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Boundary element method for general nonlinear differential operators

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In this paper, the basic ideas of homotopy in topology is applied to give a kind of high-order Boundary Element Method (BEM) formulations for strongly nonlinear problems governed by quite general nonlinear differential operators which may NOT contain any linear operators at all. As a result, the traditional BEM which treats the nonlinear parts as the inhomogeneities is only a special case of the proposed formulations. Two simple examples are used to illustrate its effectiveness. © 1997 Elsevier Science Ltd.

Key words: BEM, nonlinearity, general differential operators.

1 INTRODUCTION

Although Boundary Element Method (BEM) is in principle based on the linear superposition of fundamental solutions[1–3,5,13], many researchers[8,12–14] have applied it to solve nonlinear differential equation

$$\mathcal{A}(u)=f. \tag{1}$$

If this nonlinear operator $\mathcal{A}$ can be divided into two parts $\mathcal{L}_0$ and $\mathcal{N}_0$, where $\mathcal{L}_0$ is linear, $\mathcal{N}_0$ is nonlinear and $\mathcal{A}=\mathcal{L}_0+\mathcal{N}_0$ holds, then traditionally, writing the original eqn (1) as $\mathcal{L}_0(u)=f-\mathcal{N}_0(u)$, we can obtain the following equation of integral operator

$$c(\vec{r})u(\vec{r})=\int_\Gamma [u\mathcal{B}_0(\omega_0)-\omega_0\mathcal{B}_0(u)]\,d\Gamma + \int_\Omega [f-\mathcal{N}_0(u)]\omega_0\,d\Omega, \tag{2}$$

where ω_0 is the fundamental solution of the adjoint operator of the linear differential operator $\mathcal{L}_0$, $\mathcal{B}_0$ is its boundary operator, Γ denotes the boundary of the domain Ω. Note that the domain integral of above equation contains the unknown function $u(\vec{r})$ so that iteration is necessary.

Obviously, the operator $\mathcal{L}_0$, which denotes linear parts of the nonlinear operator $\mathcal{A}$, has special meaning for above traditional BEM: firstly, this kind of linear operator $\mathcal{L}_0$ must exist; secondly, we must know its corresponding fundamental solution ω_0. But unfortunately, both of these are not always satisfied so that this traditional BEM described above has the following restrictions:

1. Many nonlinear differential equations do NOT contain any linear terms at all, i.e., $\mathcal{A}=\mathcal{L}_0+\mathcal{N}_0$ does NOT hold, so that the traditional BEM is useless;
2. Even if the nonlinear differential operator $\mathcal{A}$ contains this kind of linear operator $\mathcal{L}_0$, $\mathcal{L}_0$ may be so complex that the corresponding fundamental solution is either unknown or very difficult to be obtained.

So, it seems necessary to develop a kind of new BEM for quite general nonlinear problems,

(I) which can be applied to solve equations governed by quite general nonlinear differential operators that may NOT contain any linear terms at all;
(II) which can give us great freedom to select a proper, simple linear operator whose fundamental solution should be familiar to us;
(III) which contains logically the traditional BEM.

We can give an example of the applications of the traditional BEM in solving nonlinear problems. We know that Navier–Stokes equations are usually very difficult to be solved. A boundary element method of solving Navier–Stokes equations in streamfunction-vorticity formulations was presented in the reference[12] in 1990, which is based on a set of fundamental solutions providing a complete coupling between the streamfunction and vorticity equations so that iteration is not needed in case $R_e=0$. In[12], the nonlinear terms of Navier–Stokes equations are considered in the traditional way as the inhomogeneities are treated by simple direct iteration, but this numerical scheme is unstable for the 2D viscous flow in a square cavity in case that the corresponding Reynold's number R_e is greater than 300, as mentioned in the reference[12].

The author has been trying to develop a kind of new nonlinear analytical technique, namely Homotopy Analysis Method[6–11], by means of the basic ideas of homotopy in topology[4]. Homotopy Analysis Method (which is called Process Analysis Method in the author's early research) is independent of the presence of small parameters of the considered nonlinear problems so that, different from the well-known perturbation techniques, it can be applied to solve nonlinear problems which contain no small parameters. As one of its applications, the author[8] gave a kind of high-order streamfunction-vorticity BEM formulation for 2D steady-state Navier–Stokes equations. The corresponding first-order formulations are the same as those given in[12] and are also unstable in case $R_e > 300$ for the 2D viscous square cavity flow, but the high-order (for instance, second-order) formulations are still stable in case $R_e = 2000$. The author's current research indicates that these high-order BEM formulations are even still stable in case $R_e = 10^4$ for 2D square cavity flow which corresponds to a very strong nonlinearity.

In this paper, the basic ideas described in the reference[8] are greatly generalized to give a new, much more general BEM for strongly nonlinear problems. And two simple examples are used to illustrate the effectiveness of the proposed method.

2 THE BASIC IDEAS OF THE PROPOSED BEM

Consider a quite general nonlinear differential operator $\mathcal{A}$ which may NOT contain any linear terms at all, and then research again the eqn (1).

Select a proper, simple linear operator $\mathcal{L}$, whose fundamental solution is familiar to us and which may be different from $\mathcal{L}_0$ even if $\mathcal{L}_0$ exists. Then, we can construct a homotopy $\nu(\vec{r},p)\quad \Omega\times[0,1]\rightarrow\mathbf{R}$, which satisfies

$$\mathcal{L}(\nu)=(1-p)\mathcal{L}(u_0)+p[\mathcal{L}(\nu)-\mathcal{A}(\nu)+f],\quad p\in[0,1], \tag{3}$$

where $u_0(\vec{r})$ is an initial solution which can be selected with great freedom, $p\in[0,1]$ is the imbedding parameter, $\nu(\vec{r},p)$ is now a function of both $p\in[0,1]$ and Ω. For simplicity, we call the eqn (3) the zero-order deformation equation.

Obviously, from eqn (3), the following two expressions

$$\nu(\vec{r},0)=u_0(\vec{r}), \tag{4}$$

$$\nu(\vec{r},1)=u(\vec{r}), \tag{5}$$

hold, where $u(\vec{r})$ is the solution of eqn (1). Therefore, $u_0(\vec{r})$ and $u(\vec{r})$ are homotopic, denotes as $\nu(\vec{r},p):u_0(\vec{r})\simeq u(\vec{r})$.

Assume that the 'continuous deformation' $\nu(\vec{r},p)$ is smooth enough about p so that

$$\nu^{[m]}(\vec{r},p)=\frac{\partial^m\nu(\vec{r},p)}{\partial p^m},\quad m=1,2,3,\ldots, \tag{6}$$

called mth-order deformation derivatives, exist. Then, according to the theory of Taylor's series, we have from (4) that

$$\begin{aligned}\nu(\vec{r},p) &= .\nu(\vec{r},0)+\sum_{m=1}^{\infty}\left.\frac{\partial^m\nu(\vec{r},p)}{\partial p^m}\right|_{p=0}\left(\frac{p^m}{m!}\right)\\ &= u_0(\vec{r})+\sum_{m=1}^{\infty}\left(\frac{p^m}{m!}\right)\nu_0^{[m]}(\vec{r}),\end{aligned} \tag{7}$$

where, $\nu_0^{[m]}(\vec{r})$ is the value of $\nu^{[m]}(\vec{r},0)$, which can be obtained in the way described later. We call the expression (7) the *Taylor's homotopy series*.

The value of the convergence radius ρ of the Taylor's series (7) is generally finite. In case $\rho\geq 1$, it holds from (5) and (7) that

$$u(\vec{r})=u_0(\vec{r})+\sum_{m=1}^{\infty}\frac{\nu_0^{[m]}(\vec{r})}{m!}. \tag{8}$$

But, in case $\rho<1$, we have only

$$\nu(\vec{r},\lambda)=u_0(\vec{r})+\sum_{m=1}^{\infty}\left[\frac{\nu_0^{[m]}(\vec{r})}{m!}\right]\lambda^m, \tag{9}$$

where $0<\lambda<\rho<1$. Note that $\nu(\vec{r},\lambda)$ obtained by above expression is usually a better approximation than the initial solution $u_0(\vec{r})$ so that expression (9) gives a family of the high-order iterative formulations:

$$u_{k+1}(\vec{r})=u_k(\vec{r})+\sum_{m=1}^{M}\left[\frac{\nu_0^{[m]}(\vec{r})}{m!}\right]\lambda^m,\quad (k=0,1,2,\ldots), \tag{10}$$

where M ($M=1,2,3,\ldots$) denotes the order of the formulation, $\nu_0^{[m]}(\vec{r})$ ($m=1,2,3,\ldots$) are dependent upon $u_k(\vec{r})$ and can be determined as follows.

Differentiating the zero-order deformation eqn (3) with respect to the imbedding parameter p, we obtain *the first-order deformation equation*

$$\begin{aligned}\mathcal{L}(\nu^{[1]}) &= -\mathcal{L}(u_0)+\mathcal{L}(\nu)\\ &\quad -\mathcal{A}(\nu)+f+p\left\{\mathcal{L}(\nu^{[1]})-\frac{\partial\mathcal{A}(\nu)}{\partial\nu}\nu^{[1]}\right\}.\end{aligned} \tag{11}$$

And similarly, differentiating above equation with respect to p gives the *second-order deformation equation*

$$\begin{aligned}\mathcal{L}(\nu^{[2]}) &= 2\left\{\mathcal{L}(\nu^{[1]})-\frac{\partial\mathcal{A}(\nu)}{\partial\nu}\nu^{[1]}\right\}\\ &\quad +p\left\{\mathcal{L}(\nu^{[2]})-\frac{\partial\mathcal{A}(\nu)}{\partial\nu}\nu^{[2]}-\frac{\partial^2\mathcal{A}(\nu)}{\partial\nu^2}(\nu^{[1]})^2\right\}.\end{aligned} \tag{12}$$

Generally, we have the *mth-order deformation equations* at $p=0$ as follows:

$$\mathcal{L}(\nu_0^{[m]})=f_m(\vec{r}), \tag{13}$$

where

$$f_1(\vec{r})=f-\mathcal{A}(u_k), \tag{14}$$

$$f_2(\vec{r}) = 2\left\{\mathcal{L}(\nu_0^{[1]}) - \frac{\partial \mathcal{A}(\nu)}{\partial \nu}\nu_0^{[1]}\right\}\Bigg|_{p=0}, \tag{15}$$

$$f_m(\vec{r}) = m\left\{\mathcal{L}(\nu_0^{[m-1]}) - \frac{d^{m-1}\mathcal{A}(\nu)}{dp^{m-1}}\Bigg|_{p=0}\right\} \quad (m > 1). \tag{16}$$

Note that $f_1(\vec{r})$ is the negative residual of the original eqn (1) and is the same for ANY linear operators $\mathcal{L}$ selected for the proposed BEM. The above linear eqn (13) can be easily solved by Boundary Element Method so that, according to (2), we obtain the corresponding boundary integral equation

$$c(\vec{r})\nu_0^{[m]}(\vec{r}) = \int_\Gamma \left[\nu_0^{[m]}\mathcal{B}(\omega) - \omega\mathcal{B}(\nu_0^{[m]})\right] d\Gamma + \int_\Omega f_m \omega \, d\Omega. \tag{17}$$

Note that we have now very great freedom to select the corresponding linear operator $\mathcal{L}$, i.e., we can now select a simple, proper linear operator whose fundamental solution is familiar to us, even if the considered nonlinear problem does NOT contain any linear operators at all! Especially, if $\mathcal{A} = \mathcal{L}_0 + \mathcal{N}_0$ holds and we select $\mathcal{L} = \mathcal{L}_0$ as the linear operator, the formulation (17) in case $m = 1$ gives the same expression as the expression (2). Obviously, for the proposed BEM, it is not important whether the nonlinear operator $\mathcal{A}$ contains the linear operator $\mathcal{L}_0$ or not; and even if this kind of linear operator $\mathcal{L}_0$ exists, we may still select other simpler linear operators, because the operator $\mathcal{L}_0$, which is very important for the traditional BEM, has now no special meaning at all for the proposed BEM – it is only one of many possible linear operators $\mathcal{L}$ suited to the proposed BEM. Thus, the three demands (I), (II), and (III) listed in the first section are completely satisfied.

As the last part of this section, let us consider some simple points. We know that homotopy technique[4] emphasizes the relations and the continuous changes between different things. As a result of it, the Taylor's homotopy series (7) gives a kind of relation between the solution $u(\vec{r})$ and the free selected initial solution $u_0(\vec{r})$ by *infinite* number of $\nu_0^{[m]}(\vec{r})$, the high-order deformation derivatives at $p = 0$. It must be emphasized that $\nu_0^{[m]}(\vec{r})$ $(m = 1,2,3,\ldots)$ satisfies the linear eqn (13) so that the Taylor's homotopy series (7) converts a nonlinear problem into infinite number of corresponding linear problems (I believe that to find a new nonlinear technique is equivalent to finding a new kind of such conversion). Note that we obtain this kind of conversion without using small parameters – perturbation techniques use small parameters supposition to obtain this kind of conversion – so that the proposed method is independent upon small parameters. This is the main basic ideas of Homotopy Analysis Method, whose effectiveness has been proved in other papers of the author[6–11]. Because the mth-order deformation eqn (13) is always linear about mth-order deformation derivatives $\nu_0^{[m]}(\vec{r})$ (a mathematical proof about it has been given by Liao[6]), it is natural to apply Boundary Element Method (BEM) to solve these linear equations. Thus, the proposed BEM is only an application of the Homotopy Analysis Method.

3 TWO SIMPLE EXAMPLES

3.1 Example 1

In order to illustrate the effectiveness of the proposed BEM, let us consider at first the following nonlinear boundary-value problem

$$x^2U_{xx} + xU_x + (x^2-1)U - \alpha(U^2+U_x^2) = x\cos(x) - \sin(x) - \alpha, \quad x \in [0, 2\pi], \ \alpha > 0, \tag{18}$$

with the two boundary conditions $U(0) = U(2\pi) = 0$.

We can use respectively the following three sorts of linear operators

$$\text{MODE 0}: \ \mathcal{L}_0(U) = x^2U_{xx} + xU_x + (x^2-1)U,$$

$$\text{MODE 1}: \ \mathcal{L}_1(U) = U_{xx} - \beta^2 U, \quad \beta > 0,$$

$$\text{MODE 2}: \ \mathcal{L}_2(U) = U_{xx} + \beta^2 U, \quad \beta > 0,$$

to construct the corresponding zero-order deformation equation as follows

$$\mathcal{L}_\gamma(V) = (1-p)\mathcal{L}_\gamma(U_0) + p\{\mathcal{L}_\gamma(V) - \mathcal{A}(V) + f\},$$
$$x \in [0, 2\pi], \ p \in [0, 1], \ (\gamma = 0, 1, 2) \tag{19}$$

which has two boundary conditions

$$V(0, p) = V(2\pi, p) = 0, \tag{20}$$

where

$$\mathcal{A}(V) = x^2V_{xx} + xV_x + (x^2-1)V - \alpha(V^2+V_x^2),$$

$$f(x) = x\cos(x) - \sin(x) - \alpha,$$

$V(x,p) : [0,2\pi] \times [0,1] \to \mathbf{R}$ is a kind of homotopy and $U_0(x)$ is a free selected initial solution satisfying the boundary conditions $U_0(0) = U_0(2\pi) = 0$.

Similarly, we can obtain the corresponding high-order iterative formulations

$$U_{k+1}(x) = U_k(x) + \sum_{m=1}^{M} \frac{\lambda^m V_0^{[m]}(x)}{m!}, \quad (k = 0, 1, 2, \ldots), \tag{21}$$

where $V_0^m(x)$ satisfies the following linear differential equation

$$\mathcal{L}_\gamma(V_0^{[m]}) = f_m(x),$$
$$x \in [0, 2\pi] \ (\gamma = 0, 1, 2, \ m = 1, 2, 3\ldots) \tag{22}$$

with the two boundary conditions

$$V_0^{[m]}(0) = V_0^{[m]}(2\pi) = 0. \tag{23}$$

Here,

$$f_1(x) = x\cos(x) - \sin(x) - \alpha - \mathcal{A}(U_k),$$

$$f_2(x) = 2\left\{\mathcal{L}_\gamma(V_0^{[1]}) - \mathcal{L}_0(V_0^{[1]}) + 2\alpha\left[U_k V_0^{[1]} + \frac{\mathrm{d}U_k}{\mathrm{d}x}\frac{\mathrm{d}V_0^{[1]}}{\mathrm{d}x}\right]\right\},$$

$$f_3(x) = 3\left\{\mathcal{L}_\gamma(V_0^{[2]}) - \mathcal{L}_0(V_0^{[2]}) + 2\alpha\left[U_k V_0^{[2]} + \frac{\mathrm{d}U_k}{\mathrm{d}x}\frac{\mathrm{d}V_0^{[2]}}{\mathrm{d}x} + (V_0^{[1]})^2 + \left(\frac{\mathrm{d}V_0^{[1]}}{\mathrm{d}x}\right)^2\right]\right\},$$

In case $\gamma = 0$ (MODE 0), we have simply the solution of linear eqn (22) with the boundary condition (23) as follows:

$$V_0^{[m]}(x) = \int_0^{2\pi} \omega_0(x,x') f_m(x')\,\mathrm{d}x', \quad x \in [0, 2\pi]\ (m \geq 1), \tag{24}$$

where

$$\omega_0(x,x') = \frac{\pi J_1(x_<)[J_1(x_>)N_1(2\pi) - N_1(x_>)J_1(2\pi)]}{2x' J_1(2\pi)},$$
$$x_< = \min(x,x'),\ x_> = \max(x,x'), \tag{25}$$

is the corresponding fundamental solution of MODE 0. Here, $J_1(x)$, $N_1(x)$ are respectively the Bessel's functions of the first and the second kind.

But, in case $\gamma = 1$ or $\gamma = 2$, we have

$$V_0^{[m]}(x) = C_m\omega_\gamma(x,0) - D_m\omega_\gamma(x,2\pi) + \int_0^{2\pi} \omega_\gamma(x,x') f_m(x')\,\mathrm{d}x', \quad (m = 1,2,3,\ldots), \tag{26}$$

where,

$$\omega_1(x,x') = \frac{1}{2\beta} e^{-\beta|x-x'|}, \tag{27}$$

$$\omega_2(x,x') = -\frac{1}{2\beta}\sin(\beta|x-x'|), \tag{28}$$

are the corresponding fundamental solutions of MODE 1 and MODE 2, respectively. The two coefficients C_m and D_m can be determined by the boundary condition (23):

$$C_m\omega_\gamma(0,0) - D_m\omega_\gamma(0,2\pi) = -\int_0^{2\pi} \omega_\gamma(0,x') f_m(x')\,\mathrm{d}x', \tag{29}$$

$$C_m\omega_\gamma(2\pi,0) - D_m\omega_\gamma(2\pi,2\pi) = -\int_0^{2\pi} \omega_\gamma(2\pi,x') f_m(x')\,\mathrm{d}x'. \tag{30}$$

Note that the linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are well-known and their corresponding fundamental solutions ω_1 (x,x') and ω_2 (x,x') are very simple. However, the fundamental solution ω_0 (x,x') of the linear operator $\mathcal{L}_0$ is not only much more complex but also unfamiliar to many researchers. Also, it is 'singular' at $x = 0$.

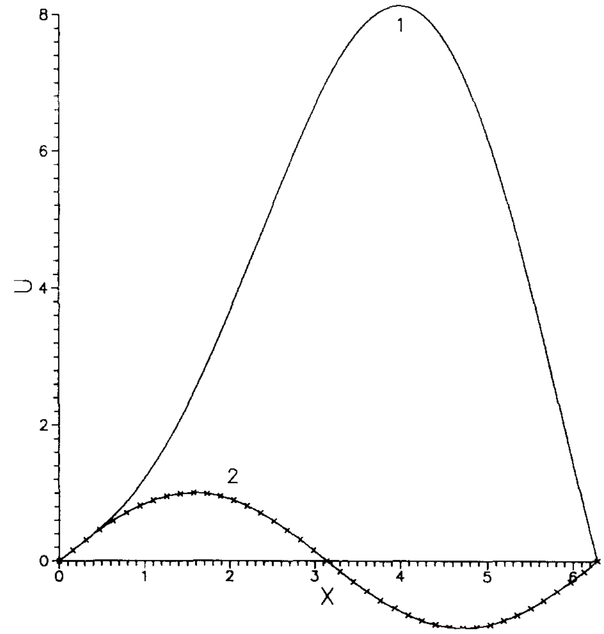

Fig. 1. The two solutions of example 1 in case $\alpha = 1.0$. Curve 1: solution $s_1(x,1.0)$; Curve 2: solution $\sin(x)$; Centered symbol: exact values of $\sin(x)$.

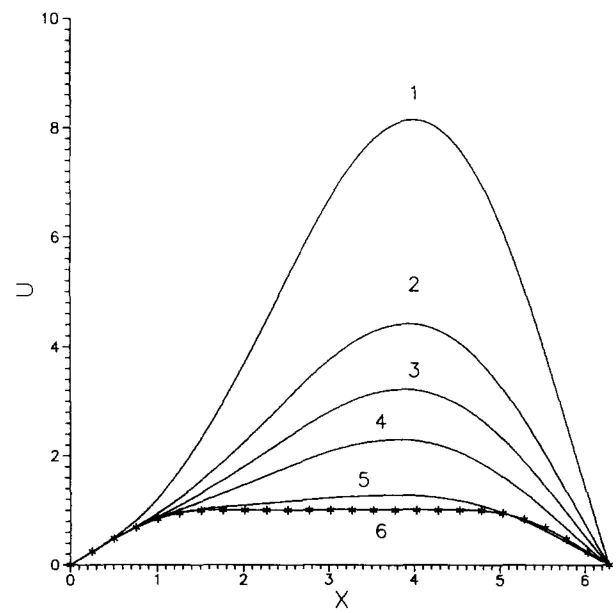

Fig. 2. The solutions s_1 (x,α) for example 1 $(1 \leq \alpha \leq 1000)$. Curve 1: $s_1(x,1)$; Curve 2: $s_1(x,2)$; Curve 3: $s_1(x,3)$; Curve 4: $s_1(x,5)$; Curve 5: $s_1(x,25)$; Curve 6: $s_1(x,1000)$; Centered symbol: $\mathcal{S}(x)$.

For the sake of numerical domain integral, we divide $[0,2\pi]$ into N equal sub-domains. Firstly, we select $N = 250$, $\lambda = 0.025$, $\alpha = 1.0$ and use the first-order formulation. Two different initial solutions, $U_0(x) = 0$, $U_0(x) = 0.25x(2\pi - x)$, and two different values of β^2, β^2

Table 1. CPU and iterative times in case $\alpha = 1$ (Example 1)

	β^2	$U_0(x) = 0$		$U_0(x) = 0.25x(2\pi - x)$		Solution
		Iterative times	CPU (s)	Iterative times	CPU (s)	
MODE 0		246	1603	298	1941	$\sin(x)$
MODE 1	0.1	357	346	357	346	$s_1(x,1.0)$
MODE 1	0.9	364	350	391	376	$s_1(x,1.0)$
MODE 2	0.1	354	312	348	308	$s_1(x,1.0)$
MODE 2	0.9	53	50	109	121	$\sin(x)$

Table 2. CPU and iterative times for different α (Example 1)

α	λ	Iterative times	CPU (s)
2	0.025	156	139
3	0.025	152	135
5	0.0125	149	133
25	0.0025	432	381
100	0.00125	136	122
1000	0.000125	152	136

$= 0.1$, $\beta^2 = 0.9$, are used respectively, and the corresponding CPU and iterative times are given in Table 1. Obviously, $\sin(x)$ is a solution of Example 1 for any values of α. But, there exists another solution for each α, which we denoted as $s_1(x,\alpha)$. The two different solutions for $\alpha = 1.0$ are shown as Fig. 1. According to Table 1, MODE 1 and MODE 2 need much less CPU to obtain convergent results than MODE 0, although MODE 1 ($\beta^2 = 0.1$, $\beta^2 = 0.9$) and MODE 2 ($\beta^2 = 0.1$) need more iterations. This is mainly because MODE 1 and MODE 2 use simpler linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and the corresponding fundamental solutions $\omega_1(x,x')$ and $\omega_2(x,x')$ are much simpler so that much less CPU is needed for the computation of them. It is interesting that MODE 2 ($\beta^2 = 0.9$) needs not only much less CPU but also less iterative times than MODE 0 for the corresponding initial solutions and numerical parameters mentioned in Table 1. It means that, for Example 1, the simpler linear operator $\mathcal{L}_2(U) = U_{xx} + 0.9U$ is much better than $\mathcal{L}_0(U) = x^2 U_x x + xU_x + (x^2 - 1) U$ itself, no matter from the view points of CPU or iterative times.

In case $\alpha = 1.0$, there exist two different solutions, one in $\sin(x)$, another is $s_1(x,1.0)$, shown as Fig. 1. It is interesting that MODE 1 and MODE 0 can give only one kind of solution, $s_1(x,1.0)$, for the two different initial solutions. But MODE 2 in case $\beta^2 = 0.1$ and $\beta^2 = 0.9$ can give respectively two different solutions for the two different initial solutions considered in Example 1. It's interesting that, for Example 1, the simpler linear operator $\mathcal{L}_2$ can give more kinds of solutions than the linear operator $\mathcal{L}_0$.

Because $\sin(x)$ is a common solution of Example 1 for all α, we are more interested in the solution $s_1(x,\alpha)$. Using MODE 2 ($\beta^2 = 0.1$, $N = 250$) and smaller values of λ, we obtain the convergent results of $s_1(x,\alpha)$ for $1 \leq \alpha \leq 1000$, shown as Fig. 2. Here, we apply the first-order formulations, i.e., $M = 1$; and $U_0(x) = 0.0$ is firstly used as the initial solution for $\alpha = 2$, then the corresponding convergent result $s_1(x,2)$ is used as the initial solution for $\alpha = 3$, and so on. The corresponding CPU and iterative times are given in Table 2. It is interesting that $s_1(x,1000)$ is very close to the function

$$\mathcal{S}(x) = \begin{cases} \sin(x) & \text{when } x \in [0, \pi/2], \\ 1.0 & \text{when } x \in [\pi/2, 3\pi/2], \\ \sin(2\pi - x) & \text{when } x \in [3\pi/2, 2\pi], \end{cases} \tag{31}$$

shown as Fig. 3. This is reasonable. Obviously, the larger α becomes, the stronger the nonlinearity of the eqn (18) is. When the value of α tends to infinity, the eqn (18) tends to the following equation

$$U_x^2 + U^2 = 1, \quad x \in [0, 2\pi], \tag{32}$$

with the two boundary conditions $U(0) = U(2\pi) = 0$. Obviously, $\mathcal{S}(x)$ is one of the solutions of above first-order nonlinear differential equation. So, we have reason to believe that the $s_1(x,\alpha)$ ($\alpha > 0$) we have obtained in this paper is indeed the solution of the Example 1. It illustrates that, applying the proposed BEM by means of a much simpler linear operator, we can indeed obtain all solutions of the Example 1 for all values of α, even if α is very large which is corresponding to a quite strong nonlinearity.

In case $N = 500$, MODE 1 and MODE 2 can give convergent results which agree very well with those obtained in case $N = 250$, but MODE 0 diverges. This may be mainly because the fundamental solution $\omega_0(x,x')$ is singular at $x = 0$. If only from the view point of this, the linear operator $\mathcal{L}_0$ is the worst among these three operators.

The above simple Example illustrates that the traditional BEM, which simply treats the nonlinear parts as the inhomogeneities and the linear parts as the linear operator and then find out the corresponding fundamental solution of this linear operator $\mathcal{L}_0$, is only a special case of the proposed BEM. The linear operator $\mathcal{L}_0$ has now no special meaning at all. It is only one of the many possible linear operators suited to the proposed BEM and is often the worst one. In most cases, other simple linear operators, if selected properly, need less CPU and sometimes even less iterative times than $\mathcal{L}_0$, as illustrated by Example 1. This is reasonable, because the properties of nonlinear differential equations are not strongly dependent on the corresponding linear operator $\mathcal{L}_0$ even if $\mathcal{L}_0$ exists. So, we have no reason to believe that the linear operator $\mathcal{L}_0$, which has special

Table 3. Iteratives times, CPU and values of λ for Example 2

α	λ	$\beta^2 = 0.1$		$\beta^2 = 0.9$	
		Iterative times	CPU (s)	Iterative times	CPU (s)
0	0.975	47	48	19	20
± 0.25	0.5	3	5	16	18
± 0.50	0.025	9	11	25	27
± 5.0	0.025	96	95	111	108
± 100.0	0.0025	238	231	248	240
± 1000	0.00025	248	248	322	319

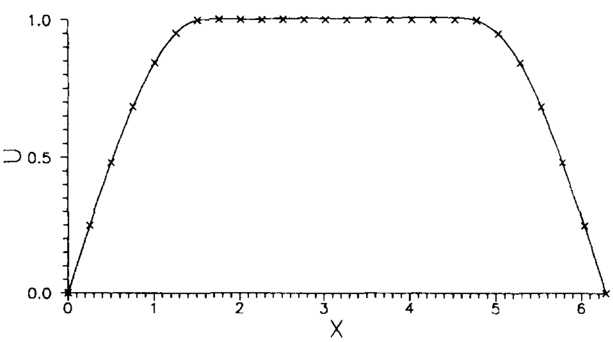

Fig. 3. The comparisons of $S(x)$ with $s_1(x,1000)$.

meaning and is very important for the traditional BEM, is better than other simple linear operators $\mathcal{L}$. The success of the traditional BEM in the limited number of nonlinear problems means ONLY that $\mathcal{L}_0$ can be used sometimes as one proper linear operator suited to Boundary Element Method, but does not mean that $\mathcal{L}_0$ is the only one and is the best one.

3.2 Example 2

As the second example, let us consider a nonlinear problem which does NOT contain any linear operators at all:

$$2W_{xx}\cos(W_{xx}) + \alpha(W_x^2 + W^2) = \alpha - \sin(x)\cos(\sin(x)),$$

$$x \in [0, 2\pi], \quad \alpha \in \mathbf{R}, \tag{33}$$

with the two boundary conditions $W(0) = W(2\pi) = 0$.

In this example, we apply the linear operator

$$\mathcal{L}_1(W) = W_{xx} - \beta^2 W, \quad (\beta > 0),$$

and also the corresponding fundamental solution

$$\omega_1(x, x') = -\frac{1}{2\beta}e^{-\beta|x-x'|},$$

which have been described in the Example 1. Similarly, we can obtain the corresponding high-order iterative formulations

$$W_{k+1}(x) = W_k(x) + \sum_{m=1}^{M} \frac{\lambda^m V_0^{[m]}(x)}{m!}, \quad (k = 0, 1, 2, \ldots), \tag{34}$$

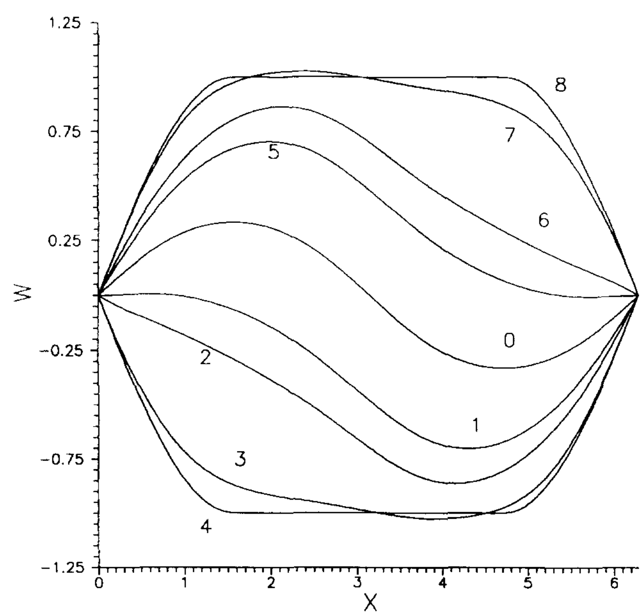

Fig. 4. The solutions $s_2(x,\alpha)$ for Example 2 ($-1000 \le \alpha \le 1000$). Curve 0: $s_2(x,0)$; Curve 1: $s_2(x,0.25)$; Curve 2: $s_2(x,0.50)$; Curve 3: $s_2(x,5.00)$; Curve 4: $s_2(x,1000)$; Curve 5: $s_2(x,-0.25)$; Curve 6: $s_2(x,-0.50)$; Curve 7: $s_2(x,-5.00)$; Curve 8: $s_2(x,-1000)$.

where, $V_o^{[m]}(x)$ satisfies the following linear differential equation

$$\mathcal{L}_1(V_0^{[m]}) = f_m(x), \quad x \in [0, 2\pi], \ (m = 1, 2, 3 \ldots), \tag{35}$$

with the two boundary conditions

$$V_0^{[m]}(0) = V_0^{[m]}(2\pi) = 0. \tag{36}$$

Here,

$$f_1(x) = \alpha - \sin(x)\cos(\sin(x)) - 2W_{xx}\cos(W_{xx}) - \alpha(W_x^2 + W^2), \tag{37}$$

$$f_2(x) = 2\left\{\mathcal{L}_1(V_0^{[1]}) - 2[\cos(W_{xx}) - W_{xx}\sin(W_{xx})]\frac{d^2V_0^{[1]}}{dx^2} - 2\alpha\left[W_x\frac{dV_0^{[1]}}{dx} + WV_0^{[1]}\right]\right\}, \tag{38}$$

Table 4. Iterative times and CPU of high-order formulations for Example 2

	Iterative times	CPU (s)	Iterative times	CPU (s)
First-order	47	48	19	20
Second-order	35	71	10	23
Third-order	20	60	7	21

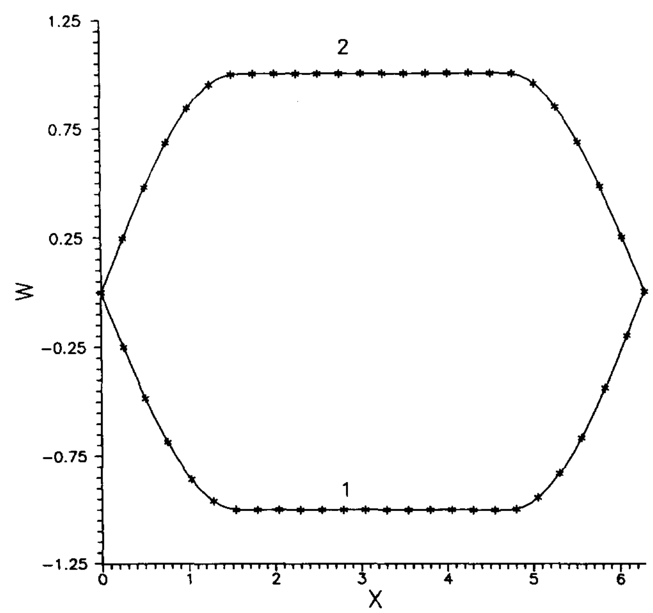

Fig. 5. The comparisons of $s_2\,(x, \pm\, 1000)$ with $\pm\, \mathcal{S}(x)$. Curve 1: $s_2(x,1000)$; Curve 2: $s_2(x, -\, 1000)$; Centered symbol: $\pm\, \mathcal{S}(x)$.

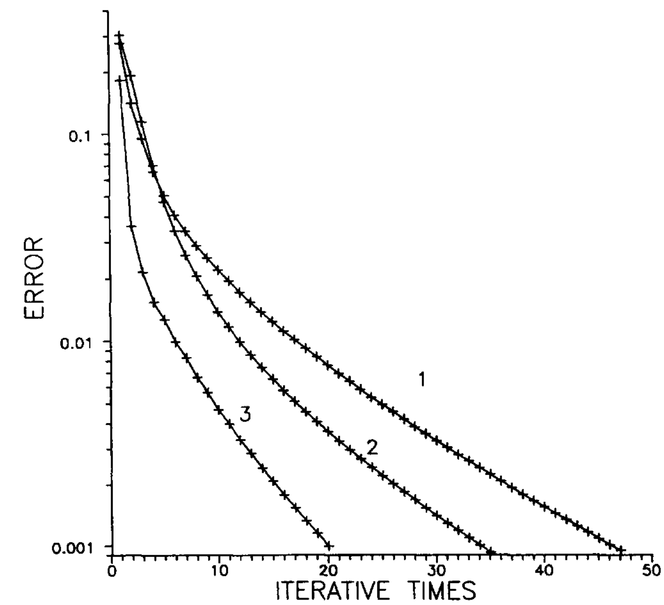

Fig. 6. The error-curve in iteration for Example 2. $\beta^2 = 0.1$, $\alpha = 0$, $W_0(x) = 0$, $\lambda = 0.975$; Curve 1: RMS_1 of first-order formulation; Curve 2: RMS_1 of second-order formulation; Curve 3: RMS_1 of third-order formulation.

$$f_3(x) = 3\left\{\mathcal{L}_1(V_0^{[2]}) - 2[\cos(W_{xx}) - W_{xx}\sin(W_{xx})]\frac{\mathrm{d}^2V_0^{[2]}}{\mathrm{d}x^2} + 2[2\sin(W_{xx}) + w_{xx}\cos(W_{xx})]\left(\frac{\mathrm{d}^2V_0^{[1]}}{\mathrm{d}x^2}\right)^2 - 2\alpha\left[W_x\frac{\mathrm{d}V_0^{[2]}}{\mathrm{d}x} + \left(\frac{\mathrm{d}V_0^{[1]}}{\mathrm{d}x}\right)^2 + WV_0^{[2]} + (V_0^{[1]})^2\right]\right\} \tag{39}$$

Similarly, we also divide $[0,2\pi]$ into N equal subdomains ($N = 250$) and use $W_0(x) = 0.0$ as the initial solutions for all values of α. We also use two different values of β^2, i.e., $\beta^2 = 0.9$, respectively, for the selected linear operator $\mathcal{L}_1$.

The iterative times, CPU and the values of λ for $\alpha = 0$, ± 0.25, $\pm\, 0.50$, $\pm\, 5.0$, $\pm\, 100$, $\pm\, 1000$ are given in Table 3. The corresponding convergent results are shown as Fig. 4, from which we can see clearly the continuous deformation of the solution about the values of α, which we denote as $s_2(x,\alpha)$. Obviously, solution $s_2(x,0)$ is similar to $\sin(x)$, although certainly $\sin(x)$ is now not a solution. It is interesting that the two solutions $s_2(x,1000)$ and $s_2(x, -\, 1000)$ are very close to the function $-\,\mathcal{S}(x)$ and $\mathcal{S}(x)$, respectively, shown as Fig. 5. This is reasonable, because $\pm\, \mathcal{S}(x)$ are obviously the two solutions of Example 2 in case that $|\alpha|$ tends to infinity. So, we have reason to believe that what we have obtained are indeed the solutions of Example 2. Note that the two solutions $s_2(x, \pm\, \alpha)$ have a kind of symmetry. In fact, we can prove that, if $s_2(x,\alpha)$ is a solution of Example 2, $-s_2(2\pi - x, -\, \alpha)$ must be also a solution. It shall be emphasized that the Example 2 does NOT contain any linear differential terms at all! But, by means of the proposed new BEM, we can still obtain very good numerical approximations for Example 2 for any value of α, even if α is very large, corresponding to a very strong nonlinearity. This means that the proposed BEM is indeed effective for quite general operators.

Until now, we have used in this paper only first-order formulation ($M = 1$). How about the high-order formulations? In case $\alpha = 0$ and using $W_0(x) = 0$, $\lambda = 0.975$, we have applied the second- and third-order formulations in Example 2, respectively. The corresponding iterative times and CPU are given in Table 4, and the error-curves about RMS_1 in iterations are shown as Fig. 6 ($\beta^2 = 0.1$) and Fig. 7 ($\beta^2 = 0.9$), respectively. It seems that higher-order formulations need fewer iterations but generally more CPU for the same value of α. Fig. 6 and Fig. 7 illustrate that the

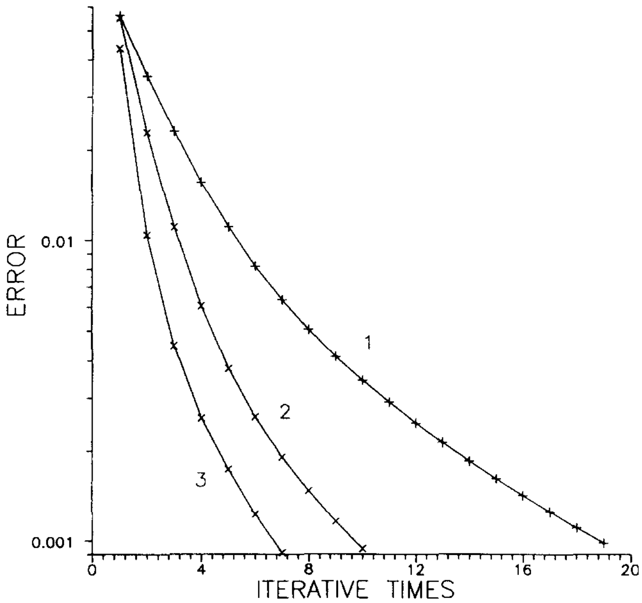

Fig. 7. The error-curve in iteration for Example 2. $\beta^2 = 0.9$, $\alpha = 0$, $W_0(x) = 0$, $\lambda = 0.975$; Curve 1: RMS_1 of first-order formulation; Curve 2: RMS_1 of second-order formulation; Curve 3: RMS_1 of third-order formulation

higher-order formulation is corresponding to a faster convergence. This can be also understood, because the high-order formulations are based on the theory of Taylor's series.

At last, let us compare the proposed BEM with that given by Tosaka[14] in 1988. Tosaka used a simple one-dimensional nonlinear equation

$$U_{xx} = \alpha^2 U^2, \quad x \in [0, 1], \tag{40}$$

which has the two boundary conditions $U(0) = 1$, $U(1) = 0.25$, to illustrate his basic ideas. Simply to say, he divided the domain [0,1] into N subdomains and then *linearized* the nonlinear eqn (40) in each subdomain. As a result, a set of $2N$ linear algebraic equations must be solved in each iteration, which certainly needs much CPU in case N is large, for instance, $N = 250$ or $N = 500$ as used in this paper. Using MODE 1 in case $\beta^2 = 1$, we have easily obtained the convergent results of eqn (40) for $1 \le \alpha^2 \le 10^8$ (Tasaka gave the results for $6 \le \alpha^2 \le 10^4$). However, we need only solve a set of TWO linear algebraic equations similar to (29) and (30) for each iteration by means of the proposed BEM. Therefore, the BEM proposed in this paper seems much more efficient than that described by Tosaka[14], because CPU is NOT directly proportional to $8N^3$ for solving a set of $2N$ linear algebraic equations but is only directly proportional to N^2 for the corresponding integral needed for the proposed BEM.

In this paper, we use COMPAQ Prolinea 4/50 as our computational tool. And double precision variables are used. The CPU given in this paper contains time for reading and writing necessary data from or to hard disk. We use, in this paper, two kinds of convergence criterion

$$RMS_1 = \sqrt{\frac{\sum_{i=0}^{N}\{\mathcal{A}[U(x_i)] - f(x_i)\}^2}{N+1}} < 10^{-3} \tag{41}$$

and

$$RMS_2 = \sqrt{\frac{\sum_{i=0}^{N}|\delta U(x_i)|^2}{N+1}} < 10^{-5}. \tag{42}$$

Iteration will be stopped if either of them is satisfied.

4 CONCLUSION

In this paper, the basic ideas published in the reference[8] are greatly generalized to give a kind of new Boundary Element Method (BEM) for quite general nonlinear differential operators which may NOT contain any linear terms at all. This kind of new BEM has the following advantages:

(A) it can be used to solve those nonlinear problems which does NOT contain any linear terms at all;

(B) in any case, we have great freedom to select a sort of proper and simple linear operator $\mathcal{L}$ whose fundamental solution is familiar to us, especially when $\mathcal{L}_0$ does not exist, or when $\mathcal{L}_0$ is so complex that its corresponding fundamental solution is unknown or difficult to obtain;

(C) the traditional BEM is only a special case of the proposed BEM so that there exists a kind of logical continuation between the traditional and the proposed BEM. This kind of logical continuation has been proved again and again to be very important in many fields of mathematics.

Two simple examples are applied to illustrate the effectiveness of the proposed method. The Example 1 indicates that, the linear operator $\mathcal{L}_0$, which is corresponding to the linear terms of the considered nonlinear problem and is very important for the traditional BEM, has now no special meaning at all – it is nothing but only a very common one of many proper linear operators suited to the proposed BEM and is mostly not the best. The Example 2 illustrates that the proposed BEM can give very good numerical approximations of a nonlinear problem which does not contain any linear operators at all, even in case that the nonlinearity is very strong. Therefore, these two examples, although very simple, illustrate that the proposed BEM is indeed effective. Other multi-dimensional examples have been done as well, e.g. the reference[8].

Note that the proposed BEM is in principle based on homotopy technique, which can generally give us greater freedom in selecting an initial solution. Now, as mentioned above, we have also great freedom in selecting a proper, simple linear operator, whose fundamental solution is familiar to us, for the proposed BEM. However, how can we use this kind of freedom? That is to say, how can we know a selected linear operator is proper and is better than another one? Certainly, for any a nonlinear problem, there should exist many proper linear operators suited to the proposed BEM, all of which would construct a mathematical space $\mathcal{S}$. It seems that there should exist the best linear operator in the space $\mathcal{S}$. But how to find out the best one? Unfortunately, we know now nearly nothing about these interesting questions. Therefore, some deep mathematical research is necessary. On the other hand, although the two examples have indeed illustrated the effectiveness of the proposed BEM, they seem to be too simple (a more complex example about 2D viscous flow has been given in the reference[8]). So, the proposed BEM must be applied to solve more complex 2D and 3D nonlinear problems in engineering so as to examine and improve it.

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APPENDIX A NOMENCLATURE

$\mathcal{A}$	quite general nonlinear differential operator;
J_1	Bessel's function of the first kind;
$\mathcal{L}$	linear differential operator;
$\mathcal{L}_0$	linear differential operator which is corresponding to all linear parts of $\mathcal{A}$;
$\mathcal{N}_0$	nonlinear differential operator which is a part of $\mathcal{A}$ but does not contain any linear operators;
N_1	Bessel's function of the second kind;
p	imbedding parameter;
R_e	Reynold's number;
$\mathcal{S}(x)$	real function defined as expression (33);
α, β, γ	parameters;
Γ	boundary of the domain Ω;
ρ	radius of convergence;
ω	fundamental solution for BEM;
Ω	integral domain.