the auxiliary linear operator, Dr. He's approach cannot provide other ways to ensure that the solution is convergent. This is the reason why one had to choose a good enough initial approximation and/or auxiliary operator when applying the so-called "homotopy perturbation method". If one luckily finds such a good enough initial guess, certainly he needs only a few terms. Dr. He's work shows some successful examples for a few simple nonlinear problems (see [13] and its cited references). However, as pointed out by Liao [2], there does not exist a general theory and an efficient approach to find a good enough initial guess for any a given nonlinear problem, especially when the nonlinear problems have multiple solutions, discontinuation, multi-value solutions, bifurcations and/or are unsteady. This is because, finding a good enough approximation is just one's purpose, i.e. the end-point, but not a beginning-point. So, finding a good enough initial guess itself is not easier too much than finding a good enough approximation of a nonlinear problem itself, in principle. Besides, solutions of many nonlinear problems are very complicated. It is well-known that perturbation solutions are mostly related with *local* properties of a nonlinear problem. However, If one would like to analyze *global* properties of a complicated nonlinear problem, it is very hard to find such a good enough initial guess in most cases (except some simple ones).

Dr. He shows some attempts of finding a good initial guess. But, it is unknown whether or not his method is general and valid for other more complicated nonlinear problems. For example, for Blasius and Falkner–Skan equation, Dr. He [13] provided an analytic approach based on avoiding the term $\eta^n \exp(-n\eta)$, which Dr. He called "secular terms". However, unlike the secular terms such as $t \sin t$ and $t \cos t$ which tends to infinity as $t \to +\infty$, the term $\eta^n \exp(-n\eta)$ is *not* such a secular term at all, because

$$\lim_{\eta \to +\infty} \eta^n \exp(-n\eta) = 0, \quad n \geqslant 1.$$

In fact, this term can appear in the solutions of Blasius and Falkner–Skan equations, as shown by Liao [2]. So, it is *not* absolutely necessary to avoid such kind of terms. Thus, Dr. He should give a more reasonable mathematical base for his approach. To generalize the idea of the secular terms, Liao provides the so-called *Rule of Solution Expression* [2, Chapter 2], which is very useful for the selection of the initial guess, the auxiliary linear operator, and the auxiliary function.

Unlike Dr. He's approach, the convergence of Liao's solution series (3) is dependent upon four terms: the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function $H(\vec{r}, t)$. If one luckily chooses a good enough initial guess and good enough auxiliary linear operator, one can get accurate approximations by only a few terms with $\hbar = -1$ and $H(\vec{r}, t) = 1$, as shown by Liao [2] and mentioned above. However, even if the initial guess and auxiliary linear operator are *not* good enough but reasonable, one can still get convergent results by properly choosing the auxiliary parameter $\hbar$ and the auxiliary function $H(\vec{r}, t)$, as shown by Liao [2]. This is mainly because, given initial guess and auxiliary linear operator, the convergence of the solution series given by the homotopy analysis method is determined by the auxiliary parameter $\hbar$. The value of $\hbar$ can be determined by plotting the so-called $\hbar$-curves, as suggested by Liao [2]. So, different from *all* previous analytic techniques (to the best of our knowledge), the homotopy analysis method provides us with a simple way to control and adjust the convergence of a solution series. Besides, Liao [2] suggested three general rules, namely *the Rule of Solution Expression*, *the Rule of Solution Existence*, and *the Rule of Coefficient Ergodicity*, to choose the initial guess, the auxiliary linear operator, and the auxiliary functions $H(\vec{r}, t)$. Therefore, the convergence of the solution series given by the homotopy analysis method is not so strongly dependent upon the initial guess and the auxiliary linear operator as the so-called "homotopy perturbation technique". So, unlike the "homotopy perturbation method", it is *not absolutely* necessary for us to find a *good* enough initial guess and a *good* enough auxiliary linear operator, but only *reasonable* ones. Therefore, the homotopy analytsis method is more general than the so-called "homotopy perturbation method", not only mathematically but also in applications. By means of the homotopy analysis method, one has larger freedom to choose initial guesses and auxiliary linear operators.

It is straightforward to generalized the zero-order deformation equation (1) in the following form:

$$(1-p)\mathscr{L}[\phi(\vec{r},t;p)-f_0(\vec{r},t)] = \hbar H(\vec{r},t)\mathscr{N}[\phi(\vec{r},t;p)] + H_2(\vec{r},t)\Pi[\phi(\vec{r},t;p);p], \tag{7}$$

where $H_2(\vec{r},t)$ is the second auxiliary function, $\Pi$ is a operator, and the term $\Pi[\phi(\vec{r},t;p);p]$ disappears when $p = 0$ and $p = 1$, i.e.

$$\Pi[\phi(\vec{r},t;0);0] = \Pi[\phi(\vec{r},t;1);1] = 0.$$

For details, please refer to Liao [2] (§3.6, §4.3, and §12.1). Obviously, there are many different ways to construct such a kind of operator $\Pi$, but up to now there does not exist a general approach to choose a better (or even the best) operator $\Pi$ for any a given nonlinear problem, as pointed by Liao [2].

Dr. He [13] made an attempt in this direction. For example, to solve the nonlinear equation

$$u' + u^2 = 1, \quad u(0) = 0, \tag{8}$$

Dr. He used such a kind of homotopy

$$(1-p)\left[u' + u\left(\sum_{k=0}^{+\infty} b_k p^k\right) - \left(\sum_{k=0}^{+\infty} c_k p^k\right)\right] + p(u' + u^2 - 1) = 0. \tag{9}$$

For details, please refer to He [13, pp. 531–532]. Based on the disappearance of his so-called "secular term" $t^n\exp(-n\,t)$ that however tends to zero at infinity and is therefore not a secular term at all, Dr. He still obtained an approximation in a few terms by solving a set of nonlinear algebraic equations about the coefficients $b_k$ and $c_k$. For example, his first-order approximation corresponds to

$$b_0 = 2, \quad b_1 = -2, \quad c_0 = 2, \quad c_1 = -1, \quad b_k = c_k = 0, \quad k \geqslant 2. \tag{10}$$

Substituting (10) into (9), one has

$$(1-p)(u' + 2u - 2) = -p(u' + u^2 - 1) + p(1-p)(2u - 1), \tag{11}$$

which is just a special case of Eq. (7) when $\hbar = -1,\ H(\vec{r},t) = H_2(\vec{r},t) = 1$, and

$$\Pi(u,p) = p(1-p)(2u-1).$$

Besides, it should be pointed out that (10) cannot be obtained from Dr. He's expression (18) (see [13, p. 531]), i.e.

$$0 = b_0 + b_1 p + b_2 p^2 + \cdots,$$

which holds for *any* $p \in [0,1]$ only when *all* coefficients $b_k$ ($k \geqslant 0$) equal to zero. So, Dr. He's approach [13] contains obvious mathematical mistakes.

## 3. The update of the concept of analytic solution

Most of concepts of human beings are updated along with the development of science and technology, so should the concept of the so-called "analytic solution". Some people think that an analytic expression more than ten terms is too complicated and thus is useless in practice. This was easy to understand hundreds of years ago when there were no computers and people wrote an "analytic solution" on paper and calculated it by hand. At that time, it was indeed a hard work to calculate an analytic expression more than 10 terms. Our traditional concept of "analytic solution" was born in such kind of situation, more or less. However, fortunately, we are now in the time of computer with huge memory and high-speed CPU: it needs only a few seconds to calculate an analytic expression even with hundreds of terms, much faster than calculating a few terms by hand. Symbolic computation software such as Mathematica, MathLab, Maple, and so on, are applied wider and wider, and more and more researchers deduce their mathematical formulas on computer without papers and pens at all. In these cases, paper and pen are replaced by hard disk and keyboard of a computer. And currently, the Internet becomes a new popular medium, similar to the appearance of the paper hundreds of years ago. The author personally believes that the traditional concept of "analytic solution" should be updated to face the computer time, and an analytic expression with many terms might be accepted by most scientists and researchers in the near future. In other words, in the time of computer, an analytic expression is *not* absolutely necessary to be only a few terms. In this meaning, the homotopy analysis method is for the time of computer, more or less.

## 4. Conclusions and discussions

As pointed out by Liao ([2], Chapter 4), the homotopy analysis method logically contains Adomian's decomposition method [14], Lyapunov's artificial small parameter method [15], and the $\delta$-expansion method [16]. In this paper, we show that the so-called "homotopy perturbation method" [13] is only a special case of the homotopy analysis method [2]. Both methods are in principle based on Taylor series in an embedding parameter. Besides, both can give very good approximations in a few terms, if initial guess and auxiliary linear operator are good enough. The difference is that, "the homotopy perturbation method" *had to* use a good enough initial guess, but this is *not* absolutely necessary for the homotopy analysis method. This is mainly because the homotopy analysis method contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region and rate of solution series. Dr. He [13] made some attempts to find good enough initial guesses and to construct a good enough homotopy for some simple nonlinear problems. Unfortunately, as mentioned above, his approach contains some mistakes and thus needs a more reasonable mathematical base, and besides should be generalized and verified for more complicated nonlinear problems.

All things have their good and bad sides, so too does the homotopy analysis method. As a new, developing analytic method for nonlinear problems in general, the homotopy analysis method has its own limitations, as pointed out by Liao [2, Chapter 5: Advantages, Limitations, and Open Questions], and thus certainly should be further modified.

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