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Two new standing solitary waves in shallow water



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HIGHLIGHTS

- Two new Faraday's standing solitary waves are found.
- Closed-form solutions.
- Some experimental phenomena are explained.

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ABSTRACT

In this paper, the **closed-form analytic solutions** of two **new Faraday's standing solitary waves** due to the parametric resonance of liquid in a vessel vibrating vertically with a constant frequency are given **for the first time**. Using a model based on the symmetry of wave elevation and the linearized Boussinesq equation, we gain the closed-form wave elevations of the two kinds of non-monotonically decaying standing solitary waves with smooth crest and the even or odd symmetry. **All of them have never been reported**, to the best of our knowledge. Besides, **they can explain some experimental phenomena well**. All of these are helpful to deepen and enrich our understanding about standing solitary waves and Faraday's wave.

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1. Introduction

As pointed out by Faraday [1] and Benjamin and Ursell [2], when a vessel containing liquid vibrates vertically with a constant driving frequency Ω , the so-called parametric resonance occurs so that standing surface waves are observed, in case that the liquid oscillates with a constant frequency ω that is half of the driving frequency Ω , say, $\omega = \Omega/2$. In fact, Faraday waves have been observed in many fields of science. For example, the experimental observation of Faraday waves in a Bose–Einstein condensate was reported by Engels et al. [3], and the Faraday instability on a free surface of superfluid ^4He was investigated by Abe et al. [4] and Ueda et al. [5]. In 2011, using a vertically vibrating Hele–Shaw cell (i.e. nearly two dimensional) partly filled with water, Rajchenbach, Leroux and Clamond [6] did an excellent experiment and observed two new standing solitary surface waves with the odd or even symmetry. These new standing waves have an unusual characteristic: their elevations *non-monotonically* decay to zero in the horizontal direction, while vibrating periodically in the vertical direction. Especially, they pointed out that “the existence of an oscillion of odd parity had never been reported in any media up to now”. To the best of our knowledge, theoretical solutions have never been found for these new standing solitary waves.

In this paper, the closed-form analytic solutions of two new Faraday's solitary waves¹ due to the parametric resonance of liquid in a vessel vibrating vertically with a constant frequency are reported. Using a model based on the symmetry of wave elevation and the linearized Boussinesq equation [7], we gain the closed-form solution of two kinds of *non-monotonically* decaying standing solitary waves with the even or odd symmetry. Both of them have never been reported, to the best of

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¹ Here, the solitary wave means the localized wave.

our knowledge. Especially, they can explain some experimental phenomena currently reported by Rajchenbach, Leroux and Clamond [6] well. All of these are helpful to deepen and enrich our understanding about standing solitary waves and Faraday's wave.

2. Closed-form solutions of the new standing waves

Consider a two-dimensional (2D) Faraday's waves in the water depth h , excited by a vertically vibrating horizontal bottom being purely sinusoidal with a single driving frequency Ω . Let ω denote the frequency of the excited standing wave, respectively. In theory, it is well-known that the parametric resonance occurs when $\omega = \Omega/2$, say, the driving frequency Ω of the bottom is twice the frequency ω of the liquid vibration [1,2], which was currently confirmed once again by the excellent experiment of Rajchenbach, Leroux and Clamond [6]. Thus, in this paper, we focus on the case $\omega = \Omega/2$ for the parametric resonance.

We use here a model based on the symmetry of wave elevation and the linearized Boussinesq equation [7]. Let $\eta(x, \tau)$ denote the dimensionless wave elevation, where $\tau = \omega t = \Omega t/2$ denote the dimensionless time and x is the dimensionless horizontal coordinate with $x = 0$ corresponding to the wave crest. According to the excellent experiment of Rajchenbach, Leroux and Clamond [6], we have many reasons to assume that the wave elevation has either the even symmetry about the wave crest $x = 0$, i.e.

$$\eta(x, \tau) = \eta(-x, \tau), \quad -\infty < x < +\infty, \tag{1}$$

or the odd symmetry

$$\eta(x, \tau) = -\eta(-x, \tau), \quad -\infty < x < +\infty, \tag{2}$$

respectively. Assuming that the crest is smooth, the even symmetry (1) gives us the boundary condition

$$\eta_x(0, \tau) = 0. \tag{3}$$

Besides, the odd symmetry (2) is equivalent to the boundary condition

$$\eta(0, \tau) = 0. \tag{4}$$

Using the above symmetry and the boundary condition at $x = 0$, we only need seek a solution $\eta(x, \tau)$ in the interval $0 < x < +\infty$. It should be emphasized that the symmetry plays an important role in our approach, as shown below.

Assumed that the fluid is inviscid, incompressible, and the flow is irrotational in $0 < x < +\infty$ (i.e. the flow is not necessarily irrotational at $x = 0$). Such kind of free surface in the interval $0 < x < +\infty$ can be modeled approximately by the famous Boussinesq equation [7], which describes many wave phenomena in shallow water. In physics, the principle of relativity requires that the equations describing the laws of physics have the same form in all admissible frames of reference. Therefore, following Boussinesq [7] and using water depth h as a characteristic length, one can gain the dimensionless Boussinesq equation in the reference-frame fixed with the vertically vibrating bottom:

$$\eta_{\tau\tau} - g' \left(\eta_{xx} + \frac{1}{3} \eta_{xxxx} + 3\eta\eta_{xx} + 3\eta_x\eta_x \right) = 0, \quad 0 < x < +\infty, \tag{5}$$

subject to the bounded condition

$$|\eta(x, \tau)| < S, \quad 0 \leq x < +\infty, \tag{6}$$

where g' is the so-called dimensionless "apparent gravity acceleration" and the S is a large enough positive constant, respectively. Note that the above equation has exactly the same form as the traditional Boussinesq equation [7], except that the "gravity acceleration" term g' has a different meaning. Obviously, according to Einstein's theory of general relativity, for an observer moving with the vertically vibrating horizontal bottom that is not an inertial frame of reference, the so-called apparent gravity acceleration reads

$$g' = G(1 - F \cos 2\tau),$$

where $G = g/(h\omega^2)$ is the dimensionless acceleration of gravity, g is the acceleration due to gravity, $F = \Gamma/g$ denotes the dimensionless driving acceleration with Γ being the amplitude of the forcing acceleration of the bottom, respectively.

Assume that the wave amplitude is so small that all nonlinear terms of (5) can be neglected. Thus, we have the linearized Boussinesq equation in the non-inertial frame of reference fixed with the vertically vibrating bottom:

$$\eta_{\tau\tau} - G(1 - F \cos 2\tau) \left(\eta_{xx} + \frac{1}{3} \eta_{xxxx} \right) = 0, \quad 0 < x < +\infty. \tag{7}$$

Our purpose is to find the solutions of Eq. (7), subject to the bounded condition (6) and either the boundary condition (3) for the standing solitary waves with the even symmetry of elevation or (4) with the odd symmetry, which oscillate periodically in time τ with the period $T = 2\pi$.

Note that $\tau = \omega t$ and $\omega = \Omega/2$, where Ω is the driving frequency of the vertically vibrating bottom. According to the excellent experiment done by Rajchenbach, Leroux and Clamond [6], the parametric resonance occurs when $\omega = \Omega/2$. Thus, we express the standing wave in the form

$$\eta(x, \tau) = f(\tau) e^{\lambda x}, \quad 0 < x < +\infty, \tag{8}$$

where $f(\tau)$ is a periodic function with the period $T = 2\pi$, and λ is an unknown eigenvalue to be determined. Note that the eigenvalue λ can be real or complex. Substituting the above expression into (7), we have a linear ordinary different equation

$$f''(\tau) - G\lambda^2 \left(1 + \frac{\lambda^2}{3}\right) (1 - F \cos 2\tau)f(\tau) = 0. \tag{9}$$

The above equation can be rewritten as the standard Mathieu equation

$$f''(\tau) + [a - 2q \cos(2\tau)]f(\tau) = 0, \tag{10}$$

where

$$a = -G\lambda^2 \left(1 + \frac{\lambda^2}{3}\right), \quad q = \left(\frac{F}{2}\right) a. \tag{11}$$

Thus, $f(\tau)$ is a periodic Mathieu function with the characteristic value a and the parameter q , denoted by $f(\tau) = M_c(\tau; a, q)$. Note that similar Mathieu-type analyses have been carried out for Faraday waves in one and two-component Bose–Einstein condensates [8–10].

It is well-known that, for a given non-zero parameter q , the corresponding Mathieu functions $f(\tau)$ of Eq. (10) are periodic in τ only for certain values of a , called Mathieu characteristic values. According to Floquet's Theorem, any Mathieu function $f(\tau)$ can be written in the form $e^{ir\tau}f^*(\tau)$, where $f^*(\tau)$ has period 2π and r is the Mathieu characteristic exponent. The Mathieu function $f(\tau)$ is periodic only when the characteristic exponent r is an integer or rational number.

For given characteristic value a and parameter $q = (aF)/2$, let us consider the even Mathieu function $f(\tau)$ of (10) with the characteristic exponent $r = 1$ so that $f(\tau)$ has the period $T = 2\pi$. As mentioned above, the characteristic value of the even Mathieu function $f(\tau)$ with characteristic exponent $r = 1$ given by the parameter q must be equal to the characteristic value a itself. This gives, by means of the computer algebra system Mathematica, the following nonlinear algebraic equation

$$\text{MathieuCharacteristicA}[1, aF/2] = a, \tag{12}$$

where the Mathematica command $\text{MathieuCharacteristicA}[r, q]$ is used to gain the characteristic value a for even Mathieu functions with characteristic exponent r and the given parameter q . Given the dimensionless driving acceleration F , the above nonlinear algebraic equation contains only the unknown characteristic value a , denoted by a^* . The corresponding solution

$$f(\tau) = M_c(\tau; a^*, Fa^*/2)$$

is an even Mathieu function with the period 2π . It should be emphasized that the characteristic value a^* depends only on the dimensionless driving acceleration F .

It is found that the nonlinear algebraic equation (12) has two solutions in general. For example, when $F = 2$, we have a positive characteristic value $a^* = 2.49527$ and a negative characteristic value $a^* = -3.47044$, respectively. For different values of the dimensionless driving acceleration F , we have different characteristic value a^* . The two curves of the characteristic value a^* versus the dimensionless driving acceleration F are as shown in Figs. 1 and 2. Note that, the maximum positive characteristic value a_{\max}^* is 2.52168, corresponding to the dimensionless driving acceleration $F = 2.28$. Note that, according to the excellent experiment of Rajchenbach, Leroux and Clamond [6], the parametric resonance was found when $F = 2.0$. So, the above theoretical result can partly explain why Rajchenbach, Leroux and Clamond [6] observed the parametric resonance in case of $F = 2.0$, since the corresponding characteristic value $a^* = 2.49527$ is rather close to $a_{\max}^* = 2.52168$ so that the parametric resonance more easily created and observed. We will discuss this later in detail.

According to (11), as long as the characteristic value a^* is known, it is easy to gain the unknown eigenvalue λ by solving the nonlinear algebraic equation

$$\lambda^2 \left(1 + \frac{\lambda^2}{3}\right) + \mu = 0, \tag{13}$$

where

$$\mu = \frac{a^*}{G}.$$

Thus, λ is dependent upon the dimensionless gravity acceleration G and the dimensionless driving acceleration F , since a^* is determined by F only. When $\mu > 3/4$, the above nonlinear algebraic equation has four complex roots

$$\lambda_{1,2,3,4} = \pm \sqrt{-\frac{3}{2} \pm i\sqrt{3\left(\mu - \frac{3}{4}\right)}}, \tag{14}$$

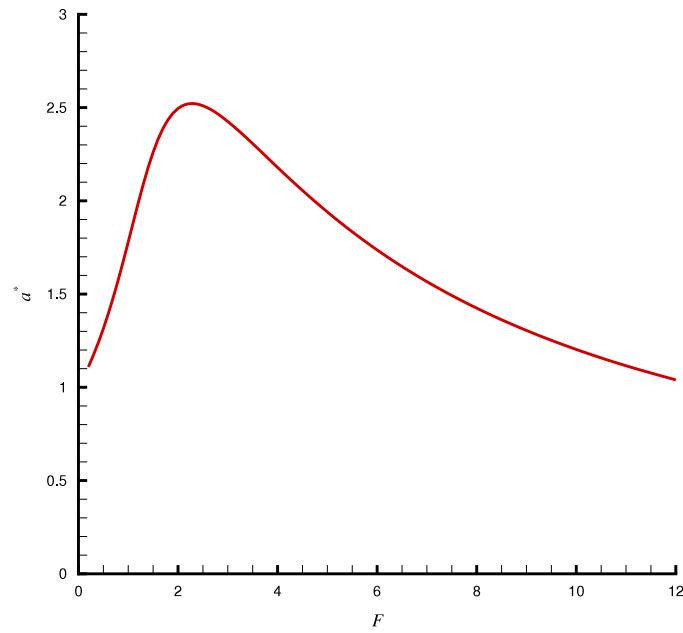


Fig. 1. The positive characteristic values of a^* versus the dimensionless driving acceleration F given by the linearized Boussinesq equation (7).

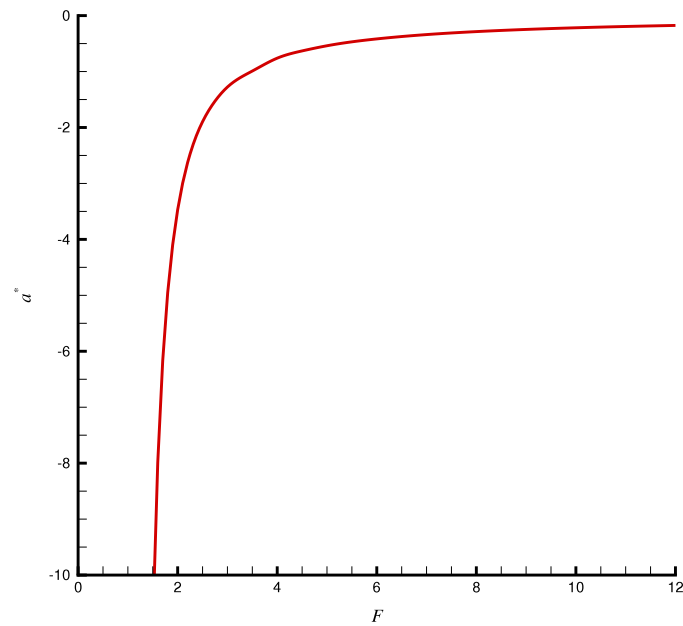


Fig. 2. The negative characteristic values of a^* versus the dimensionless driving acceleration F given by the linearized Boussinesq equation (7).

where $i = \sqrt{-1}$ denotes the imaginary unit. When $0 < \mu < 3/4$, there exist four pure imaginary roots

$$\lambda_{1,2} = \pm i \sqrt{\frac{3}{2} \pm \sqrt{3 \left(\frac{3}{4} - \mu \right)}}. \tag{15}$$

When $\mu < 0$, there are two pure imaginary roots

$$\lambda_{3,4} = \pm i \sqrt{\frac{3}{2} \left(\sqrt{1 - \frac{4}{3}\mu} + 1 \right)} \tag{16}$$

and two real roots

$$\lambda_{3,4} = \pm \sqrt{\frac{3}{2} \left(\sqrt{1 - \frac{4}{3}\mu} - 1 \right)}. \tag{17}$$

Thus, when $\mu > 3/4$, we have four complex roots $\lambda = \pm\alpha \pm \beta i$ with $\alpha > 0$ and $\beta > 0$, corresponding to the wave elevation in a general form

$$\eta(x, \tau) = A_1 Mc(\tau; a^*, q^*) e^{-\alpha x} (\cos \beta x + i \sin \beta x) + A_2 Mc(\tau; a^*, q^*) e^{-\alpha x} (\cos \beta x - i \sin \beta x) + A_3 Mc(\tau; a^*, q^*) e^{+\alpha x} (\cos \beta x + i \sin \beta x) + A_4 Mc(\tau; a^*, q^*) e^{+\alpha x} (\cos \beta x - i \sin \beta x), \quad (18)$$

where A_1, A_2, A_3 and A_4 are constants. However, restricted by the bounded condition (6), only the solution in the form

$$\eta(x, \tau) = A_1 Mc(\tau; a^*, q^*) e^{-\alpha x} (\cos \beta x + i \sin \beta x) + A_2 Mc(\tau; a^*, q^*) e^{-\alpha x} (\cos \beta x - i \sin \beta x) = Mc(\tau; a^*, q^*) e^{-\alpha x} (A_0 \cos \beta x + B_0 \sin \beta x), \quad 0 < x < +\infty, \quad (19)$$

has physical meanings, where $A_0 = A_1 + A_2$ and $B_0 = (A_1 - A_2)i$ are real constants.

Thus, using the boundary condition (4) for the odd symmetry and enforcing $A_0 = 0$, we have the wave elevation

$$\eta(x, \tau) = B_0 Mc(\tau; a^*, q^*) \sin(\beta x) e^{-\alpha x}, \quad 0 < x < +\infty. \quad (20)$$

Then, due to the odd symmetry (2), we have the odd-pattern elevation

$$\eta(x, \tau) = B_0 Mc(\tau; a^*, q^*) \sin(\beta x) e^{-\alpha|x|}, \quad -\infty < x < +\infty. \quad (21)$$

Similarly, using the boundary condition (3) for the even symmetry (1), we have the even-pattern wave elevation

$$\eta(x, \tau) = A_0 Mc(\tau; a^*, q^*) e^{-\alpha|x|} \left[\cos(\beta x) + \left(\frac{\alpha}{\beta}\right) \sin(\beta|x|) \right], \quad (22)$$

which has a smooth crest and is valid in the whole domain $-\infty < x < +\infty$. Note that the standing solitary wave elevations (21) and (22) decay non-monotonically in the x direction, and have no peaked crest.

For example, in case of the driving frequency $\Omega = 11$ Hz with the vibration amplitude 4.1 mm and water depth 5 cm, which were used by Rajchenbach, Leroux and Clamond [6] in their excellent experiment, we have the dimensionless driving acceleration $F \approx 2$ and the dimensionless gravity acceleration $G \approx 0.164$. When $F = 2$, there exist one positive characteristic $a^* = 2.49527$ and one negative characteristic $a^* = -3.47044$, corresponding to $\mu = a^*/G = 15.2151$ and $\mu = -21.1612$, respectively. Especially, when $a^* = 2.49527$, i.e. $\mu = 15.2151$, we have four complex eigenvalues

$$\lambda = \pm 1.62113 \pm 2.03126i,$$

corresponding to a non-monotonically decaying standing solitary wave with the odd symmetry and the smooth crest

$$\eta(x, \tau) = A_0 Mc(\tau; 2.49527, 2.49527) \sin(2.03126x) e^{-1.62113|x|}, \quad (23)$$

as shown in Fig. 3, and the wave elevation with even symmetry and smooth crest

$$\eta(x, \tau) = A_0 Mc(\tau; 2.49527, 2.49527) e^{-1.62113|x|} [\cos(2.03126x) + 0.798091 \sin(2.03126|x|)], \quad (24)$$

as shown in Fig. 4, respectively.

It should be emphasized that the standing solitary wave (23) has the odd parity about $x = 0$. Note that Rajchenbach, Leroux and Clamond [6] found a similar standing solitary wave with the odd parity in their excellent experiment, and pointed out that “the existence of an oscillon of odd parity had never been reported in any media up to now”. Thus, the closed-form solution (23) might provide a theoretical explanation for this experimental phenomenon.

Note that the standing solitary waves (23) and (24) do not decay monotonically, as shown in Figs. 3 and 4, which are qualitatively similar to those experimentally found by Rajchenbach, Leroux and Clamond [6]. Note that these non-monotonically decaying standing solitary waves (23) and (24) are not exactly the same as those found by the excellent experiment of Rajchenbach, Leroux and Clamond [6]. Such a kind of difference may likely be attributed to the probe motion in their experiment [6], and also to the neglect of the nonlinearity of the Boussinesq equation that is valid for fairly long waves with small-amplitude in shallow water. The nonlinearity of the Boussinesq equation might affect the eigenvalue λ , which determines the decay-rate of wave elevation. However, the nonlinear terms should not qualitatively influence the shape of wave elevation. Since we mainly focus on the shape of wave elevation in this article, the neglect of the nonlinear terms is acceptable. Obviously, better analytic approximations of the two new standing solitary waves should be gained, if the exact Boussinesq equation (5) or the fully nonlinear wave equation is solved.

3. Discussions and concluding remarks

In this paper, some new Faraday’s waves due to the parametric resonance of liquid in a vessel vibrating vertically with a constant frequency are reported. Using a model based on the symmetry of wave elevation and the linearized Boussinesq equation, we gain the closed-form solutions of two kinds of non-monotonically decaying standing solitary waves with the even or odd symmetry. They can explain well, although partly, some experimental phenomena currently reported by Rajchenbach, Leroux and Clamond [6].

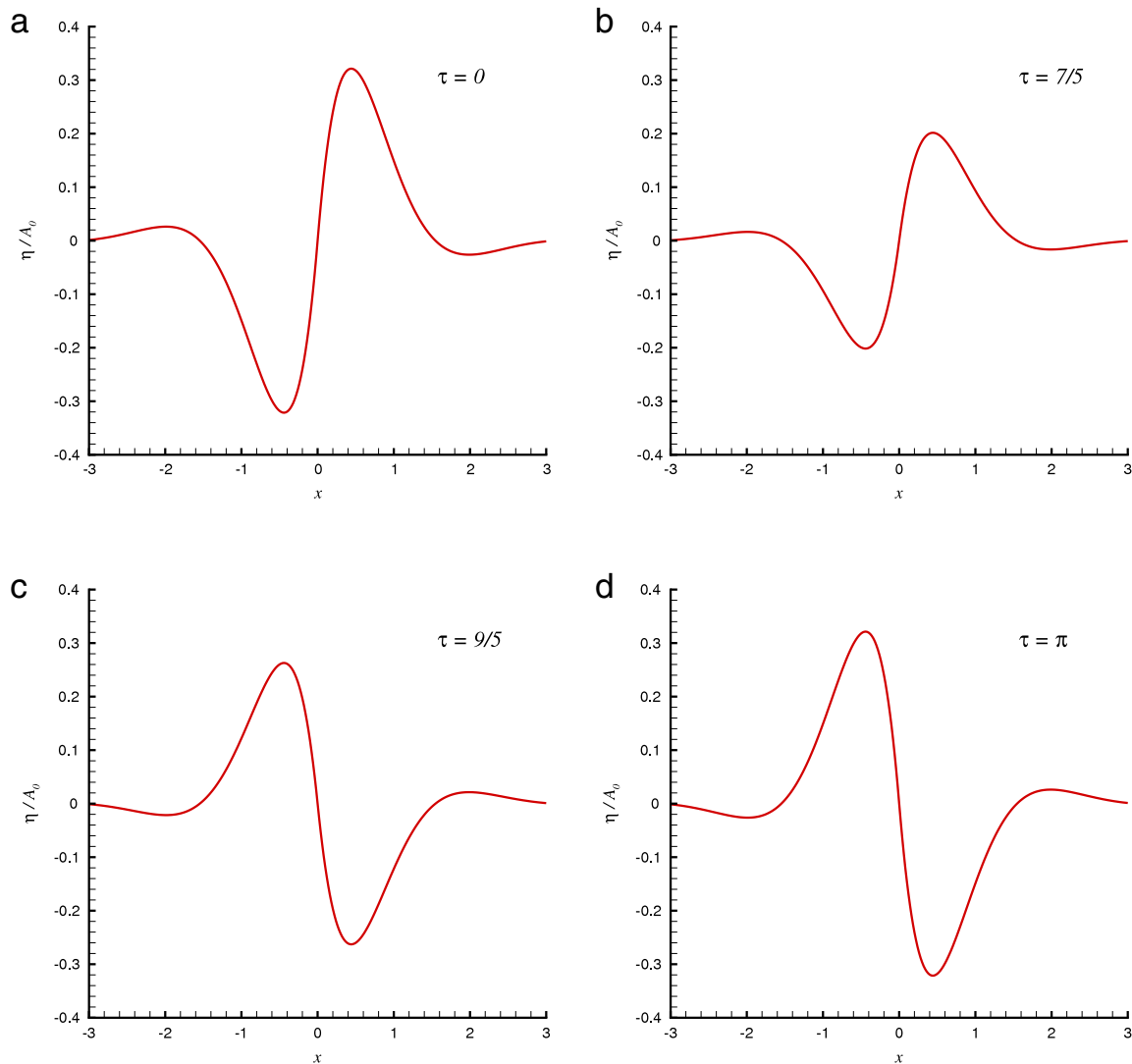


Fig. 3. Non-monotonically decaying standing solitary wave (23) with the odd symmetry and a smooth crest. (a): $\tau = 0$; (b): $\tau = 7/5$; (c): $\tau = 9/5$; (d): $\tau = \pi$.

First, our closed-form solution (21) has the odd parity about $x = 0$, as shown in Fig. 3. Note that Rajchenbach, Leroux and Clamond [6] found a similar standing solitary wave with the odd parity in their excellent experiment, and pointed out that “the existence of an oscillon of odd parity had never been reported in any media up to now”. So, our closed-form solution (21) supports this experimental phenomenon.

Secondly, based on the linearized Boussinesq equation, the characteristic value a^* is dependent upon the dimensionless driving acceleration F only. Thus, for given dimensionless gravity acceleration $G = g/(h\omega^2) = 4g/(h\Omega^2)$, the occurrence of the non-monotonically decaying standing solitary waves mainly depends on the dimensionless driving acceleration $F = \Gamma/g$ of the vertically vibrating bottom: the larger a^* , the larger possibility of the occurrence of the non-monotonically decaying standing solitary waves, since $\mu = a^*/G > 3/4$ is the criterion for the linearized Boussinesq equation. According to Fig. 1, the non-monotonically decaying standing solitary waves occur with the maximum possibility at $F \approx 2.28$, corresponding to the maximum characteristic value $a^*_{\max} = 2.52168$. Note that Rajchenbach, Leroux and Clamond [6] observed the two non-monotonically decaying standing solitary waves at $F = 2$, corresponding to the characteristic value $a^* = 2.49527$ that is rather close to $a^*_{\max} = 2.52168$. So, our theoretical result can explain this experimental phenomena quite well.

Thirdly, Rajchenbach, Leroux and Clamond [6] found experimentally that the two non-monotonically decaying standing solitary waves occur in an interval $F_L < F < F_R$. They gave it a theoretical explanation using Meron’s stability theory [11]. Based on the linearized Boussinesq equation, we gain the criterion of occurrence of the two non-monotonically decaying standing solitary waves: $\mu = a^*/G > 3/4$. According to Fig. 1, a^* has a maximum $a^*_{\max} = 2.52168$ at $F = 2.28$. So, given a proper value of G , one might find a closed interval of F for the occurrence of the two non-monotonically decaying standing solitary waves. Thus, our theoretical result about the criterion $\mu > 3/4$ can explain this experimental phenomenon, too.

Seriously speaking, the profile of the standing waves should be dependent upon not only the dimensionless accelerations G , F but also the wave height. However, based on the linearized Boussinesq equation, the profile of the two standing waves is dependent on G and F only. This is similar to the periodic traveling waves, whose wave profile is sinusoidal and independent

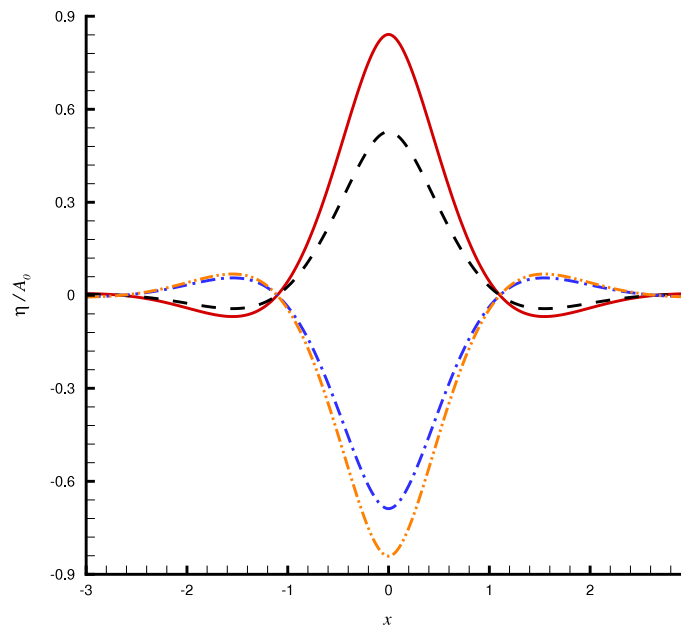


Fig. 4. Non-monotonically decaying standing solitary wave (24) with even symmetry and a smooth crest. Solid line: $\tau = 0$; Dashed line: $\tau = 7/5$; Dash-dotted line: $\tau = 3\pi/4$; Dash-dot-dotted line: $\tau = \pi$.

of wave height. Besides, the detailed evolution of the standing waves should be also influenced by wave height. So, if the influence of wave height is considered, the nonlinear Boussinesq equation or even the fully nonlinear water wave equations should be used for a more accurate wave profile and better understanding about such a kind of standing waves.

Note that, using the even or odd symmetry, we have either the boundary condition (3) or (4) at $x = 0$, so that it is *enough* for the governing equation to be satisfied in the interval $0 < x < +\infty$ except $x = 0$. This is well-known and widely used in the theory of differential equations.

Traditionally, one needs to give a global expression of a solution in the whole domain. However, this is difficult in many cases. Fortunately, this traditional idea is out of date. In modern mathematics, we often express a smooth function by lots of local simple functions in a finite number of sub-domains: this idea is widely used in the Finite Element Method (FEM). Although there exists singularity at each boundary of the sub-domain where the solution is not smooth, such kinds of solutions are widely accepted and used. In this paper, we use different base functions to express the solution of a new type of standing waves found by experiment in two sub-domains only, i.e. $(-\infty, 0)$ and $(0, +\infty)$, and connect them by the symmetry and the smoothness condition at $x = 0$. Compared to the FEM, the mathematical approach used in this article is more traditional: we use the symmetry to divide the whole domain into only two sub-domains, and besides the solutions are smooth at $x = 0$.

Finally, it should be emphasized that the closed-form wave elevations of the two non-monotonically decaying standing solitary waves (21) and (22) are obtained under the assumption of the even or odd symmetry of wave elevation by means of the linearized Boussinesq equation with the neglect of viscosity of fluid in the interval $0 < x < +\infty$. The symmetry has an important role in our approach. The fact that the two closed-form solutions explain well some phenomenon of the excellent experiment of Rajchenbach et al. [6] indicates the validity of this model. In addition, since the linearized Boussinesq equation is only a simplified model for shallow water waves, all conclusions and theoretical predictions reported in this article should be further checked and verified by fine numerical simulations and physical experiments in future, even though our closed-form solutions explain well some experimental phenomena of Rajchenbach et al. [6]. All of these are helpful to deepen and enrich our understanding about standing solitary waves and Faraday's wave.

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